A note on integral representation of Feller kernels

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Abstract. We consider integral representations of Feller probability kernels from a Tikhonov space $X$ into a Hausdorff space $Y$ by continuous functions from $X$ into $Y$. From the existence of such a representation for every kernel it follows that the space $X$ has to be 0-dimensional. Moreover, both types of representations coincide in the metrizable case when in addition $X$ is compact and $Y$ is complete. It is also proved that the representation of a single kernel is equivalent to the existence of some non-direct product measure on the product space $Y^N$.

Introduction. Let $X$ and $Y$ be Hausdorff spaces and let $\mathcal{B}_Y$ be the Borel $\sigma$-algebra in $Y$. A Feller kernel $p$ on $X \times \mathcal{B}_Y$ is a continuous mapping $x \to p(x, A)$ from $X$ into the space of all Radon probabilities on $Y$ endowed with the weak* topology. The set of all Feller kernels on $X \times \mathcal{B}_Y$ will be denoted by $\Phi$.

The space $C(X,Y)$ of all continuous functions from $X$ into $Y$ can be embedded as a subspace of $\Phi$. Indeed, every $\varphi$ in $C(X,Y)$ defines the deterministic Feller kernel $p_\varphi(x, A) = 1_A(\varphi(x))$. It is obvious that $\Phi$ is convex and $p_\varphi$ is an extreme point of $\Phi$ for every $\varphi$ in $C(X,Y)$. If in addition $X$ is separable metrizable and $Y$ is Polish then the extreme points of $\Phi$ are exactly the deterministic Feller kernels (see [4] for details).

We endow $\Phi$ with the least $\sigma$-algebra for which all the mappings $p \to p(x, A)$ ($x \in X, A \in \mathcal{B}_Y$) are measurable. In $C(X,Y)$ we define the least $\sigma$-algebra $\Sigma$ for which the embedding $\varphi \to p_\varphi$ is measurable. In other words, $\Sigma$ is the least $\sigma$-algebra which makes measurable all the evaluation mappings $\varphi \to \varphi(x)$ ($x \in X$).

We say that the Feller kernel $p \in \Phi$ has an integral representation on $\Sigma$ if there exists a probability measure $\mu$ on $\Sigma$ such that

$$p(x, A) = \int p_\varphi(x, A) \, d\mu(\varphi) \quad (x \in X, A \in \mathcal{B}_Y).$$

Equivalently, $p(x, \cdot) = \pi_x(\mu)$ where $\pi_x$ is the evaluation map $\pi_x(\varphi) = \varphi(x)$

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on \( C(X,Y) \). The above formula gives a Choquet-type integral representation for \( p \in \Phi \).

In \( C(X,Y) \) we can also consider the \( \sigma \)-algebra \( C \) of Borel sets for the compact-open topology in \( C(X,Y) \). Clearly \( \Sigma \subset C \).

The integral representation problem for Feller kernels has been considered by Blumenthal and Corson in [1,2] (see also [3]–[5]). In [1] they proved the following theorem:

Let \( X \) be a 0-dimensional compact Hausdorff space and let \( Y \) be complete metrizable. Then for every Feller kernel \( p \) on \( X \times B_Y \) there is a Radon measure \( \mu \) on \( C \) such that \( p(x,\cdot) = \pi_x(\mu) \) for all \( x \) in \( X \).

Hence if \( X \) and \( Y \) satisfy the assumptions of the above theorem, the existence of the integral representation on \( \Sigma \) also follows for every \( p \in \Phi \).

In Section 1 we show that the 0-dimensionality assumption on \( X \) is in fact necessary in the Blumenthal–Corson integral representation theorem and we prove that the representation of every \( p \in \Phi \) by means of a Radon measure on \( C \) is in fact equivalent to the integral representation on \( \Sigma \) for every \( p \in \Phi \) under rather mild conditions on \( X \) and \( Y \).

Section 2 shows that the existence of an integral representation on \( \Sigma \) for a single Feller kernel is equivalent to the existence of a certain non-direct product measure on \( Y^N \).

1. Necessary conditions for integral representation. We begin by showing that the 0-dimensionality assumption on \( X \) in the Blumenthal–Corson integral representation theorem is in fact necessary. This makes precise a remark in [1], p. 194.

Indeed, assume that \( X \) is a Tikhonov space and \( Y \) is a Hausdorff space containing at least two points. We prove that if every Feller kernel \( p \) on \( X \times B_Y \) has an integral representation by a Radon measure \( \mu \) on \( C \) then \( X \) is 0-dimensional. To this end, take an open neighbourhood \( U \) of \( x_0 \) in \( X \). Without loss of generality we may assume \( U \neq X \). Fix a continuous function \( g \) from \( X \) into the unit interval such that \( g(x_0) = 1 \) and \( g(x) = 0 \) on \( X \setminus U \). Then \( x_0 \in Z(1-g) \subset U \) and \( Z(g) \cap Z(1-g) = \emptyset \), where \( Z(h) \) denotes the zero set of \( h \).

For any two different points \( y \) and \( z \) in \( Y \) we define a Feller kernel \( p \) by

\[
p(x,\cdot) = g(x)\delta_y + (1-g(x))\delta_z
\]

and take a probability Radon measure \( \mu \) on \( C \) which represents \( p \). Now, since \( \mu \) is Radon, we have \( \mu(\{\varphi : \varphi(X) \subset \{y,z\}\}) = 1 \) and clearly \( \mu(\{\varphi : \varphi(x) = y\}) = 1 \) on \( Z(1-g) \) while \( \mu(\{\varphi : \varphi(x) = z\}) = 1 \) on \( Z(g) \). Hence there is a mapping \( \varphi \in C(X,Y) \) such that \( \varphi(Z(g)) = \{z\} \), \( \varphi(Z(1-g)) = \{y\} \) and \( \varphi(X) = \{y,z\} \). This gives a partition of \( X \) into two closed-and-open sets \( V \),
$W$ such that $Z(1-g) \subset V$ and $Z(g) \subset W$. Finally, since $x_0 \in V \subset U$, we see that $X$ is 0-dimensional.

In general $\Sigma \neq C$, so there is no reason for the measure $\mu$ on $\Sigma$ which represents $p \in \Phi$ to have an extension to some Radon measure on the larger $\sigma$-algebra $C$. Nevertheless, we have a similar result for integral representation on $\Sigma$ under an additional separability condition.

**Theorem 1.** Let $X$ be a separable metrizable space and let $Y$ be Hausdorff with at least two elements. If every Feller kernel on $X \times B_Y$ has an integral representation on $\Sigma$ then $X$ is 0-dimensional.

**Proof.** Let $g$ and $p$ be as in the above proof and assume that $p$ has an integral representation on $\Sigma$. By using, instead of the Radon property, the fact that $X, Z(g)$ and $Z(1-g)$ are separable, we obtain as before $\varphi(X) = \{y, z\}$, $\varphi(Z(1-g)) = \{y\}$ and $\varphi(Z(g)) = \{z\}$ for some $\varphi \in C(X, Y)$. This yields the 0-dimensionality of $X$.

Now by combining the Blumenthal–Corson theorem and Theorem 1 we have

**Corollary.** Let $X$ and $Y$ be metric spaces with $X$ compact and $Y$ complete. Assume $Y$ has at least two elements. Then the following conditions are equivalent:

1. $X$ is 0-dimensional.
2. Every $p \in \Phi$ has an integral representation on $C$ by a Radon measure.
3. Every $p \in \Phi$ has an integral representation on $\Sigma$.

2. Integral representation of Feller kernels. Let $X$ be an infinite separable Hausdorff space and let $Y$ be metrizable. For every $\varphi \in C(X, Y)$ let $T\varphi = (\varphi(x_1), \varphi(x_2), \ldots) \in Y^N$, where $\{x_n\}$ is a fixed dense subset of $X$ with $x_i \neq x_j$ for $i \neq j$. Then $T$ is 1-1 but need not be onto $Y^N$ and we denote by $\text{im}(T)$ the image of $C(X, Y)$ in $Y^N$ under $T$. It is easy to check that $T^{-1}(B_{Y^n}) = \Sigma$, where $B_{Y^n}$ denotes the Borel $\sigma$-algebra in $Y^N$ endowed with the product topology.

The last observation allows us to give an alternative description of the representing measure in terms of a non-direct product measure on $B_{Y^n}$.

**Theorem 2.** Let $X$ be an infinite separable Hausdorff space and let $Y$ be metrizable. For every Feller kernel $p$ on $X \times B_Y$ the following conditions are equivalent:

1. $p$ has an integral representation on $\Sigma$.
2. There exists a probability measure $\lambda$ on $B_{Y^n}$ with $n$-th marginal $\lambda_n$ equal to $p(x_n, \cdot)$ and the outer measure $\lambda^*(\text{im}T)$ equal to one.
Proof. (1)⇒(2). The equality \( \Sigma = T^{-1}(\mathcal{B}_Y) \) implies \( \lambda^*(\text{im} T) = 1 \) for \( \lambda := T(\mu) \). Since for every \( n = 1, 2, \ldots \) and \( A \in \mathcal{B}_Y \) we have \( \lambda_n(A) = (T(\mu))_n(A) = p(x_n, A) \), the condition (2) is satisfied.

(2)⇒(1). Note that the condition \( \lambda^*(\text{im} T) = 1 \) allows us to define a probability measure \( \mu \) on \( \Sigma \) such that \( T(\mu) = \lambda \). In particular, for every Borel set \( A \) in \( Y \) and every \( n \) we have \( \mu(\{ \varphi : \varphi(x_n) \in A \}) = \lambda_n(A) = p(x_n, A) \).

Fix \( x_0 \in X \) and choose a sequence \( z_n \to x_0 \) selected from \( \{x_n\} \).

For any nonempty closed subset \( F \) in \( Y \) define \( V_n = \{ y : d(y, F) < 1/n \} \) and \( F_n = \{ y : d(y, F) \leq 1/n \} \) where \( d(y, F) \) is the distance of \( y \) from \( F \).

Since for every open (closed) set \( A \) the function \( x \to p(x, A) \) is lower (upper) semicontinuous, the Fatou lemma implies

\[
\int 1_F(\varphi(x_0)) \, d\mu(\varphi) \leq \int 1_{V_n}(\varphi(x_0)) \, d\mu(\varphi) \leq \liminf_k \int 1_{V_n}(\varphi(z_k)) \, d\mu(\varphi)
\]

\[
\leq \limsup_k \int 1_{V_n}(\varphi(z_k)) \, d\mu(\varphi) \leq \limsup_k p(z_k, F_n) \leq p(x_0, F_n)
\]

for every \( n \). Consequently, for every closed set \( F \) in \( Y \) and every \( x \) in \( X \) we have \( \int 1_F(\varphi(x)) \, d\mu(\varphi) \leq p(x, F) \). Since the left hand side is a probability measure on the metric space \( Y \), this implies that \( \mu \) in fact represents \( p \).

References


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