ANNALES
POLONICI MATHEMATICI
LVI.1 (1991)

Approximation of relaxed solutions for lower semicontinuous differential inclusions

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Abstract. We construct a guided continuous selection for lsc multifunctions with decomposable values in $L^1[0,T]$. We then apply it to obtain a new result on the uniform approximation of relaxed solutions for lsc differential inclusions.

Introduction. Let K be a compact metric space. We construct a "guided" continuous selection for multifunctions $G : K \to L^1$ which are lsc (lower semicontinuous) with closed decomposable values contained in a ball. This result refines a selection theorem proved by Fryszkowski [9].

As a consequence of this abstract result we obtain an approximation property for the solution set $S(\xi)$ of a differential inclusion

$$\text{(CP)} \qquad x' \in F(t,x)\,, \quad x(0) = \xi\,,$$

where $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable multifunction which is (at least) lsc in x and has values $F(t,x)$ compact and integrably bounded. A special case of this approximation property is the well known density result of Filippov [8] and Ważewski [19], which says that provided $F(t,x)$ is Lipschitz in x the solution set $S(\xi)$ is dense in the relaxed solution set associated to the convexified problem

$$\text{(CPR)} \qquad x' \in \operatorname{co} F(t,x), \quad x(0) = \xi\,.$$

This density result was extended by Pianigiani [16] to cover the case in which F has a modulus of continuity relative to x of the Kamke type (i.e. implying uniqueness of solution for differential equations). Tolstonogov–Finogenko [18] extended further this result in order to allow measurable dependence of $F(t,x)$ on t. Bressan [4] treated the locally Lipschitz case. However, in all these papers the relationship between the density property and the uniqueness condition is somewhat hidden. In this paper we bring to light

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1991 *Mathematics Subject Classification*: 34A60, 49A50.
* Supported by Instituto Nacional de Investigação Científica, Portugal.

this relationship, by showing that the density result is a straightforward consequence of the uniqueness condition via the above-mentioned "guided" selection theorem in L^1 and a uniformly continuous selection theorem in $\mathbb{R}^n$ ([15]) applied to $\operatorname{co} F(t,x)$.

In case $F(t,x)$ is just continuous in x, a counterexample of Pliś [17] shows that the density property does not hold anymore. However, a weaker approximation property holds. In fact, let $f(t,x)$ be a selection from $\operatorname{co} F(t,x)$ which is measurable in t and continuous in x. We show in the present paper that there exists a solution $\underline{x}$ of the differential equation $x' = f(t,x)$, $x(0) = \xi$, and a sequence (x_i) of solutions to the differential inclusion $x' \in F(t,x)$, $x(0) = \xi$, such that the sequence (x_i) converges uniformly to $\underline{x}$. This result was found by Pianigiani [16] under stronger assumptions, namely for F continuous in (t,x) with values $F(t,x)$ contained in a ball of $\mathbb{R}^n$.

We use the method of continuous selections in L^1 which was initiated by Antosiewicz–Cellina [1]. For other applications and refinements of this method see Pianigiani [16], Bressan [3], Fryszkowski [9], Bressan–Colombo [5], Cellina–Marchi [6], Tolstonogov–Finogenko [18] and also [7]. Łojasiewicz jr. [13] also treats the problem (CP) with $F(t,x)$ lsc, but he uses a different method based on polygonal approximate solutions. General information on multifunctions and differential inclusions can be found in [2]. For the history of decomposable sets see Hiai–Umegaki [11], Olech [14] and [5].

After completion of this paper, I have received paper [10] in which a result similar to our Theorem 1 is proved (with different applications).

Assumptions and the selection theorem. Let I be the interval $[0,T]$ and let K be a compact metric space with distance d. Denote by L^1 the space $L^1(I,\mathbb{R}^n)$, with norm $|\cdot|_1$. A set $D \subset L^1$ is said to be *decomposable* provided the following property holds: whenever u, v are in D and χ is the characteristic function of a measurable set $S \subset I$ then the function $w := \chi u + (1-\chi)v$ is also in D.

Hypothesis (G). $G : K \to L^1$ is a multifunction and $g_* : K \to L^1$ is a function satisfying:

(a) each value $G(u)$ is closed decomposable;
(b) $\exists M : I \to \mathbb{R}^+$ integrable such that: $v \in G(u) \Rightarrow |v(t)| \le M(t)$ a.e.;
(c) $g_*(u)(t)$ is in the closed convex hull of $G(u)(t)$, $\forall u \in K$, for a.e. t;
(d) G is lsc and g_* is continuous.

Theorem 1. *Let G and g_* satisfy hypothesis* (G). *Then there exists a sequence (g_i) of continuous selections from the multifunction G such that*

$$\Big| \int_0^t [g_*(u) - g_i(u)]\, ds \Big| \le 1/i \quad \forall i \in \mathbb{N}\ \forall t \in I\ \forall u \in K\,.$$

Intermediate results and proofs. Set $L^1_+ := \{\delta \in L^1(I,\mathbb{R}) : \delta(t) \geq 0 \text{ a.e.}\}$.

Proposition 1. *Let Δ be a nonempty bounded decomposable subset of L^1_+. Then there exists a uniquely determined element δ_0 in L^1_+ such that*:

(i) $\delta \in \Delta \Rightarrow \delta_0 \leq \delta$ *a.e.*
(ii) *if $\delta_1 \in L^1_+$ satisfies "$\delta \in \Delta \Rightarrow \delta_1 \leq \delta$ a.e." then $\delta_1 \leq \delta_0$ a.e.*

Proof. Follows from Proposition 1 of Bressan–Colombo [5]. ■

For a set Δ as in Proposition 1, we define $\operatorname{Inf}\Delta$ as the unique element δ_0 in L^1_+ as stated.

Proposition 2. *Fix some element v and some closed bounded decomposable set V in L^1. Define*

$$D : L^1 \times L^1 \to L^1_+\,, \quad D(u,v)(t) := |u(t) - v(t)| \quad a.e.\,,$$
$$D(u,V) := \operatorname{Inf}\{D(u,v) : v \in V\}\,,$$
$$d_1(u,v) := \int D(u,v)(t)\,dt\,, \quad d_1(u,V) := \int D(u,V)(t)\,dt\,.$$

Then there exists a measurable multifunction $\Gamma : I \to \mathbb{R}^n$ with closed values such that $\Gamma(t) = \{v(t) : v \in V\}$. Moreover, there exists a measurable selection γ from Γ such that

$$d(u(t),\Gamma(t)) = |u(t) - \gamma(t)|\,, \quad D(u,V) = D(u,\gamma)\,, \quad d_1(u,V) = d_1(u,\gamma)\,.$$

Proof. The existence of Γ follows from Hiai–Umegaki [11]. For the existence of γ, see [2] or [12]. Finally, it is clear that if $v \in V$ then $v(t) \in \Gamma(t)$ a.e., hence $|u(t)-v(t)| \geq d(u(t),\Gamma(t)) = |u(t)-\gamma(t)|$, i.e. $D(u,V) \geq D(u,\gamma)$ a.e. Since the opposite inequality is obvious, the equality holds. ■

Proposition 3. *Let G satisfy hypothesis* (G) *and fix some $(u_0,v_0) \in \operatorname{graph} G$. Then there exists a continuous map $\varrho_{u_0v_0} : K \to L^1_+$ such that*

$$\varrho_{u_0,v_0}(u_0) = 0\,, \quad D(v_0,G(u)) \leq \varrho_{u_0v_0}(u) \quad \forall u \in K\,.$$

Proof. See Fryszkowski [9, Proposition 2.2, Lemma 3.1] or Bressan–Colombo [5, Propositions 4 and 5]. ■

To simplify the statement of the next proposition, we define a set $\Lambda^m \subset L^1(I,\mathbb{R}^m)$ which represents a partition of I into m disjoint measurable subsets. Namely, we set

$$\Lambda^m := \Big\{\lambda \in L^1(I,\mathbb{R}^m) : \lambda_i(t) \in \{0,1\} \text{ and } \sum_{i=1}^m \lambda_i(t) = 1 \text{ a.e.}\Big\}\,.$$

Proposition 4. *Let $p : K \to [0,1]^m$ be a continuous partition of unity, let $\varphi : K \to L^1(I,\mathbb{R}^l)$ be a continuous map, and fix $\varepsilon > 0$. Then there exists a continuous map $\lambda : K \to \Lambda^m$ satisfying*:

(i) $\int \lambda(u)\,d\tau = p(u)\cdot T$;
(ii) $|\int \lambda_i(u)(\tau)\varphi(u)(\tau)\,d\tau - p_i(u)\int \varphi(u)(\tau)\,d\tau| \le \varepsilon/m$;
(iii) $p_i(u)=1 \Rightarrow \lambda_i(u)\equiv 1$; $p_i(u)=0 \Rightarrow \lambda_i(u)\equiv 0$, *a.e.* $\forall u\in K\ \forall i$.

Proof. See Fryszkowski [9, Proposition 1.2]. ■

LEMMA 1. *Let G satisfy hypothesis* (G). *Then for each $\varepsilon>0$ there exists a continuous map $g: K\to L^1$ such that*

$$d_1(g(u),G(u))\le\varepsilon,\qquad \Big|\int_0^t (g(u)(\tau)-g_*(u)(\tau))\,d\tau\Big|\le\varepsilon\qquad \forall t\in I\ \forall u\in K\,.$$

Proof. Using the integrable boundedness of G we can find a partition of I into subintervals $I_j=[t_{j-1},t_j)$, $j=1,\ldots,m_1$, such that

$$\forall u\in K\ \forall v\in G(u),\qquad \Big|\int_{I_j} v\,ds\Big|\le\varepsilon/4\,,\quad j=1,\ldots,m_1\,.$$

Since g_* is continuous on K, we can find ε' such that, denoting by d the distance in K,

$$u_1,u_2\in K,\ d(u_1,u_2)<\varepsilon'\ \Rightarrow\ d_1(g_*(u_1),g_*(u_2))<\varepsilon/4\,.$$

Set $\varepsilon_1 := \frac14\min\{\varepsilon,\varepsilon'\}$, and:

$$V_j(u) := \Big\{v|_{I_j} : v\in G(u),\ \int_{I_j}(g_*(u)-v)\,ds=0\Big\},\qquad j=1,\ldots,m_1\,,$$
$$V(u) := \{v\in G(u) : v|_{I_j}\in V_j(u),\ \forall j=1,\ldots,m_1\}\,.$$

By Lyapunov's theorem on the range of vector measures (see [9]), $V_j(u)$ is nonempty $\forall j$, and since $G(u)$ is decomposable, we have $V(u)\ne\emptyset, \forall u\in K$. If we fix some $u_0\in K$ and some $v_0\in V(u_0)$ then by Proposition 3 there exists a continuous map $\varrho_{u_0v_0}$ such that $\varrho_{u_0v_0}(u_0)=0$ and $D(v_0,G(u))\le\varrho_{u_0v_0}(u)$, $\forall u\in K$; therefore the set

$$U(u_0,v_0) := \{u\in K : d(u,u_0)<\varepsilon_1, |\varrho_{u_0v_0}(u)|_1<\varepsilon_1\}$$

is an open nbd of u_0. By compactness of K, the open cover $\{U(u_0,v_0) : u_0\in K,\ v_0\in V(u_0)\}$ has a finite subcover $\{U_1,\ldots,U_m\}$, where $U_i=U(u_i,v_i)$, and:

$$u\in U_i\ \Rightarrow\ d(u,u_i)<\varepsilon_1,\ d_1(g_*(u),g_*(u_i))<\varepsilon/4\,,$$
$$v_i\in G(u_i),\quad D(v_i,G(u))\le\varrho_i(u) := \varrho_{u_iv_i}(u),\quad |\varrho_i(u)|_1<\varepsilon_1\,,$$
$$\sum_{j=1}^{m_1}\Big|\int_{I_j}(g_*(u)-v_i)\,ds\Big|\le d_1(g_*(u),g_*(u_i))+\sum_{j=1}^{m_1}\Big|\int_{I_j}(g_*(u_i)-v_i)\,ds\Big|<\varepsilon/4$$

for $i=1,\dots,m$. Let $p:K\to[0,1]^m$ be a subordinate continuous partition of unity, and apply Proposition 4 to $\varphi=(\varphi_1,\dots,\varphi_{m+mm_1n})$ defined by

$$\varphi_i=\varrho_i:=\varrho_{u_iv_i}\,,\qquad \varphi_{m+k}(u)(t)=\chi_{I_j}(t)[g_*(u)(t)-v_i(t)]_r\,,$$

for $i=1,\dots,m$, $j=1,\dots,m_1$, $r=1,\dots,n$, $k=1,\dots,mm_1n$, where $[\cdot]_r$ denotes the rth component of the vector $[\cdot]$, with ε_1/m_1 in place of ε, obtaining a continuous map $\lambda:K\to\Lambda^m$ satisfying

$$\int \lambda(u)\,d\tau=p(u)\cdot T\,;$$

$$\int \lambda_i(u)D(v_i,G(u))(\tau)\,d\tau\le p_i(u)|\varrho_i(u)|_1+\varepsilon_1/mm_1\le (p_i(u)+1/m)\varepsilon/4\,;$$

$$\Big|\int_0^{t_j}\lambda_i(u)(g_*(u)-v_i)\,ds\Big|\le\sum_{j=1}^{m_1}\Big(p_i(u)\Big|\int_{I_j}(g_*(u)-v_i)\,ds\Big|+\varepsilon_1/mm_1\Big)$$

$$\le (p_i(u)+1/m)\varepsilon/4$$

$$p_i(u)=1\ \Rightarrow\ \lambda_i(u)\equiv 1;\qquad p_i(u)=0\ \Rightarrow\ \lambda_i(u)\equiv 0\,,$$

$$\text{a.e. }\forall u\in K\text{ for }i=1,\dots,m\,.$$

Define now $g:K\to L^1$, $g(u):=\sum_{i=1}^m\lambda_i(u)v_i$. To see that g is continuous, it is enough to note that

$$|g(u)-g(\underline{u})|_1\le\sum_{i=1}^m\int|\lambda_i(u)-\lambda_i(\underline{u})||v_i|\,ds$$

$$\le\sum_{i=1}^m\int|\lambda_i(u)-\lambda_i(\underline{u})|M(s)\,ds\,,$$

and each term in this sum is the integral of M over a set of measure $\int|\lambda_i(u)-\lambda_i(\underline{u})|\,ds$, which clearly tends to 0 as $u\to\underline{u}$, since $\lambda_i:K\to L^1$ is continuous, $\forall j$. Moreover,

$$\Big|\int_0^t(g_*(u)-g(u))\,ds\Big|\le\Big|\int_{t_{j(t)}}^t g_*(u)\,ds\Big|+\Big|\int_{t_{j(t)}}^t g(u)\,ds\Big|$$

$$+\sum_{i=1}^m\Big|\int_0^{t_{j(t)}}\lambda_i(u)[g_*(u)-v_i]\,ds\Big|$$

$$\le\frac14\varepsilon+\frac14\varepsilon+\frac14\varepsilon\sum_{i=1}^m(p_i(u)+1/m)=\varepsilon\,.$$

To see that g is an ε-approximate selection from G, recall that by Proposition 2, $\forall u\in K\ \exists v_i(u)\in G(u)$ such that $D(v_i,v_i(u))=D(v_i,G(u))$, $i=1,\dots,m$, so that, setting $v(u) := \sum_{i=1}^m \lambda_i(u) v_i(u) \in G(u)$, $\forall u \in K$, we have

$$D(g(u), G(u)) \le D(g(u), v(u)) = \sum_{i=1}^{m} \lambda_i(u) D(v_i, v_i(u))$$
$$= \sum_{i=1}^{m} \lambda_i(u) D(v_i, G(u)) .$$

Therefore

$$d_1(g(u), G(u)) \le \frac{1}{4}\varepsilon \sum_{i=1}^{m} [p_i(u) + 1/m] \le \varepsilon \qquad \forall u \in K . \ \blacksquare$$

LEMMA 2. *Let G satisfy hypothesis* (G). *Let $g^{k-1} : K \to L^1$ be a continuous map satisfying $d_1(g^{k-1}(u), G(u)) \le \varepsilon_{k-1}$ for some $\varepsilon_{k-1} > 0$. Then for any $0 < \varepsilon_k < \varepsilon_{k-1}$ there exists a continuous map $g^k : K \to L^1$ such that*

$$d_1(g^k(u), G(u)) \le \varepsilon_k , \qquad d_1(g^k(u), g^{k-1}(u)) \le \varepsilon_k + \varepsilon_{k-1} .$$

Proof. Since g^{k-1} is continuous on K, we can find ε' such that

$$u, \underline{u} \in K,\ d(u, \underline{u}) < \varepsilon' \Rightarrow\ d_1(g^{k-1}(u), g^{k-1}(\underline{u})) \le \varepsilon_k/2 .$$

Set $\varepsilon = \frac{1}{2}\min\{\varepsilon_k, \varepsilon'\}$ and $V(u) := \{v \in G(u) : d_1(g^{k-1}(u), v) = d_1(g^{k-1}(u), G(u))\}$; then, by Proposition 2, $V(u) \ne \emptyset$, $\forall u \in K$. As in Lemma 1, for each $u_0 \in K$ and each $v_0 \in V(u_0)$, the set

$$U(u_0, v_0) = \{u \in K : d(u, u_0) < \varepsilon , |\varrho_{u_0 v_0}(u)|_1 < \varepsilon\}$$

is an open nbd of u_0, and the rest of the proof follows the steps of the proof of Lemma 1. $\blacksquare$

Proof of Theorem 1. Choose a positive decreasing sequence (ε_k) such that $\sum \varepsilon_k = 1/(2i)$, and apply Lemma 1 with ε_0 replacing ε, obtaining a continuous map $g^0 : K \to L^1$ such that

$$d_1(g^0(u), G(u)) \le \varepsilon_0 , \qquad \Big| \int_0^t (g_*(u) - g^0(u))\, ds \Big| \le \varepsilon_0$$

$\forall t \in I$, $\forall u \in K$. For $k = 1, 2, \ldots$, apply Lemma 2, obtaining a continuous $g^k : K \to L^1$ such that

$$d_1(g^k(u), G(u)) \le \varepsilon_k , \qquad d_1(g^k(u), g^{k-1}(u)) \le \varepsilon_k + \varepsilon_{k-1} .$$

In particular, the sequence $(g^k(u))$ is Cauchy, uniformly in $u \in K$, i.e. the sequence (g^k) is a Cauchy sequence of continuous maps converging uniformly to some continuous $g_1 : K \to L^1$ satisfying

$$d_1(g_1(u), G(u)) \le d_1(g_1(u), g^k(u)) + d_1(g^k(u), G(u))$$
$$\le d_1(g_1(u), g^k(u)) + \varepsilon_k \to 0$$

as $k \to \infty$, hence $g_1(u) \in G(u)$, $\forall u \in K$. This means that g_1 is a continuous selection from G, and

$$\Big| \int_0^t [g_*(u) - g_1(u)]\, ds \Big| \le \Big| \int_0^t [g_*(u) - g^0(u)]\, ds \Big| + d_1(g^0(u), g^1(u)) + \ldots + \\ + d_1(g^{k-1}(u), g^k(u)) + d_1(g^k(u), g_1(u)) \\ \le \varepsilon_0 + (\varepsilon_0 + \varepsilon_1) + \ldots + (\varepsilon_{k-1} + \varepsilon_k) + d(g^k(u), g_1(u)) \\ \to 2 \sum \varepsilon_k = 1/i \quad \forall t \in I \ \forall u \in K\,. \ \blacksquare$$

Application to differential inclusions. Let I be the interval $[0, T]$, let Ξ be a compact convex set in $\mathbb{R}^n$ and Ω an open or closed set in $\mathbb{R}^n$.

HYPOTHESIS (F). $F : I \times \Omega \to \mathbb{R}^n$ is a multifunction such that:

(a′) the values $F(t, x)$ are compact;

(b′) $\exists I_0 \subset I$ such that $I \setminus I_0$ is a null set and $F|_{I_0 \times \Omega}$ is $\mathcal{L} \otimes \mathcal{B}$-measurable;

(c′) $\exists M : I \to \mathbb{R}^+$ integrable such that: $y \in F(t, x) \Rightarrow |y| \le M(t)$ a.e. and $d(y, \Xi) \le |M|_1 \Rightarrow y \in \Omega$;

(d′) $F(t, \cdot)$ is lsc.

COROLLARY 1. *Let F satisfy hypothesis* (F). *Let (ξ_i) be a sequence converging to some ξ_* in Ξ. Let $f(t, x)$ be a selection from* $\operatorname{co} F(t, x)$ *which is measurable in t and continuous in x. Then there exists a solution x_* of $x' = f(t, x)$, $x(0) = \xi_*$, and a sequence (x_i) of solutions of* (CP) *with $x_i(0) = \xi_i$ such that (x_i) converges uniformly to x_*.*

Consider the compact convex subset of C^0 defined by

$$K_\infty := \{x \in C^0 : x \in \mathrm{AC},\ x(0) \in \Xi,\ |x'(t)| \le M(t) \text{ a.e.}\}\,.$$

LEMMA 3. *Let F satisfy hypothesis* (F). *Let $f(t, x)$ be a selection from* $\operatorname{co} F(t, x)$, *measurable in t and continuous in x. Then the function $g_* : K_\infty \to L^1$ and the multifunction $G : K_\infty \to L^1$ defined by*

$$g_*(x)(t) := f(t, x(t)), \qquad G(x) := \{v \in L^1 : v(t) \in F(t, x(t)) \ \textit{a.e.}\}$$

satisfy hypothesis (G).

Proof. Using the results of Hiai–Umegaki [11], it is clear that we need only prove that G is lsc. To prove this notice first that for each $u \in K_\infty$ the multifunction $\Phi(t) := F(t, u(t))$ is measurable. In fact, for each closed set C in $\mathbb{R}^n$ we can write

$$\Phi^-(C) = \{t \in I : F(t, u(t)) \cap C \ne \emptyset\} \\ = \{t \in I : F(t, \xi) \cap C \ne \emptyset \text{ for some } \xi \text{ with } (t, \xi) \in \operatorname{graph}(u)\} \\ = \text{projection of } F^-(C) \cap \operatorname{graph}(u) \text{ on } I\,.$$

But, apart from a null set, this is the projection of an $\mathcal{L}\otimes\mathcal{B}$-measurable set, hence is measurable. Let C be a closed set in L^1, and consider a sequence $(u_k)\to u_0$ such that $G(u_k)\subset C$, $\forall k\in\mathbb{N}$. Fix any $v_0\in G(u_0)$; since $G(u_k)$ is closed decomposable, by Proposition 2 there exists $v_k\in G(u_k)$ such that $D(v_0,v_k)=D(v_0,G(u_k))$, hence for a.e. t we have

$$|v_0(t)-v_k(t)|=D(v_0,v_k)(t)=D(v_0,G(u_k))(t)=d(v_0(t),F(t,u_k(t)))\,;$$

but $F(t,\cdot)$ is lsc, $(u_k(t))\to u_0(t)$, and $v_0(t)\in F(t,u_0(t))$, hence $|v_0(t)-v_k(t)|\to 0$ as $k\to\infty$. This means that $d_1(v_0,v_k)\to 0$, and since $(v_k)\subset C$, we have $v_0\in C$. ∎

Proof of Corollary 1. Define g_* and G as in Lemma 3. Then by Theorem 1 there exists a sequence (g_i) of continuous selections from the multivalued Nemytskiĭ operator G associated to F such that, setting

$$h_i,h_*:K_\infty\to K_\infty,\quad h_i(x)(t)=\xi_i+\int_0^t g_i(x)(\tau)\,d\tau,$$
$$h_*(x)(t)=\xi_*+\int_0^t g_*(x)(\tau)\,dt\,,$$

we have $(h_i)\to h_*$ uniformly.

It is clear that $h_i(K_\infty)\subset K_\infty$, and that h_i is continuous. By the Schauder theorem, for each $i\in\mathbb{N}$ there exists a fixed point $x_i=h_i(x_i)$, i.e. $x_i'=g_i(x_i)\in G(x_i)$, $x_i(0)=\xi_i$. This means that $x_i'(t)\in F(t,x(t))$ a.e. Since (x_i) is a sequence in the compact K_∞, a subsequence, which we denote again by (x_i), converges to some x_*. It is clear that $x_*=h_*(x_*)$, so that $x_*'(t)=f(t,x_*(t))$ a.e. ∎

Hypothesis (K). $F:I\times\mathbb{R}^n\to\mathbb{R}^n$ is a multifunction such that:

(a) the values $F(t,x)$ are compact;

(b) $F(\cdot,x)$ is measurable;

(c) $\exists M:I\to\mathbb{R}^+$ integrable such that: $y\in F(t,x)\Rightarrow|y|\le M(t)$ for a.e. t;

(d) $\exists w:I\times\mathbb{R}^+\to\mathbb{R}^+$ such that $d(F(t,x),F(t,\underline{x}))\le w(t,|x-\underline{x}|)$, with $w(\cdot,r)$ measurable, $w(t,\cdot)$ continuous concave, $w(t,0)=0$ and $w(t,r)\le 2M(t)$, for a.e. $t\in I$;

(e) the differential equation $r'(t)=12nw(t,r)$, $r(0)=0$, has a unique AC solution on $[0,\underline{t}\,]$, for each $\underline{t}$ in $[0,T]$.

Condition (f). $f_*:I\times\mathbb{R}^n\to\mathbb{R}^n$ is a function such that:

(b′) $f_*(\cdot,x)$ is measurable;

(c′) $\exists M$ as in (c) such that $|f_*(t,x)|\le M(t)$ for a.e. t;

(d′) $\exists w$ as in (d), (e) of hypothesis (K) such that $|f_*(t,x) - f_*(t,\underline{x})| \le 12nw(t, |x - \underline{x}|)$ for a.e. t.

COROLLARY 2. *Let F satisfy hypothesis* (K). *Then for each solution x_* of the relaxed Cauchy problem* (CPR) *there exists a selection f_* from* $\operatorname{co} F(t,x)$ *satisfying condition* (f) *such that x_* is the unique solution of the differential equation*

$$x' = f_*(t,x), \qquad x(0) = \xi\,.$$

In particular, *the solution set $S(\xi)$ of* (CP) *is dense in the solution set of the relaxed Cauchy problem* (CPR).

P r o o f. As in [15, Theorem 1 and Proposition 3], we can find a function f such that $f(t,x,B) = \operatorname{co} F(t,x)$, B the unit ball in $\mathbb{R}^n$; and $u_* : I \to B$ such that $f(t, x_*(t), u_*(t)) = x_*'(t)$ a.e., in such a way that the function f_* defined by $f_*(t,x) := f(t,x,u_*(t))$ satisfies condition (f). Now apply Corollary 1 and notice that (e) of hypothesis (K) holds. ■

Acknowledgement. I wish to thank Professor Arrigo Cellina for suggesting the problem and for useful discussions.

R e m a r k. The referee pointed out the following:

(a) the interval $[0,T]$ can be replaced by a general separable measure space, just by using an isomorphism theorem, in the selection theorem;

(b) the δ_0 constructed in Proposition 1 is usually called "$\operatorname{ess\,inf} \Delta$";

(c) an interesting consequence of Theorem 1 is that the set of continuous selections from the multifunction G is weakly dense in the set of continuous selections from $\overline{\operatorname{co}}\, G$, i.e. for every g_* there exists a sequence (g_i) such that

$$\int_0^T \phi g_i(u)\, ds \to \int_0^T \phi g_*(u)\, ds$$

for every measurable bounded $\phi : [0,T] \to \mathbb{R}$ (indeed, by Theorem 1 this certainly holds when ϕ is piecewise constant).

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Reçu par la Rédaction le 15.7.1989
Révisé le 10.9.1990