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Holomorphic non-holonomic differential systems on complex manifolds

by S. Dimiev (Sofia)

Abstract. We study coherent subsheaves $\mathcal{D}$ of the holomorphic tangent sheaf of a complex manifold. A description of the corresponding $\mathcal{D}$-stable ideals and their closed complex subspaces is sketched. Our study of non-holonomicity is based on the Noetherian property of coherent analytic sheaves. This is inspired by the paper [3] which is related with some problems of mechanics.

1. Systems of holomorphic vector fields and integral subspaces. Let M be a complex ν-manifold ($\dim_{\mathbb{C}} M = \nu$). Let $\mathcal{O}_M$ be the structure sheaf of M and let $\mathcal{T}_M$ be the holomorphic tangent sheaf of M ($\mathcal{T}_M = \mathrm{Der}_{\mathbb{C}}\, \mathcal{O}_M$). We say that each submodule $\mathcal{D}$ of $\mathcal{T}_M$ of finite type defines a *holomorphic differential system of first order* or a *system of holomorphic vector fields* on M. In fact, $\mathcal{D}$ is a coherent ($\mathcal{O}_M$-coherent) subsheaf of $\mathcal{T}_M$, as $\mathcal{T}_M$ is a locally free sheaf. The local sections of $\mathcal{D}$ are differential operators of first order with holomorphic coefficients, i.e. holomorphic vector fields. For each open subset U of M, $\mathcal{D}(U)$ is an $\mathcal{O}_M$-module, i.e. if $\Delta \in \mathcal{D}(U)$ and $f \in \mathcal{O}_M(U)$ then $f\Delta \in \mathcal{D}(U)$ etc.

We denote by $\mathcal{LD}$ the minimal Lie algebra subsheaf of $\mathcal{T}_M$ which contains $\mathcal{D}$, i.e. $\mathcal{D} \subset \mathcal{LD} \subset \mathcal{T}_M$. This means that for every $p \in M$ the stalk $\mathcal{D}_p$ is contained in the stalk $(\mathcal{LD})_p$ and the following condition is satisfied: if $\mathfrak{J}$ is a Lie algebra subsheaf of $\mathcal{T}_M$ such that $\mathcal{D}_p \subset \mathfrak{J}_p$ for each $p \in M$, then $(\mathcal{LD})_p \subset \mathfrak{J}_p$ for each $p \in M$.

Let G be a subset of M. We say that the differential system $\mathcal{D}$ is *holonomic* on G iff $\mathcal{D}|G = \mathcal{LD}|G$. In the case $\mathcal{D}|G \neq \mathcal{LD}|G \subsetneq \mathcal{T}_M|G$, we say that $\mathcal{D}$ is a non-holonomic differential system. In the case $\mathcal{D}|G \neq \mathcal{LD}|G = \mathcal{T}_M|G$, we say that $\mathcal{D}$ is *completely non-holonomic*. Recall that $\mathcal{D}|G$ denotes the restriction of $\mathcal{D}$ on G.

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We also recall that a complex space X is called a *closed complex subspace* of M if there is a coherent ideal I of $\mathcal{O}_M$, $I \subset \mathcal{O}_M$, such that $X = \operatorname{supp}(\mathcal{O}_M/I)$ and $\mathcal{O}_x = (\mathcal{O}_M/I)|X$. In this case there is a canonical holomorphic map determined by the injection and denoted by $X \subset M$. The tangent space of X, denoted by TX, is defined as usual [1]. If G is an open subset of M and $\mathcal{O}_G$ is the induced structure sheaf, we assume that the ideal I is generated on G by $f_1, \ldots, f_\nu \in \mathcal{O}_G(G)$. If $(z_1, \ldots, z_\nu, s_1, \ldots, s_\nu)$ are coordinates on $G \times \mathbb{C}^\nu$, then $TX \subset G \times \mathbb{C}^\nu$ is defined as the closed subspace generated by $f_1, \ldots, f_\nu$, $(\partial f_k/\partial z_1)s_1, \ldots, (\partial f_k/\partial z_\nu)s_\nu$, $k = 1, \ldots, \nu$, where f_k and $\partial f_k/\partial z_j$ are viewed as holomorphic functions on $G \times \mathbb{C}^\nu$ via the canonical projection $G \times \mathbb{C}^\nu \to G$.

We say that X is an *integral subspace* for $\mathcal{D}$ or a *singular integral* of $\mathcal{D}$ if each vector field $\Delta \in \mathcal{D}$ admits a restriction to a vector field on X, i.e. to a vector field of the type $X \to TX$. The following proposition is well known.

Proposition 1.1. *The closed complex subspace X defined by I is an integral subspace for the differential system $\mathcal{D}$ iff the ideal I is stable relative to $\mathcal{D}$, i.e. $\mathcal{D}(I) \subset I$, which means that $\Delta(I) \subset I$ for every vector field $\Delta \in \mathcal{D}$.*

Such an ideal will be called a *$\mathcal{D}$-stable ideal*.

2. Involutive completion. If $\mathcal{A}_U$ and $\mathcal{B}_U$ are submodules of $\mathcal{T}_M$, U being an open subset of M, we denote by $[\mathcal{A}_U, \mathcal{B}_U]$ the submodule of $\mathcal{T}_M(U)$ generated by all vector fields $\Delta \in \mathcal{A}_U$, $\Delta' \in \mathcal{B}_U$ and all brackets $[\Delta, \Delta']$.

We shall consider the following increasing sequence of submodules of $\mathcal{T}_M(U)$

$$\begin{aligned} &\mathcal{D}_1(U) := \mathcal{D}(U), \quad \mathcal{D}_2(U) := [\mathcal{D}_1(U), \mathcal{D}_1(U)], \ldots, \\ &\mathcal{D}_j(U) := [\mathcal{D}_{j-1}(U), \mathcal{D}_1(U)], \ldots \end{aligned} \tag{2.1}$$

For each $j \in \mathbb{N}$ the presheaf $\mathcal{D}_j = \{\mathcal{D}_j(U), \rho_V^U\}$ (where ρ_V^U is as usual the restriction operator from U to V, $V \subset U$) is a (canonical) sheaf, which is a subsheaf of $\mathcal{T}_M$.

Since by assumption $\mathcal{D}_1$ is of finite type, the same is true for $\mathcal{D}_2$. One proves by induction that for each $j \in \mathbb{N}$ the sheaf $\mathcal{D}_j$ is of finite type. It follows that $\mathcal{D}_j$ is also an $\mathcal{O}_M$-coherent subsheaf of $\mathcal{T}_M$.

Proposition 2.2. *Every increasing sequence of coherent sheaves $\{\mathcal{D}_j\}$ on a complex space Y is stationary over any relatively compact subset of Y.*

The proof is by induction (see [2]). The proposition holds for empty spaces (of dimension less than 0). Assume it is true for all complex spaces of dimension less than $\mu \geq 0$. As $\dim_y Y \leq \mu$ is equivalent to there being an open neighborhood U of y and a finite holomorphic map $f : U \to D$, where D is a connected open set in $\mathbb{C}^\mu$, by using the reduction steps of [2]

(Ch. 5) it is enough to verify the proposition for the structure sheaf $\mathcal{O}_Y$ and for connected domains D in $\mathbb{C}^\mu$, i.e. for $\mathcal{O}_D$. Finally, we use the fact that closed complex subspaces of D are nowhere dense in D, which implies that their dimension is strictly less than μ. Indeed, in this case all sheaves $\mathcal{D}_j$ are coherent ideals. Let $\mathcal{D}_{j_0} \neq 0$. The complex space Y_{j_0} of D defined by the ideal $\mathcal{D}_{j_0}$ is different from D and according to the above remark we have $\dim Y_{j_0} < \mu$. Taking the sequence of all ideals $\mathcal{D}_j$ such that $\mathcal{D}_j \supset \mathcal{D}_{j_0}$ we conclude by the induction hypothesis that the family $\{\mathcal{D}_j\}$ is stationary over any relatively compact subset of Y. Of course, we have in mind that all ideals $\mathcal{D}_j$ with $\mathcal{D}_j \supset \mathcal{D}_{j_0}$ are coherent over Y_{j_0} in a natural way. The proof is finished.

So, for a compact subset K of M there exist integers j such that for every $p \in K$, $\mathcal{D}_j(p) = \mathcal{D}_{j+1}(p) = \ldots$ The minimal such j will be denoted by $h(K)$. In the case $h(K) = 1$, the system $\mathcal{D}$ is holonomic (or involutive) on K. If $h(K) = 1$ for all compact subsets of M, the system is holonomic on M in the usual sense. If $h(K) > 1$, the system $\mathcal{D}$ is non-holonomic on K. The integer $h(K)$ is called the index of non-holonomicity on K. In the case $(\mathcal{L}\mathcal{D})_p = (\mathcal{T}_M)_p$ for every $p \in K$, the system $\mathcal{D}$ is completely non-holonomic on K.

PROPOSITION 2.3. *Let U be an open connected domain in M and let $\{\mathcal{D}_j\}$ be the sequence* (2.1), *which is by assumption non-holonomic on U. Then the subset $\mathcal{Q}_n(U,\mathcal{D}) = \{p \in U : h(\{p\}) = n\}$, where n is a positive integer*, *is an analytic subset of U.*

Proof. Denoting by $\{\Delta_1, \ldots, \Delta_k\} = B_1$ the base of $\mathcal{D}_1(U) = \mathcal{D}(U)$, we consider the following base B_2 for $\mathcal{D}_1(U)$:

$$B_2 = \{\Delta_1, \ldots, \Delta_k, [\Delta_{j_1}, \Delta_{j_2}] : j_1, j_2 = 1, \ldots, k\}$$

(the order of the vector fields included in B_2 is fixed), etc. The base B_l is defined by induction:

$$B_l = \{\Delta_1, \ldots, \Delta_k, [\Delta_{j_1}, \Delta_{j_2}], [[\Delta_{j_1}, \Delta_{j_2}], \Delta_{j_3}], \ldots\}, \quad l \in \mathbb{N}.$$

In such a way we obtain an increasing sequence of bases $\{B_l\}$.

Now, the condition that $h(\{p\}) = n$ can be formulated by means of the last member of the base B_{n+1}. In fact, $B_{n+1} = B_n$ implies the equality

$$[\ldots[[\Delta_{j_1}, \Delta_{j_2}], \Delta_{j_3}] \ldots \Delta_{j_{n+1}}] \ldots] = \sum_{j_1,\ldots,j_n} C_{j_1\ldots j_n} \Delta_{j_1\ldots j_n}, \tag{2.4}$$

where $\Delta_{j_1\ldots j_n} \in B_n$. Having in mind that (2.4) is satisfied for every $f \in \mathcal{O}(U)$, and calculating the explicit coordinate representation of all relevant vector fields, we conclude that the coefficients on the right and left side are zero at p. But they are holomorphic functions on U and this zero-set is an analytic set in U.

3. Power $\mathcal{D}$-expansions. Locally we shall work with power $\mathcal{D}$-expansions. For this purpose we can assume that U is an open neighborhood of the origin O in $\mathbb{C}^\nu$ with coordinates $(z_1, \ldots, z_\nu)$. As in the previous paragraph, $\Delta_1, \ldots, \Delta_k$ are holomorphic vector fields on U which are generators for the $\mathcal{O}_M$-module $\mathcal{D}(U)$. The notion of power $\mathcal{D}$-expansion or power $\mathcal{D}$-series is based on the coordinate representation of the generators Δ_j:

$$\Delta_j = \sum_{i=1}^{\nu} \Delta_j^i(z) \frac{\partial}{\partial z_i}, \qquad (z_i) = z \in U,\ \Delta_j^i(z) \in \mathcal{O}(U)\,. \tag{3.1}$$

For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$ we denote by Δ^α the composition

$$\Delta^\alpha := \Delta_1^{\alpha_1} \ldots \Delta_k^{\alpha_k}\,. \tag{3.2}$$

In the case $\Delta_j = \partial/\partial z_j$ we write D^α instead of Δ^α.

We assume in the sequel that the vector fields Δ_j appear in a fixed order in the sequence $\Delta_1, \ldots, \Delta_k$.

In the case when U is a polydisc in $\mathbb{C}^\nu$ with center at the origin O and equal radii $(r_1 = \ldots = r_\nu = r)$ we have the Cauchy inequality

$$|D^\alpha g(0)| \le C\alpha!/r^{|\alpha|} \tag{3.3}$$

for every holomorphic function g on U.

Lemma 3.4 (Cauchy inequality for $\Delta^\alpha g$). *Under the above assumptions we have*

$$|(\Delta^\alpha g)(0)| \le C^{|\alpha|+1} \nu^{|\alpha|} (|\alpha|!)^2 / r^{|\alpha|}\,.$$

Proof. It is not difficult to prove by induction on the length of the multi-index α that $\Delta^\alpha(f)$ contains $|\alpha|!\nu^{|\alpha|}$ summands of the type

$$\Delta_{i_0}^{j_0}(z)(D^{\beta^1}\Delta_{i_1}^{j_1}(z)) \ldots (D^{\beta^{n-1}}\Delta_{i_{n-1}}^{j_{n-1}}(z)) D^{\beta^n} f(z)\,. \tag{3.5}$$

where $n := |\alpha|$ and $\beta^1, \ldots, \beta^n$ is a multi-index such that $|\beta^1| + \ldots + |\beta^n| = |\alpha|$. From (3.3) it follows that

$$|\Delta_{i_0}^{j_0}(0) D^{\beta^1}\Delta_{i_1}^{j_1}(0) \ldots D^{\beta^{n-1}}\Delta_{i_{n-1}}^{j_{n-1}}(0) D^{\beta^n} f(0)| \le C^{n+1}|\alpha|!/r^{|\alpha|}\,,$$

where C is the common constant in (3.3) for every pair (i, j), $j = 1, \ldots, k$ and $i = 1, \ldots, n$, i.e. for every $g = \Delta_j^i$.

Combining the above remark on the number of summands of $\Delta^\alpha(f)$ with the last inequality we obtain (3.4).

In the sequel we also need the inequality

$$|\alpha|!/\alpha! \le C_1|\alpha|\nu^{|\alpha|}\,, \tag{3.6}$$

where $\alpha! = \alpha_1! \ldots \alpha_k!$, which can be proved with the help of the Stirling formula.

Having a differential system $\mathcal{D}$ on U, we consider the formal power series

$$T_{\mathcal{D}}(f) := \sum_{\alpha} \frac{\Delta^{\alpha}(f)(0)}{|\alpha|!\nu^{|\alpha|}\alpha!} z^{2\alpha}, \tag{3.7}$$

where 2α is the multi-index $(2\alpha_1, \ldots, 2\alpha_\nu)$. On the polydisc with common radius r (i.e. for $|z_j| < r$, $j = 1, \ldots, \nu$) we have $|2\alpha| = 2|\alpha|$, etc.

In the classical case of the Frobenius system $(\partial/\partial z_1, \ldots, \partial/\partial z_\nu)$ the following remark holds. Since the ordinary Taylor expansion of f about the origin is

$$\sum_{\alpha} \frac{D^{\alpha}(f)(0)}{\alpha!} z^{\alpha},$$

we see that in the case of convergent series, $T_{\mathcal{D}}(f)$ converges faster than the ordinary Taylor expansion. In fact, in this case we have

$$T_{\mathcal{D}}(f) := \sum_{\alpha} \frac{D^{\alpha}(f)(0)}{\alpha!} z^{2\alpha}.$$

LEMMA 3.8 (Convergence lemma). *The formal power series* (3.7) *is convergent near the origin*, *i.e. on polydiscs with common radius r sufficiently small.*

Proof. The series (3.7) can be represented as an expansion into homogeneous polynomials

$$\sum_{n} \Big(\sum_{|\alpha|=n} \frac{\Delta^{\alpha}(f)(0)}{|\alpha|!\nu^{|\alpha|}\alpha!} z^{2\alpha} \Big).$$

First, we give an estimate for each homogeneous member. Having in mind that if $|A_\alpha| \le A$ then $|\sum_{|\alpha|=n} A_\alpha| \le n^{\nu} A$ (recall that ν is the number of components of the multi-index α) we get

$$\Big| \sum_{|\alpha|=n} \frac{\Delta^{\alpha}(f)(0)}{|\alpha|!\nu^{|\alpha|}\alpha!} z^{2\alpha} \Big| \le n^{\nu} \frac{C^{|\alpha|+1}|\alpha|!}{\alpha!} r^{|\alpha|} \le CC_1 n^{\nu+1}(C\nu r)^n$$

in view of (3.6).

Finally, the series (3.7) is convergent on the mentioned polydiscs with $r < 1/(C\nu)$.

4. Construction of $\mathcal{D}$-stable ideals. According to (1.1) the integral subspaces of the holomorphic differential system $(M, \mathcal{D})$ are defined by $\mathcal{D}$-stable ideals of $\mathcal{O}_M$. Denote by $(f_{r+1}, \ldots, f_\nu)$ the ideal generated by $\nu - r$ holomorphic functions f_j defined on a neighborhood U of the origin in $\mathbb{C}^\nu$. We suppose that this ideal defines a germ of integral subspace which passes through the origin ($f_{r+1}(0) = \ldots = f_\nu(0) = 0$). Let $J_{\mathcal{D}}$ be the ideal of all $f \in \mathcal{O}_M(U)$ such that $\Delta^\alpha(f)(0) = 0$ for all multi-indices α.

PROPOSITION 4.1. *Every $\mathcal{D}$-stable ideal $(f_{r+1}, \ldots, f_\nu)$ is contained in the ideal $J_{\mathcal{D}}$.*

Proof. If $(f_{r+1}, \ldots, f_\nu)$ is $\mathcal{D}$-stable, then for every $f \in (f_{r+1}, \ldots, f_\nu)$ we have $\Delta^\alpha(f) \in (f_{r+1}, \ldots, f_\nu)$, which implies that $\Delta^\alpha(f)(0) = 0$ for all multi-indices α.

In the sequel we need the notion of embedding dimension of a closed complex subspace X, and also the well known Jacobi Criterion. For every $p \in X$ there exists a smallest positive integer, denoted by $\mathrm{emb}_p X$, such that a neighborhood V of p is holomorphic to a closed complex subspace of a domain in $\mathbb{C}^{\mathrm{emb}_p X}$.

LEMMA 4.2 (Jacobi criterion). *Let X be a closed subspace of a domain $D \in \mathbb{C}^v$. If $p \in X$ and $f_1, \ldots, f_l \in \mathcal{O}(D)$ are such that*

$$\mathcal{O}_{X,p} = \mathcal{O}_{D,p}/(f_{1p}, \ldots, f_{lp})\mathcal{O}_{D,p}$$

then

$$\mathrm{emb}_p X + \mathrm{rank}_p(f_1, \ldots, f_l) = \mu\,.$$

(*Here f_{jp}, $j = 1, \ldots, l$, denote the germs of f_j at p.*)

The proof is based on the implicit function theorem.

In general, $\dim_p X \le \mathrm{emb}_p X$ for all $p \in X$. The following proposition is also well known.

LEMMA 4.3 (Criterion of smoothness). *A point $p \in X$ is smooth iff $\mathrm{emb}_p X = \dim_p X$.*

Recall that $p \in X$ is *smooth* if there exists a neighborhood of p in X which is biholomorphic to an open neighborhood in $\mathbb{C}^\mu$ for some μ.

PROPOSITION 4.4. *If the rank at the origin of the (globally non-holonomic) differential system $(M, \mathcal{D})$ is r, then*

1) *there exist $\nu - r$ convergent power $\mathcal{D}$-expansions $g_j(z_1, \ldots, z_r)$, $j = r+1, \ldots, \nu$, such that the ideal generated by $w_j - g_j(z_1, \ldots, z_r)$, $j = r+1, \ldots, \nu$, is $\mathcal{D}$-stable only if*

$$\Delta^\alpha(g_j)(0) = \Delta^\alpha(z_j)(0), \qquad j = r+1, \ldots, \nu\,,$$

for all multi-indices α,

2) *the closed complex subspace defined by the above ideal $(w_j - g_j(z_1, \ldots, z_r))$ is a complex manifold.*

Proof. As the dimension of the stalk $\mathcal{D}(0)$ of $\mathcal{D}$ does not depend on the chosen coordinates $(z_1, \ldots z_r, w_{r+1}, \ldots, w_\nu)$, we can suppose that after some renumbering, the vectors $(\Delta_1(0),\dots,\Delta_r(0))$ form a base for $\mathcal{D}(0)$. This means that the matrix

$$||\Delta^i_j(0)||,\qquad i,j=1,\dots,r\,,$$

is non-singular. By means of a suitable change of coordinates the following canonical form for the generators Δ_j can be obtained:

$$\begin{aligned}
\Delta_1 &= \frac{\partial}{\partial z_1}+\Delta^1_{r+1}\frac{\partial}{\partial z_{r+1}}+\Delta^1_\nu\frac{\partial}{\partial z_\nu}\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\Delta_r &= \frac{\partial}{\partial z_r}+\Delta^r_{r+1}\frac{\partial}{\partial z_{r+1}}+\Delta^r_\nu\frac{\partial}{\partial z_\nu}\,,\\
\Delta_{r+1} &= \qquad\Delta^{r+1}_{r+1}\frac{\partial}{\partial z_{r+1}}+\Delta^{r+1}_\nu\frac{\partial}{\partial z_\nu}\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\Delta_k &= \qquad\Delta^k_{r+1}\frac{\partial}{\partial z_{r+1}}+\Delta^k_\nu\frac{\partial}{\partial z_\nu}\,,\ z_j=w_j\,,
\end{aligned}\tag{4.5}$$

where $\Delta^i_j(0)=0$ for every $i=r+1,\dots,k$ and $j=r+1,\dots,\nu$.

Indeed, taking the inverse matrix of $||\Delta^i_j(0)||$, i.e. $||\Delta^i_j(0)||^{-1}:=||\delta^i_j||$, we introduce the new vector fields

$$\Delta'_j=\sum_{j+1}^{r}\delta^i_j\Delta_j,\qquad i=1,\dots,r\,,$$

as generators. After easy calculations, we obtain the required form for the generators.

Now, set

$$f(z_1,\dots,z_r,w_{r+1},\dots,w_\nu)=z_j-g_j(z_1,\dots,z_r),\qquad z_j=w_j\,,$$

for $j=r+1,\dots,\nu$, where the g_j are formal power series

$$g_j(z)=\sum a_\alpha\zeta^\alpha,\qquad z^2=(z_1^2,\dots,z_r^2),\ \zeta=z^2\,,$$

$\alpha:=(\alpha_1,\dots,\alpha_r,0,\dots,0)$ and $a_\alpha\in\mathbb{C}$. Having in mind (4.5) we obtain

$$\Delta_i(g_j)=\partial g_j/\partial z_i\qquad\text{for } 1\le i\le r\,.$$

Then in view of the ordinary Taylor formula, we set

$$\frac{\partial^{\alpha_1+\dots+\alpha_r}g}{\partial\zeta_1^{\alpha_1}\dots\partial\zeta_r^{\alpha_r}}(0)=a_\alpha\alpha!|\alpha|!\nu^{|\alpha|}\,.$$

It follows that

$$\Delta^\alpha(g_j)(0)=a_\alpha\alpha!|\alpha|!\nu^{|\alpha|}\,.\tag{4.6}$$

On the other hand, $\Delta^\alpha(f_j)=\Delta^\alpha(z_j)-\Delta^\alpha(g_j)$. Thus by (4.1) the ideal $(f_{r+1},\dots,f_\nu)$ is $\mathcal{D}$-stable only if $\Delta^\alpha(z_j)(0)-\Delta^\alpha(g_j)(0)=0$ for each α. In view of (4.6) $g_j(z)$ obtains the form

$$g_j(z) = \sum_\alpha \frac{\Delta^\alpha(g_j)(0)}{|\alpha|!\alpha!\nu^{|\alpha|}}\zeta^\alpha = \sum_\alpha \frac{\Delta^\alpha(z_j)(0)}{|\alpha|!\alpha!\nu^{|\alpha|}} z^{2\alpha} .$$

The local convergence follows from (3.8).

For the second statement we consider the product

$$\mathbb{C}^{\nu-r}(w_{r+1},\ldots,w_\nu)\times U ,$$

where U is an open neighborhood in $\mathbb{C}^r(z_1,\ldots,z_r)$ on which the holomorphic functions g_j are defined. Denote by Z the closed complex subspace of the above product, defined by the ideals generated by f_j. Then $\operatorname{rank}_x(f_{r+1},\ldots,f_\nu) = \nu - r$ and $\dim_x Z = r$ for all $x \in Z$. The functions $f_{r+1},\ldots,f_\nu$ generate all ideals of Z, i.e. all $J(Z)_x$ for $x \in Z$, as every analytic set A is in a canonical way a closed complex subspace with structure sheaf $(\mathcal{O}_Z/J(A))|A$. By (4.2) we get $\operatorname{emb}_x Z + \nu - r = \nu - r + r$ for all $x \in Z$. Hence $\operatorname{emb}_x Z = r = \dim_x Z$ for all $x \in Z$. By (4.3) the statement is proved.

5. Local holonomicity. Having the differential system $(M,\mathcal{D})$ take the sequence of subsheaves of $\mathcal{T}_M$

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_{h(K)} = \ldots$$

To each system $\mathcal{D}_j$ we assign the ideal of all germs f at the points p such that $\Delta^\alpha(f)(p) = 0$, where Δ^α is constructed from $\mathcal{D}_j$. We get $J_{\mathcal{D}_1} \supset J_{\mathcal{D}_2} \supset \ldots \supset J_{\mathcal{D}_{h(K)}} = \ldots$

PROPOSITION 5.1. *If an ideal I is $\mathcal{D}$-stable, then it is also $\mathcal{D}_j$-stable*, $j = 1,\ldots,h(K)$.

P r o o f. If Δ, $\Delta' \in \mathcal{D}$, we have $\Delta(I) \subset I$ and $\Delta'(I) \subset I$, which implies $(\Delta\circ\Delta' - \Delta'\circ\Delta)(I) \subset I$. So we obtain $\Delta''(I) \subset I$ for every $\Delta'' \in \mathcal{D}_2$, etc.

PROPOSITION 5.2. *The ideal $J_{\mathcal{D}_{h(K)}}$ is $\mathcal{D}_{h(K)}$-stable.*

P r o o f. It is not difficult to see that

$$\Delta^\alpha(\Delta_i(f)) = \Delta^{\alpha+\gamma}(f) + P\Delta(f) ,$$

where $\gamma = (0,\ldots,1,\ldots,0)$ (1 is in the ith position) and $P\Delta$ is a polynomial of $\Delta_1,\ldots,\Delta_k$ of degree less than $|\alpha+\gamma|$. The above equality is true because $\mathcal{D}_{h(K)}$ is a Lie algebra.

Now let $\widetilde{\Delta}_1,\ldots,\widetilde{\Delta}_k$ be a base of $\mathcal{D}_{h(K)}$. It is enough to show that $\widetilde{\Delta}_i(f) \in J_{\mathcal{D}_{h(K)}}$, $i = 1,\ldots,k$, for every $f \in J_{\mathcal{D}_{h(K)}}$. But this follows by induction, on the length of the multi-index α, from the equality

$$\Delta^\alpha(\widetilde{\Delta}_i(f))(p) = \Delta^{\alpha+\gamma}(f)(p) + P\Delta(f)(p) ,$$

since $\Delta^{\alpha+\gamma}(f)(p) = P\Delta(f)(p) = 0$.

From Propositions 5.1 and 4.1 we conclude that $(f_{r+1}, \ldots, f_\nu) \subset J_{\mathcal{D}_{h(K)}}$. Using the Weierstrass division theorem we can also prove the inverse inclusion. Indeed, if $f \in J_{\mathcal{D}_{h(K)}}$ we divide it by f_{r+1}. Since f_{r+1} is of order 1 in z_{r+1} we get $f = Q_{r+1}f_{r+1} + R_{r+1}$, where the remainder R_{r+1} does not depend on z_{r+1}. Dividing R_{r+1} by f_{r+2} and so on, we get finally $f = Q_{r+1}f_{r+1} + Q_{r+2}f_{r+2} + \ldots + Q_\nu f_\nu + R_\nu$, where R_ν is 0.

Recapitulating, we find that $(f_{r+1}, \ldots, f_\nu)$ is a $\mathcal{D}_{h(K)}$-stable ideal.

Remark. In general, the obtained result is of local character. It is interesting to construct a maximal integral subspace.

EXAMPLES 5.3. 1) Consider $(\mathbb{C}^3, \Delta)$, where $\Delta = \partial/\partial z_2 + z_1\partial/\partial z_3$. This is a holonomic holomorphic differential system whose singular integral is the closed subspace defined by $z_3 - z_1 z_2 = 0$.

2) Now we take the holomorphic differential system $(\mathbb{C}^3, \mathcal{D})$ where $\mathcal{D}$ is defined globally by the vector fields $\Delta_1 = \partial/\partial z_1$ and $\Delta_2 = \partial/\partial z_2 + z_1 z_3 \partial/\partial z_3$. It is easy to calculate that $[\Delta_1, \Delta_2] = z_3\partial/\partial z_3$ and, following the method of 4.4, that $\Delta^\alpha(g_3) = z_3\Delta^\alpha(z_1)$. So, we see that on every compact K in the vector subspace defined by $z_3 = 0$, the series g_3 is zero and $f_3 = z_3 - g_3$ is even zero on the whole subspace $z_3 = 0$. This means that $h(K) = h(z_3 = 0) = 1$, or that the maximal integral subspace is the complex manifold defined by $z_3 = 0$.

3) The system $(\mathbb{C}^3, \Delta_1 = \partial/\partial z_1, \Delta_2 = \partial/\partial z_2 + z_1\partial/\partial z_3)$ is not holonomic as $[\Delta_1, \Delta_2] = \partial/\partial z_3$. The completed system $\mathcal{D}_1 = \{\Delta_1, \Delta_2, \partial/\partial z_3\}$ defines a Lie algebra sheaf, i.e. the index of non-holonomicity is 1.

References

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INSTITUTE OF MATHEMATICS
BULGARIAN ACADEMY OF SCIENCES
P.O. BOX 373
1090 SOFIA, BULGARIA

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