Absolute Nörlund summability factors of power series and Fourier series

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Abstract. Four theorems of Ahmad [1] on absolute Nörlund summability factors of power series and Fourier series are proved under weaker conditions.

1. Introduction. Let \( \sum a_n \) be a given infinite series with the sequence of partial sums \( (s_n) \) and \( w_n = na_n \). By \( u^\alpha_n \) and \( t^\alpha_n \) we denote the \( n \)th Cesàro means of order \( \alpha (\alpha > -1) \) of the sequences \( (s_n) \) and \( (w_n) \), respectively. The series \( \sum a_n \) is said to be summable \(|C, \alpha|\) if (see [3])

\[
\sum_{n=1}^{\infty} |u^\alpha_n - u^\alpha_{n-1}| < \infty.
\]

Since \( t^\alpha_n = n(u^\alpha_n - u^\alpha_{n-1}) \) (see [5]) condition (1.1) can also be written as

\[
\sum_{n=1}^{\infty} \frac{1}{n} |t^\alpha_n| < \infty.
\]

Let \( (p_n) \) be a sequence of constants, real or complex, and let us write

\[
P_n = p_0 + p_1 + p_2 + \ldots + p_n \neq 0 \quad (n \geq 0).
\]

The sequence-to-sequence transformation

\[
z_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_{\nu} \quad (P_n \neq 0)
\]

defines the sequence \( (z_n) \) of Nörlund means of the sequence \( (s_n) \), generated by the sequence of coefficients \( (p_n) \). The series \( \sum a_n \) is said to be summable \(|N, p_n|\) if (see [6])

\[
\sum_{n=1}^{\infty} |z_n - z_{n-1}| < \infty.
\]

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In the special case where
\[
p_n = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)\Gamma(n + 1)} , \quad \alpha \geq 0 ,
\]
the Nörlund mean reduces to the \((C, \alpha)\) mean and \(|N, p_n|\) summability becomes \(|C, \alpha|\) summability. For \(p_n = 1\) and \(P_n = n\), we get the \((C, 1)\) mean and then \(|N, p_n|\) summability becomes \(|C, 1|\) summability.

The series \(\sum a_n\) is said to be bounded \([C, 1]\) if
\[
\sum_{\nu=1}^{n} |s_{\nu}| = O(n) \quad \text{as } n \to \infty ,
\]
and it is said to be bounded \([R, \log n, 1]\) if (see [8])
\[
\sum_{\nu=1}^{n} \frac{1}{n} |s_{\nu}| = O(\log n) \quad \text{as } n \to \infty .
\]

Let \(f(t)\) be a periodic function, with period \(2\pi\), Lebesgue integrable over \((-\pi, \pi)\), and let
\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} B_n(t).
\]

For any sequence \(X_n\) we write \(\Delta X_n = X_n - X_{n+1}, \Delta^2 X_n = \Delta(\Delta X_n)\).

2. Concerning \([C, 1]\) and \([N, p_n]\) summability Kishore [4] proved the following theorem.

**Theorem A.** Let \(p_0 > 0, p_n \geq 0\) and let \((p_n)\) be a non-increasing sequence. If \(\sum a_n\) is summable \([C, 1]\), then the series \(\sum a_n P_n(n+1)^{-1}\) is summable \([N, p_n]\).

Later Ahmad [1] proved the following theorems related to the absolute Nörlund summability factors of power series and Fourier series.

**Theorem B.** Let \((p_n)\) be as in Theorem A. If
\[
\sum_{\nu=1}^{n} \frac{1}{\nu} |t_{\nu}| = O(X_n) \quad \text{as } n \to \infty ,
\]
where \((X_n)\) is a positive non-decreasing sequence, and if the sequence \((\lambda_n)\) is such that
\[
X_n \lambda_n = O(1) ,
\]
\[
n \Delta X_n = O(X_n) ,
\]
\[
\sum n X_n |\Delta^2 \lambda_n| < \infty ,
\]
then \(\sum a_n P_n \lambda_n(n+1)^{-1}\) is summable \([N, p_n]\).
**Theorem C.** Let \((p_n)\) be as in Theorem A. If
\begin{align}
\lambda_n \log n &= O(1), \tag{2.5} \\
\sum n \log n |\Delta^2 \lambda_n| &< \infty, \tag{2.6}
\end{align}
then \(\sum B_n(x) P_n \lambda_n(n + 1)^{-1}\) is summable \(|N, p_n|\) for almost all \(x\).

**Theorem D.** Let \((p_n)\) be as in Theorem A. If \(F\) is even, \(F \in L^2(-\pi, \pi)\),
\begin{equation}
\int_0^t |F(x)|^2 \, dx = O(t) \quad \text{as} \quad t \to +0, \tag{2.7}
\end{equation}
and if \((\lambda_n)\) satisfies the same conditions as in Theorem C, then the sequence \((A_n)\) of Fourier coefficients of \(F\) has the property that \(\sum A_n P_n \lambda_n(n + 1)^{-1}\) is summable \(|N, p_n|\).

**Theorem E.** If \(f(z) = \sum c_n z^n\) is a power series of complex class \(L\) such that
\begin{equation}
\int_0^t |f(e^{i\theta})| \, d\theta = O(|t|) \quad \text{as} \quad t \to +0, \tag{2.8}
\end{equation}
and if \((\lambda_n)\) satisfies the same conditions as in Theorem C, then \(\sum c_n P_n \lambda_n(n + 1)^{-1}\) is summable \(|N, p_n|\).

3. The aim of this paper is to prove Theorems B–E under weaker conditions. Also our proofs are shorter and different from Ahmad’s [1].

Now, we shall prove the following theorems.

**Theorem 1.** Let \((p_n)\) be as in Theorem A. Let \((X_n)\) be a positive non-decreasing sequence. If conditions (2.1) and (2.2) of Theorem B are satisfied and the sequences \((\lambda_n)\) and \((\beta_n)\) are such that
\begin{align}
|\Delta \lambda_n| &\leq \beta_n, \tag{3.1} \\
\beta_n &\to 0, \tag{3.2} \\
\sum n X_n |\Delta \beta_n| &< \infty, \tag{3.3}
\end{align}
then \(\sum a_n P_n \lambda_n(n + 1)^{-1}\) is summable \(|N, p_n|\).

**Remark.** We note that it may be possible to choose \((\beta_n)\) satisfying (3.1) so that \(\Delta \beta_n\) is much smaller than \(|\Delta^2 \lambda_n|\); roughly speaking, when \((\Delta \lambda_n)\) oscillates it may be possible to choose \((\beta_n)\) so that \(|\Delta \beta_n|\) is significantly smaller than \(|\Delta^2 \lambda_n|\) so that \(\sum n X_n |\Delta \beta_n| < \infty\) is a weaker requirement than \(\sum n X_n |\Delta^2 \lambda_n| < \infty\). This fact can be verified by the following example.
Take
\[ \Delta \lambda_n = \begin{cases} 
\frac{1}{n(n+1)} & (n \text{ even}), \\
0 & (n \text{ odd}). 
\end{cases} \]
Then
\[ \Delta^2 \lambda_n = \begin{cases} 
\frac{1}{n(n+1)} & (n \text{ even}), \\
-\frac{1}{(n+1)(n+2)} & (n \text{ odd}). 
\end{cases} \]
But we can take \( \beta_n = \frac{1}{n(n+1)} \), so that \( \Delta \beta_n = 2/(n(n+1)(n+2)) \). Thus the condition (2.4) of Ahmad [1] is stronger than the condition (3.3) of our theorem.

**Theorem 2.** Let \( (p_n) \) be as in Theorem A. Suppose that \( (\lambda_n) \) and \( (\beta_n) \) satisfy conditions (3.1)–(3.2) of Theorem 1 and
\[
(3.4) \quad \lambda_n \log n = O(1), \\
(3.5) \quad \sum n \log n |\Delta \beta_n| < \infty.
\]
Then \( \sum B_n(x) P_n \lambda_n(n+1)^{-1} \) is summable \( |N,p_n| \) for almost all \( x \).

**Theorem 3.** Let \( (p_n) \) be as in Theorem A. If \( F \) is even, \( F \in L^2(-\pi,\pi) \),
\[
(3.6) \quad \int_0^t |F(x)|^2 \, dx = O(t) \quad \text{as } t \to +0,
\]
and if \( (\lambda_n) \) and \( (\beta_n) \) satisfy the same conditions as in Theorem 2, then the sequence \( (A_n) \) of Fourier coefficients of \( F \) has the property that \( \sum A_n P_n \lambda_n(n+1)^{-1} \) is summable \( |N,p_n| \).

**Theorem 4.** If \( f(z) = \sum c_n z^n \) is a power series of complex class \( L \) such that
\[
(3.7) \quad \int_0^t |f(e^{i\theta})| \, d\theta = O(|t|) \quad \text{as } t \to +0,
\]
and if \( (\lambda_n) \) and \( (\beta_n) \) satisfy the same conditions as in Theorem 2, then \( \sum c_n P_n \lambda_n(n+1)^{-1} \) is summable \( |N,p_n| \).

**4.** We need the following lemmas for the proof of our theorems.

**Lemma 1 ([7]).** Let \( (X_n) \) be a positive non-decreasing sequence and suppose that \( (\lambda_n) \) and \( (\beta_n) \) satisfy conditions (3.1)–(3.2) of Theorem 1. Then
\[
(4.1) \quad nX_n \beta_n = o(1) \quad \text{as } n \to \infty, \\
(4.2) \quad \sum X_n \beta_n < \infty.
\]
**Lemma 2 ([1]).** Let

\( t_n(x) = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu B_\nu(x) \).

Then

\( \sum_{\nu=1}^{n} \frac{1}{\nu} |t_\nu(x)| = o(\log n) \quad \text{as} \ n \to \infty, \)

for almost all \( x \).

**Lemma 3 ([9]).** Let \( F \) be even, \( F \in L^2(-\pi, \pi) \), and let \( S_n \) denote the \( n \)-th partial sum of its Fourier series at the origin. If

\( \theta \int_0^{\theta} |F(x)|^2 \, dx = O(\theta) \quad \text{as} \ \theta \to +0, \)

then \( (S_n) \) is bounded \([C, 1]\).

**Lemma 4 ([1]).** If \( \sum a_\nu \) is bounded \([C, 1]\), it is bounded \([R, \log n, 1]\).

**Lemma 5 ([8]).** If \( \sum a_\nu \) is bounded \([R, \log n, 1]\), then

\( \sum_{\nu=1}^{n} \frac{1}{\nu} |t_\nu| = O(\log n) \quad \text{as} \ n \to \infty \).

**Lemma 6 ([9]).** If \( f(z) = \sum c_\nu z^\nu \) is a power series of complex class \( L \) such that

\( \int_0^{t} |f(e^{i\theta})| \, d\theta = O(|t|) \quad \text{as} \ t \to +0, \)

then \( \sum c_\nu \) is bounded \([R, \log n, 1]\).

5. **Proof of Theorem 1.** We need only consider the special case where \((N, p_n) \) is \((C, 1)\), that is, we shall prove that \( \sum a_\nu \lambda_\nu \) is summable \([C, 1]\). Theorem 1 will then follow from Theorem A.

Let \( T_n \) be the \( n \)-th \((C, 1)\) mean of the sequence \((na_\nu \lambda_\nu)\), that is,

\( T_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu \).

Applying Abel’s transformation, we get

\[ T_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu = \frac{1}{n+1} \sum_{\nu=1}^{n} \Delta \lambda_\nu (\nu + 1) t_\nu + t_n \lambda_n = T_{n,1} + T_{n,2}, \quad \text{say.} \]
By (1.2), to complete the proof of Theorem 1, it is sufficient to show that

\[(5.2) \quad \sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}| < \infty \quad \text{for } r = 1, 2.\]

Now, we have

\[
\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}| \leq \sum_{n=2}^{m+1} \frac{1}{n(n+1)} \left\{ \sum_{\nu=1}^{n-1} \frac{\nu + 1}{\nu} \nu |\Delta \lambda_{\nu}| |t_{\nu}| \right\}
\]

\[
= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{\nu=1}^{n-1} \nu \beta_{\nu} |t_{\nu}| \right\}
\]

\[
= O(1) \sum_{\nu=1}^{m} \nu \beta_{\nu} |t_{\nu}| \sum_{n=\nu+1}^{m+1} \frac{1}{n^2} = O(1) \sum_{\nu=1}^{m} \nu \beta_{\nu} \nu^{-1} |t_{\nu}|
\]

\[
= O(1) \sum_{\nu=1}^{m-1} \Delta(\nu \beta) \sum_{r=1}^{\nu} r^{-1} |t_{r}| + O(1) m \beta_{m} \sum_{\nu=1}^{m} \nu^{-1} |t_{\nu}|
\]

\[
= O(1) \sum_{\nu=1}^{m-1} |\Delta(\nu \beta)_{\nu}| X_{\nu} + O(1) m \beta_{m} m
\]

\[
= O(1) \quad \text{as } m \to \infty,
\]

by (2.1), (3.1), (3.3), (4.1) and (4.2). Also,

\[
\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}| = \sum_{n=1}^{m} |\lambda_{n}| n^{-1} |t_{n}|
\]

\[
= \sum_{n=1}^{m-1} |\Delta \lambda_{n}| \sum_{\nu=1}^{n} \nu^{-1} |t_{\nu}| + |\lambda_{m}| \sum_{n=1}^{m} n^{-1} |t_{n}|
\]

\[
= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_{n}| X_{n} + O(1) |\lambda_{m}| X_{m}
\]

\[
= O(1) \sum_{n=1}^{m-1} \beta_{n} X_{n} + O(1) |\lambda_{m}| X_{m} = O(1) \quad \text{as } m \to \infty
\]

by (2.1), (2.2), (3.1) and (4.2). This completes the proof of Theorem 1.

6. Proof of Theorems 2–4. We obtain Theorem 2 from Theorem 1, with $X_{n} = \log n$, by an appeal to Lemma 2. Theorem 3 can be easily obtained from Theorem 1, with $X_{n} = \log n$, by successive application of
Lemmas 3, 4, and 5. Finally, we obtain Theorem 4 from Theorem 1, with $X_n = \log n$, by appealing to Lemmas 6 and 5.

References


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