ANNALES
POLONICI MATHEMATICI
LVI.1 (1991)

Absolute Nörlund summability factors of power series and Fourier series

by Hüseyin Bor (Kayseri)

Abstract. Four theorems of Ahmad [1] on absolute Nörlund summability factors of power series and Fourier series are proved under weaker conditions.

1. Introduction. Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the nth Cesàro means of order α $(\alpha > -1)$ of the sequences (s_n) and (w_n), respectively. The series $\sum a_n$ is said to be *summable* $|C, \alpha|$ if (see [3])

$$\sum_{n=1}^{\infty} |u_n^\alpha - u_{n-1}^\alpha| < \infty\,. \tag{1.1}$$

Since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [5]) condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha| < \infty\,. \tag{1.2}$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \neq 0 \qquad (n \geq 0)\,. \tag{1.3}$$

The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} s_\nu \qquad (P_n \neq 0) \tag{1.4}$$

defines the sequence (z_n) of *Nörlund means* of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be *summable* $|N, p_n|$ if (see [6])

$$\sum_{n=1}^{\infty} |z_n - z_{n-1}| < \infty\,. \tag{1.5}$$

1991 *Mathematics Subject Classification*: 40G05, 42A28, 42A24, 42A45.

In the special case where

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \qquad \alpha \geq 0, \tag{1.6}$$

the Nörlund mean reduces to the (C,α) mean and $|N,p_n|$ summability becomes $|C,\alpha|$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C,1)$ mean and then $|N,p_n|$ summability becomes $|C,1|$ summability.

The series $\sum a_n$ is said to be *bounded* $[C,1]$ if

$$\sum_{\nu=1}^{n} |s_\nu| = O(n) \qquad \text{as } n \to \infty, \tag{1.7}$$

and it is said to be *bounded* $[R, \log n, 1]$ if (see [8])

$$\sum_{\nu=1}^{n} \frac{1}{\nu}|s_\nu| = O(\log n) \qquad \text{as } n \to \infty. \tag{1.8}$$

Let $f(t)$ be a periodic function, with period 2π, Lebesgue integrable over $(-\pi,\pi)$, and let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} B_n(t). \tag{1.9}$$

For any sequence X_n we write $\Delta X_n = X_n - X_{n+1}$, $\Delta^2 X_n = \Delta(\Delta X_n)$.

2. Concerning $|C,1|$ and $|N,p_n|$ summability Kishore [4] proved the following theorem.

THEOREM A. *Let $p_0 > 0$, $p_n \geq 0$ and let (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C,1|$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N,p_n|$.*

Later Ahmad [1] proved the following theorems related to the absolute Nörlund summability factors of power series and Fourier series.

THEOREM B. *Let (p_n) be as in Theorem* A. *If*

$$\sum_{\nu=1}^{n} \frac{1}{\nu}|t_\nu| = O(X_n) \qquad \text{as } n \to \infty, \tag{2.1}$$

where (X_n) is a positive non-decreasing sequence, and if the sequence (λ_n) is such that

$$X_n \lambda_n = O(1), \tag{2.2}$$

$$n\Delta X_n = O(X_n), \tag{2.3}$$

$$\sum n X_n |\Delta^2 \lambda_n| < \infty, \tag{2.4}$$

then $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|$.

THEOREM C. *Let* (p_n) *be as in Theorem* A. *If*

$$\lambda_n \log n = O(1)\,, \tag{2.5}$$

$$\sum n \log n |\Delta^2 \lambda_n| < \infty\,, \tag{2.6}$$

then $\sum B_n(x) P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$ *for almost all* x.

THEOREM D. *Let* (p_n) *be as in Theorem* A. *If* F *is even*, $F \in L^2(-\pi,\pi)$,

$$\int_0^t |F(x)|^2\,dx = O(t) \qquad \text{as } t \to +0\,, \tag{2.7}$$

and if (λ_n) *satisfies the same conditions as in Theorem* C, *then the sequence* (A_n) *of Fourier coefficients of* F *has the property that* $\sum A_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

THEOREM E. *If* $f(z) = \sum c_n z^n$ *is a power series of complex class* L *such that*

$$\int_0^t |f(e^{i\theta})|\,d\theta = O(|t|) \qquad \text{as } t \to +0, \tag{2.8}$$

and if (λ_n) *satisfies the same conditions as in Theorem* C, *then* $\sum c_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

3. The aim of this paper is to prove Theorems B–E under weaker conditions. Also our proofs are shorter and different from Ahmad's [1].

Now, we shall prove the following theorems.

THEOREM 1. *Let* (p_n) *be as in Theorem* A. *Let* (X_n) *be a positive non-decreasing sequence. If conditions* (2.1) *and* (2.2) *of Theorem* B *are satisfied and the sequences* (λ_n) *and* (β_n) *are such that*

$$|\Delta \lambda_n| \le \beta_n\,, \tag{3.1}$$

$$\beta_n \to 0\,, \tag{3.2}$$

$$\sum n X_n |\Delta \beta_n| < \infty\,, \tag{3.3}$$

then $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

R e m a r k. We note that it may be possible to choose (β_n) satisfying (3.1) so that $\Delta\beta_n$ is much smaller than $|\Delta^2\lambda_n|$: roughly speaking, when $(\Delta\lambda_n)$ oscillates it may be possible to choose (β_n) so that $|\Delta\beta_n|$ is significantly smaller than $|\Delta^2\lambda_n|$ so that $\sum nX_n|\Delta\beta_n| < \infty$ is a weaker requirement than $\sum nX_n|\Delta^2\lambda_n| < \infty$. This fact can be verified by the following example.

Take

$$\Delta\lambda_n = \begin{cases} \dfrac{1}{n(n+1)} & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases}$$

Then

$$\Delta^2\lambda_n = \begin{cases} \dfrac{1}{n(n+1)} & (n \text{ even}), \\ \dfrac{-1}{(n+1)(n+2)} & (n \text{ odd}). \end{cases}$$

But we can take $\beta_n = 1/(n(n+1))$, so that $\Delta\beta_n = 2/(n(n+1)(n+2))$. Thus the condition (2.4) of Ahmad [1] is stronger than the condition (3.3) of our theorem.

THEOREM 2. *Let* (p_n) *be as in Theorem* A. *Suppose that* (λ_n) *and* (β_n) *satisfy conditions* (3.1)–(3.2) *of Theorem* 1 *and*

$$\lambda_n \log n = O(1)\,, \tag{3.4}$$

$$\sum n \log n |\Delta\beta_n| < \infty\,. \tag{3.5}$$

Then $\sum B_n(x)\ P_n\lambda_n(n+1)^{-1}$ *is summable* $|N,p_n|$ *for almost all* x.

THEOREM 3. *Let* (p_n) *be as in Theorem* A. *If* F *is even*, $F \in L^2(-\pi,\pi)$,

$$\int_0^t |F(x)|^2\, dx = O(t) \qquad \text{as } t \to +0, \tag{3.6}$$

and if (λ_n) *and* (β_n) *satisfy the same conditions as in Theorem* 2, *then the sequence* (A_n) *of Fourier coefficients of* F *has the property that* $\sum A_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N,p_n|$.

THEOREM 4. *If* $f(z) = \sum c_n z^n$ *is a power series of complex class* L *such that*

$$\int_0^t |f(e^{i\theta})|\, d\theta = O(|t|) \qquad \text{as } t \to +0\,, \tag{3.7}$$

and if (λ_n) *and* (β_n) *satisfy the same conditions as in Theorem* 2, *then* $\sum c_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N,p_n|$.

4. We need the following lemmas for the proof of our theorems.

LEMMA 1 ([7]). *Let* (X_n) *be a positive non-decreasing sequence and suppose that* (λ_n) *and* (β_n) *satisfy conditions* (3.1)–(3.2) *of Theorem* 1. *Then*

$$nX_n\beta_n = o(1) \qquad \text{as } n \to \infty\,, \tag{4.1}$$

$$\sum X_n \beta_n < \infty\,. \tag{4.2}$$

LEMMA 2 ([1]). *Let*

$$t_n(x) = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu B_\nu(x)\,. \tag{4.3}$$

Then

$$\sum_{\nu=1}^{n} \frac{1}{\nu}|t_\nu(x)| = o(\log n) \qquad \text{as } n \to \infty\,, \tag{4.4}$$

for almost all x.

LEMMA 3 ([9]). *Let* F *be even,* $F \in L^2(-\pi,\pi)$, *and let* S_n *denote the* n*-th partial sum of its Fourier series at the origin. If*

$$\int_0^\theta |F(x)|^2\,dx = O(\theta) \qquad \text{as } \theta \to +0\,, \tag{4.5}$$

then (S_n) *is bounded* $[C,1]$.

LEMMA 4 ([1]). *If* $\sum a_n$ *is bounded* $[C,1]$, *it is bounded* $[R,\log n,1]$.

LEMMA 5 ([8]). *If* $\sum a_n$ *is bounded* $[R,\log n,1]$, *then*

$$\sum_{\nu=1}^{n} \frac{1}{\nu}|t_\nu| = O(\log n) \qquad \text{as } n \to \infty\,. \tag{4.6}$$

LEMMA 6 ([9]). *If* $f(z) = \sum c_n z^n$ *is a power series of complex class* L *such that*

$$\int_0^t |f(e^{i\theta})|d\theta = O(|t|) \qquad \text{as } t \to +0\,, \tag{4.7}$$

then $\sum c_n$ *is bounded* $[R,\log n,1]$.

5. Proof of Theorem 1. We need only consider the special case where (N,p_n) is $(C,1)$, that is, we shall prove that $\sum a_n\lambda_n$ is summable $|C,1|$. Theorem 1 will then follow from Theorem A.

Let T_n be the nth $(C,1)$ mean of the sequence $(na_n\lambda_n)$, that is,

$$T_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu\,. \tag{5.1}$$

Applying Abel's transformation, we get

$$\begin{aligned} T_n &= \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu\lambda_\nu = \frac{1}{n+1}\sum_{\nu=1}^{n-1} \Delta\lambda_\nu(\nu+1)t_\nu + t_n\lambda_n \\ &= T_{n,1} + T_{n,2}\,, \quad \text{say.} \end{aligned}$$

By (1.2), to complete the proof of Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n}|T_{n,r}| < \infty \qquad \text{for } r = 1, 2\,. \tag{5.2}$$

Now, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n}|T_{n,1}| &\le \sum_{n=2}^{m+1} \frac{1}{n(n+1)}\Big\{\sum_{\nu=1}^{n-1} \frac{\nu+1}{\nu}\nu|\Delta\lambda_\nu|\,|t_\nu|\Big\} \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n^2}\Big\{\sum_{\nu=1}^{n-1} \nu\beta_\nu|t_\nu|\Big\} \\
&= O(1)\sum_{\nu=1}^{m} \nu\beta_\nu|t_\nu| \sum_{n=\nu+1}^{m+1} \frac{1}{n^2} = O(1)\sum_{\nu=1}^{m} \nu\beta_\nu\nu^{-1}|t_\nu| \\
&= O(1)\sum_{\nu=1}^{m-1} \Delta(\nu\beta_\nu)\sum_{r=1}^{\nu} r^{-1}|t_r| + O(1)m\beta_m\sum_{\nu=1}^{m} \nu^{-1}|t_\nu| \\
&= O(1)\sum_{\nu=1}^{m-1} |\Delta(\nu\beta_\nu)|X_\nu + O(1)m\beta_m X_m \\
&= O(1)\sum_{\nu=1}^{m-1} \nu X_\nu|\Delta\beta_\nu| + O(1)\sum_{\nu=1}^{m-1} |\beta_{\nu+1}|X_{\nu+1} + O(1)m\beta_m X_m \\
&= O(1) \qquad \text{as } m \to \infty\,,
\end{aligned}$$

by (2.1), (3.1), (3.3), (4.1) and (4.2). Also,

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n}|T_{n,2}| &= \sum_{n=1}^{m} |\lambda_n|n^{-1}|t_n| \\
&= \sum_{n=1}^{m-1} \Delta|\lambda_n|\sum_{\nu=1}^{n} \nu^{-1}|t_\nu| + |\lambda_m|\sum_{n=1}^{m} n^{-1}|t_n| \\
&= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m|X_m = O(1) \qquad \text{as } m \to \infty
\end{aligned}$$

by (2.1), (2.2), (3.1) and (4.2). This completes the proof of Theorem 1.

6. Proof of Theorems 2–4. We obtain Theorem 2 from Theorem 1, with $X_n = \log n$, by an appeal to Lemma 2. Theorem 3 can be easily obtained from Theorem 1, with $X_n = \log n$, by successive application of

Lemmas 3, 4, and 5. Finally, we obtain Theorem 4 from Theorem 1, with $X_n = \log n$, by appealing to Lemmas 6 and 5.

References

[1] Z. U. Ahmad, *Absolute Nörlund summability factors of power series and Fourier series*, Ann. Polon. Math. 27 (1972), 9–20.

[2] H. C. Chow, *On the summability factors of Fourier series*, J. London Math. Soc. 16 (1941), 215–220.

[3] M. Fekete, *Zur Theorie der divergenten Reihen*, Math. és Termes Ersitö (Budapest) 29 (1911), 719–726.

[4] N. Kishore, *On the absolute Nörlund summability factors*, Riv. Mat. Univ. Parma (2) 6 (1965), 129–134.

[5] E. Kogbetliantz, *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, Bull. Sci. Math. 49 (1925), 234–256.

[6] F. M. Mears, *Some multiplication theorems for the Nörlund mean*, Bull. Amer. Math. Soc. 41 (1935), 875–880.

[7] K. N. Mishra, *On the absolute Nörlund summability factors of infinite series*, Indian J. Pure Appl. Math. 14 (1983), 40–43.

[8] T. Pati, *Absolute Cesàro summability factors of infinite series*, Math. Z. 78 (1962), 293–297.

[9] C. T. Rajagopal, *On* $|C,1|$ *summability factors of power series and Fourier series*, ibid. 80 (1963), 265–268.

DEPARTMENT OF MATHEMATICS
ERCIYES UNIVERSITY
KAYSERI 38039, TURKEY

Reçu par la Rédaction le 10.8.1989
Révisé le 11.12.1989