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A note on integral representation of Feller kernels

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Abstract. We consider integral representations of Feller probability kernels from a Tikhonov space X into a Hausdorff space Y by continuous functions from X into Y. From the existence of such a representation for every kernel it follows that the space X has to be 0-dimensional. Moreover, both types of representations coincide in the metrizable case when in addition X is compact and Y is complete. It is also proved that the representation of a single kernel is equivalent to the existence of some non-direct product measure on the product space $Y^{\mathbb{N}}$.

Introduction. Let X and Y be Hausdorff spaces and let $\mathcal{B}_Y$ be the Borel σ-algebra in Y. A *Feller kernel* p on $X \times \mathcal{B}_Y$ is a continuous mapping $x \to p(x, \cdot)$ from X into the space of all Radon probabilities on Y endowed with the weak* topology. The set of all Feller kernels on $X \times \mathcal{B}_Y$ will be denoted by Φ.

The space $C(X, Y)$ of all continuous functions from X into Y can be embedded as a subspace of Φ. Indeed, every φ in $C(X, Y)$ defines the deterministic Feller kernel $p_\varphi(x, A) = 1_A(\varphi(x))$. It is obvious that Φ is convex and p_φ is an extreme point of Φ for every φ in $C(X, Y)$. If in addition X is separable metrizable and Y is Polish then the extreme points of Φ are exactly the deterministic Feller kernels (see [4] for details).

We endow Φ with the least σ-algebra for which all the mappings $p \to p(x, A)$ $(x \in X, A \in \mathcal{B}_Y)$ are measurable. In $C(X, Y)$ we define the least σ-algebra Σ for which the embedding $\varphi \to p_\varphi$ is measurable. In other words, Σ is the least σ-algebra which makes measurable all the evaluation mappings $\varphi \to \varphi(x)$ $(x \in X)$.

We say that the Feller kernel $p \in \Phi$ *has an integral representation on* Σ if there exists a probability measure μ on Σ such that

$$p(x, A) = \int p_\varphi(x, A)\, d\mu(\varphi) \quad (x \in X,\ A \in \mathcal{B}_Y)\,.$$

Equivalently, $p(x, \cdot) = \pi_x(\mu)$ where π_x is the evaluation map $\pi_x(\varphi) = \varphi(x)$

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on $C(X,Y)$. The above formula gives a Choquet-type integral representation for $p \in \Phi$.

In $C(X,Y)$ we can also consider the σ-algebra $\mathcal{C}$ of Borel sets for the compact-open topology in $C(X,Y)$. Clearly $\Sigma \subset \mathcal{C}$.

The integral representation problem for Feller kernels has been considered by Blumenthal and Corson in [1,2] (see also [3]–[5]). In [1] they proved the following theorem:

Let X be a 0-dimensional compact Hausdorff space and let Y be complete metrizable. Then for every Feller kernel p on $X \times \mathcal{B}_Y$ there is a Radon measure μ on $\mathcal{C}$ such that $p(x,\cdot) = \pi_x(\mu)$ for all x in X.

Hence if X and Y satisfy the assumptions of the above theorem, the existence of the integral representation on Σ also follows for every $p \in \Phi$.

In Section 1 we show that the 0-dimensionality assumption on X is in fact necessary in the Blumenthal and Corson theorem and we prove that the representation of every $p \in \Phi$ by means of a Radon measure on $\mathcal{C}$ is in fact equivalent to the integral representation on Σ for every $p \in \Phi$ under rather mild conditions on X and Y.

Section 2 shows that the existence of an integral representation on Σ for a single Feller kernel is equivalent to the existence of a certain non-direct product measure on $Y^{\mathbb{N}}$.

1. Necessary conditions for integral representation. We begin by showing that the 0-dimensionality assumption on X in the Blumenthal–Corson integral representation theorem is in fact necessary. This makes precise a remark in [1], p. 194.

Indeed, assume that X is a Tikhonov space and Y is a Hausdorff space containing at least two points. We prove that if every Feller kernel p on $X \times \mathcal{B}_Y$ has an integral representation by a Radon measure μ on $\mathcal{C}$ then X is 0-dimensional. To this end, take an open neighbourhood U of x_0 in X. Without loss of generality we may assume $U \neq X$. Fix a continuous function g from X into the unit interval such that $g(x_0) = 1$ and $g(x) = 0$ on $X \setminus U$. Then $x_0 \in Z(1-g) \subset U$ and $Z(g) \cap Z(1-g) = \emptyset$, where $Z(h)$ denotes the zero set of h.

For any two different points y and z in Y we define a Feller kernel p by

$$p(x,\cdot) = g(x)\delta_y + (1-g(x))\delta_z$$

and take a probability Radon measure μ on $\mathcal{C}$ which represents p. Now, since μ is Radon, we have $\mu(\{\varphi : \varphi(X) \subset \{y,z\}\}) = 1$ and clearly $\mu(\{\varphi : \varphi(x) = y\}) = 1$ on $Z(1-g)$ while $\mu(\{\varphi : \varphi(x) = z\}) = 1$ on $Z(g)$. Hence there is a mapping $\varphi \in C(X,Y)$ such that $\varphi(Z(g)) = \{z\}$, $\varphi(Z(1-g)) = \{y\}$ and $\varphi(X) = \{y,z\}$. This gives a partition of X into two closed-and-open sets V,

W such that $Z(1-g) \subset V$ and $Z(g) \subset W$. Finally, since $x_0 \in V \subset U$, we see that X is 0-dimensional.

In general $\Sigma \neq \mathcal{C}$, so there is no reason for the measure μ on Σ which represents $p \in \Phi$ to have an extension to some Radon measure on the larger σ-algebra $\mathcal{C}$. Nevertheless, we have a similar result for integral representation on Σ under an additional separability condition.

THEOREM 1. *Let X be a separable metrizable space and let Y be Hausdorff with at least two elements. If every Feller kernel on $X \times \mathcal{B}_Y$ has an integral representation on Σ then X is* 0*-dimensional.*

Proof. Let g and p be as in the above proof and assume that p has an integral representation on Σ. By using, instead of the Radon property, the fact that $X, Z(g)$ and $Z(1-g)$ are separable, we obtain as before $\varphi(X) = \{y, z\}$, $\varphi(Z(1-g)) = \{y\}$ and $\varphi(Z(g)) = \{z\}$ for some $\varphi \in C(X, Y)$. This yields the 0-dimensionality of X.

Now by combining the Blumenthal–Corson theorem and Theorem 1 we have

COROLLARY. *Let X and Y be metric spaces with X compact and Y complete. Assume Y has at least two elements. Then the following conditions are equivalent*:

(1) *X is* 0*-dimensional.*

(2) *Every $p \in \Phi$ has an integral representation on $\mathcal{C}$ by a Radon measure.*

(3) *Every $p \in \Phi$ has an integral representation on Σ.*

2. Integral representation of Feller kernels. Let X be an infinite separable Hausdorff space and let Y be metrizable. For every $\varphi \in C(X, Y)$ let $T\varphi = (\varphi(x_1), \varphi(x_2), \ldots) \in Y^{\mathbb{N}}$, where $\{x_n\}$ is a fixed dense subset of X with $x_i \neq x_j$ for $i \neq j$. Then T is 1-1 but need not be onto $Y^{\mathbb{N}}$ and we denote by $\mathrm{im}(T)$ the image of $C(X, Y)$ in $Y^{\mathbb{N}}$ under T. It is easy to check that $T^{-1}(\mathcal{B}_{Y^{\mathbb{N}}}) = \Sigma$, where $\mathcal{B}_{Y^{\mathbb{N}}}$ denotes the Borel σ-algebra in $Y^{\mathbb{N}}$ endowed with the product topology.

The last observation allows us to give an alternative description of the representing measure in terms of a non-direct product measure on $\mathcal{B}_{Y^{\mathbb{N}}}$.

THEOREM 2. *Let X be an infinite separable Hausdorff space and let Y be metrizable. For every Feller kernel p on $X \times \mathcal{B}_Y$ the following conditions are equivalent*:

(1) *p has an integral representation on Σ.*

(2) *There exists a probability measure λ on $\mathcal{B}_{Y^{\mathbb{N}}}$ with n-th marginal λ_n equal to $p(x_n, \cdot)$ and the outer measure $\lambda^*(\mathrm{im}\, T)$ equal to one.*

P r o o f. (1)⇒(2). The equality $\Sigma = T^{-1}(\mathcal{B}_{Y^{\mathbb{N}}})$ implies $\lambda^*(\operatorname{im} T) = 1$ for $\lambda := T(\mu)$. Since for every $n = 1, 2, \ldots$ and $A \in \mathcal{B}_Y$ we have $\lambda_n(A) = (T(\mu))_n(A) = p(x_n, A)$, the condition (2) is satisfied.

(2)⇒(1). Note that the condition $\lambda^*(\operatorname{im} T) = 1$ allows us to define a probability measure μ on Σ such that $T(\mu) = \lambda$. In particular, for every Borel set A in Y and every n we have $\mu(\{\varphi : \varphi(x_n) \in A\}) = \lambda_n(A) = p(x_n, A)$. Fix $x_0 \in X$ and choose a sequence $z_n \to x_0$ selected from $\{x_n\}$. For any nonempty closed subset F in Y define $V_n = \{y : d(y, F) < 1/n\}$ and $F_n = \{y : d(y, F) \le 1/n\}$ where $d(y, F)$ is the distance of y from F. Since for every open (closed) set A the function $x \to p(x, A)$ is lower (upper) semicontinuous, the Fatou lemma implies

$$\int 1_F(\varphi(x_0))\, d\mu(\varphi) \le \int 1_{V_n}(\varphi(x_0))\, d\mu(\varphi) \le \int \liminf_k 1_{V_n}(\varphi(z_k))\, d\mu(\varphi)$$
$$\le \limsup_k \int 1_{V_n}(\varphi(z_k))\, d\mu(\varphi) \le \limsup_k p(z_k, F_n) \le p(x_0, F_n)$$

for every n. Consequently, for every closed set F in Y and every x in X we have $\int 1_F(\varphi(x))\, d\mu(\varphi) \le p(x, F)$. Since the left hand side is a probability measure on the metric space Y, this implies that μ in fact represents p.

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