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Diagonal series of rational functions

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Abstract. Some representations of Nash functions on continua in $\mathbb{C}$ as integrals of rational functions of two complex variables are presented. As a simple consequence we get close relations between Nash functions and diagonal series of rational functions.

1. Introduction. Let Ω be an open subset of $\mathbb{C}^m$. We shall use the following notation:

$\mathcal{O}(\Omega)$ – the space of all holomorphic functions on Ω,
$\mathcal{N}(\Omega)$ – the space of all Nash functions on Ω,
$\mathcal{R}(\Omega)$ – the space of all rational holomorphic functions on Ω.

For any compact subset K of $\mathbb{C}^m$ we denote by $\mathcal{O}(K)$ the space of all functions defined on K which have a holomorphic extension to an open neighbourhood of K. In the same way we define $\mathcal{N}(K)$ and $\mathcal{R}(K)$. We denote by U and T the unit disc and unit circle in $\mathbb{C}$, respectively.

The paper is organized as follows:

Section 2 is of preparatory nature. We collect in it some special properties of Nash functions of one complex variable.

In Section 3, for a continuum $K \subset \mathbb{C}$, we consider the operator

$$S : \mathcal{O}(K \times T) \ni f \mapsto S(f) = f_0 \in \mathcal{O}(K),$$

where $f(z, w) = \sum_{n\in\mathbb{Z}} f_n(z)w^n$. In particular, we prove that $S(\mathcal{R}(K \times T)) = \mathcal{N}(K)$.

In Section 4 we consider the diagonal operator

$$I : \mathcal{O}(T \times T) \ni f \mapsto I(f) \in \mathcal{O}(T)$$

defined by $I(f)(z) = \sum_{n\in\mathbb{Z}} a_{n,n}z^n$, where $f(x, y) = \sum_{p,q\in\mathbb{Z}} a_{p,q}x^p y^q$. We show that $I(\mathcal{R}(T \times T)) = \mathcal{N}(T)$ and $I(\mathcal{R}(\overline{U} \times \overline{U})) = \mathcal{N}(\overline{U})$.

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Our results were inspired by [2], [3], [4] and [6]. In particular, the last section of our paper gives a more quantitative version of Safonov's result ([6], Th. 1).

2. Simple Nash functions. Let Ω be an open subset of $\mathbb{C}^m$ and let $g \in \mathcal{O}(\Omega)$.

Definition 1. We say that g is a *Nash function at* $x_0 \in \Omega$ if there exist an open neighbourhood $U \subset \Omega$ of x_0 and a polynomial $P : \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$, $P \neq 0$, such that $P(x, g(x)) = 0$ for $x \in U$. A function g is said to be a *Nash function in* Ω if it is a Nash function at each point of Ω. We denote by $\mathcal{N}(\Omega)$ the space of all Nash functions on Ω.

We recall some basic properties of Nash functions (see e.g. [7]). The following remark is a simple consequence of the identity principle for holomorphic functions and some known facts in algebraic geometry.

Remark 1. Let D be an open connected subset of $\mathbb{C}^m$. If $g \in \mathcal{O}(D)$ and $x_0 \in D$ then the following statements are equivalent:

(1) g is a Nash function at x_0,

(2) $g \in \mathcal{N}(D)$,

(3) there exists a proper algebraic subset Z of $\mathbb{C}^m \times \mathbb{C}$ such that $g = \{(x, g(x)) \in \mathbb{C}^m \times \mathbb{C} : x \in D\} \subset Z$,

(4) there exists a unique irreducible algebraic hypersurface X in $\mathbb{C}^m \times \mathbb{C}$ such that $g \subset X$,

(5) there exists an irreducible polynomial $Q : \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$, unique up to scalars, such that $Q(x, g(x)) = 0$ for $x \in D$.

Moreover, it can be seen that X in (4) is equal to the Zariski closure $\overline{g}^Z$ of g in $\mathbb{C}^m \times \mathbb{C}$.

Now, suppose that D is an open connected subset of $\mathbb{C}^m$ and $g \in \mathcal{N}(D)$. Then

$$X_g = \overline{g}^Z \cap (D \times \mathbb{C})$$

is an analytic subset of $D \times \mathbb{C}$ of pure dimension m. It is easy to see that g is an irreducible component of X_g. We denote by Y_g the union of the other components of X_g.

Definition 2. A function $g \in \mathcal{N}(D)$ is said to be a *simple Nash function* if $g \cap Y_g = \emptyset$. We denote by ${}^\circ\mathcal{N}(D)$ the family of all simple Nash functions on D.

Observe that $g \cap Y_g = \emptyset$ if and only if each point of g is a regular point of the algebraic set $\overline{g}^Z$, and so

$${}^\circ\mathcal{N}(D) = \{g \in \mathcal{N}(D) : g \subset \operatorname{Reg}(\overline{g}^Z)\},$$

where $\operatorname{Reg}(\overline{g}^Z)$ denotes the set of regular points of $\overline{g}^Z$.

LEMMA 1. *Let D be an open connected subset of $\mathbb{C}^m$, $R \in \mathcal{R}(D)$ and $g \in \mathcal{N}(D)$. If $F_R : D \times \mathbb{C} \ni (z, w) \mapsto (z, w + R(z)) \in D \times \mathbb{C}$, then*

$$X_{g+R} = F_R(X_g) \quad \textit{and} \quad Y_{g+R} = F_R(Y_g).$$

Moreover, if $g \in {}^\circ\mathcal{N}(D)$ then $g + R \in {}^\circ\mathcal{N}(D)$.

Proof. It is easy to verify that F_R is a biholomorphism and that $X_{g+R} \subset F_R(X_g)$ for each $R \in \mathcal{R}(D)$ and $g \in \mathcal{N}(D)$.

Now, fix R and g. Suppose on the contrary that $X_{g+R} \subsetneq F_R(X_g)$. Then $X_g = X_{(g+R)+(-R)} \subset F_{-R}(X_{g+R}) \subsetneq F_{-R}(F_R(X_g)) = X_g$, which is impossible, and so $X_{g+R} = F_R(X_g)$.

The mapping F_R is a biholomorphism and $g + R = F_R(g)$, hence the second assertion of the lemma follows.

If $g \in {}^\circ\mathcal{N}(D)$ then, by definition, $g \cap Y_g = \emptyset$. We have $(g + R) \cap Y_{g+R} = F_R(g) \cap F_R(Y_g) = F_R(g \cap Y_g) = \emptyset$, and the proof is complete.

The aim of this section is to give a special characterization of Nash functions on open connected subsets of $\mathbb{C}$. We can now formulate our main result in this direction.

LEMMA 2. *Let D be an open connected subset of $\mathbb{C}$ and let $g \in \mathcal{N}(D)$. Then there exist two polynomials P, $Q \in \mathbb{C}[z]$ and $h \in {}^\circ\mathcal{N}(D)$ such that $g = Ph + Q$.*

Proof. We can certainly assume that $g \notin {}^\circ\mathcal{N}(D)$, since otherwise $g = 1 \cdot g + 0$. The set $g \cap Y_g$ is contained in the set of singular points of $\overline{g}^Z$, and so is finite.

Let $g \cap Y_g = \{(z_1, g(z_1)), \ldots, (z_k, g(z_k))\}$, $k \geq 1$. We can take radii $r_1, \ldots, r_k > 0$ and positive integers $\alpha_1, \ldots, \alpha_k$ such that:

(1) $D_j = \{z \in \mathbb{C} : |z - z_j| < r_j\} \subset D$ for $j = 1, \ldots, k$,
(2) if $z \in D_j$ and $(z, w) \in Y_g$ then $|w - g(z)| \geq |z - z_j|^{\alpha_j}$ for $j = 1, \ldots, k$.

Choose a polynomial $Q \in \mathbb{C}[z]$ satisfying

$$Q^{(s)}(z_j) = g^{(s)}(z_j) \quad \text{for } s = 0, 1, \ldots, \alpha_j,\ j = 1, \ldots, k.$$

Now, we consider the function $g_1 = g - Q$. By the definition of Q we get

(3) $g_1^{(s)}(z_j) = 0$ for $s = 0, \ldots, \alpha_j$, $j = 1, \ldots, k$.

Moreover, (1), (2) and Lemma 1 imply $g_1 \cap Y_{g_1} = \{(z_1, 0), \ldots, (z_k, 0)\}$ and

(4) there exist $\rho_j \in (0, r_j)$ such that $|w| \geq \frac{1}{2}|z - z_j|^{\alpha_j}$, provided $|z - z_j| < \rho_j$ and $(z, w) \in Y_{g_1}$ for $j = 1, \ldots, k$.

From (3) we deduce that the function

$$h(z) = g_1(z)(z - z_1)^{-(\alpha_1+1)} \dots (z - z_k)^{-(\alpha_k+1)}$$

has a holomorphic extension to D. An easy computation, based on (4), shows that $h \cap Y_h = \emptyset$ and so $h \in {}^\circ\mathcal{N}(D)$. Hence $g = Ph + Q$ where $P(z) = (z - z_1)^{\alpha_1+1} \dots (z - z_k)^{\alpha_k+1}$, which ends the proof.

We conclude this section with a useful lemma.

LEMMA 3. *Let D be an open connected subset of $\mathbb{C}$, and let G be an open relatively compact subset of D. If $a \in G$ and $g \in \mathcal{N}(D)$ then there exist $P \in \mathbb{C}[z]$, $R \in \mathcal{R}(D)$ and $h \in \mathcal{N}(D)$ such that*

(1) $h(a) = 0$,
(2) $h(G) \subset U$,
(3) $\overline{h}^Z \cap (G \times \overline{U}) = h|G$,
(4) $g = Ph + R$.

Proof. By Lemma 2, $g = P_1 h_1 + Q_1$ where P_1, $Q_1 \in \mathbb{C}[z]$ and $h_1 \in {}^\circ\mathcal{N}(D)$. By compactness of $E = \overline{G} \subset D$, there exists $d > 0$ such that $|w_1 - w_2| \geq 2d$, provided $z \in E$, $w_1 = h_1(z)$ and $(z, w_2) \in Y_{h_1}$.

The Runge Theorem shows that there exists $R_1 \in \mathcal{R}(D)$ such that $R_1(a) = h_1(a)$ and $|h_1(z) - R_1(z)| < d$ for $z \in E$. Define $h_2 = h_1 - R_1$ and observe that

(a) $h_2(a) = 0$,
(b) $|h_2(z)| < d$ for $z \in E$,
(c) $|w_1 - w_2| \geq 2d$, provided $z \in E$, $w_1 = h_2(z)$ and $(z, w_2) \in Y_{h_2}$.

Indeed, (a), (b) are obvious and (c) is a simple consequence of Lemma 1.

Now, it is easy to verify that the function $h = d^{-1} h_2$ satisfies the assertions (1)–(3) of Lemma 3, and that $P = dP_1 \in \mathbb{C}[z]$, $R = P_1 R_1 + Q_1 \in \mathcal{R}(D)$ are functions required in (4). This ends the proof.

3. Integral representations of Nash functions. Let $K \subset \mathbb{C}$ be a continuum. In this section we consider the operator

$$S : \mathcal{O}(K \times T) \mapsto \mathcal{O}(K)$$

defined by $S(f)(z) = f_0(z)$, where $f(z, w) = \sum_{n\in\mathbb{Z}} f_n(z) w^n$ is the Hartogs–Laurent series of the function f. This operator admits the following integral representation:

$$S(f)(z) = \frac{1}{2\pi i} \int_T f(z, w) \, \frac{dw}{w}.$$

The main result of this section is

THEOREM 1. $S(\mathcal{R}(K \times T)) = \mathcal{N}(K)$.

P r o o f. Let $g \in \mathcal{N}(K)$. There exist an open connected neighbourhood D of K and a function $\widetilde{g} \in \mathcal{N}(D)$ such that $g = \widetilde{g}|K$.

Let G be an open neighbourhood of K relatively compact in D. By Lemma 3 we have $\widetilde{g} = P\widetilde{h} + R$ (P, $\widetilde{h}$ and R fulfill the assertions of that lemma). Let Q be an irreducible polynomial describing the graph of $\widetilde{h}$.

As $\widetilde{h}(z)$ is the only zero in $\overline{U}$ of the holomorphic function $\mathbb{C} \ni w \mapsto Q(z,w) \in \mathbb{C}$ (with multiplicity one), we have

$$\widetilde{h}(z) = \frac{1}{2\pi i} \int_T w \frac{Q_w(z,w)}{Q(z,w)}\, dw \quad \text{for } z \in G.$$

Define

$$F(z,w) = P(z) w^2 \frac{Q_w(z,w)}{Q(z,w)} + R(z) \quad \text{for } (z,w) \in K \times T.$$

Then $S(F) = g$, $F \in \mathcal{R}(K \times T)$ and consequently $g \in S(\mathcal{R}(K \times T))$.

Now, let $f = P/Q \in \mathcal{R}(K \times T)$. There exists an open connected neighbourhood D of K such that $Q^{-1}(0) \cap (\overline{D} \times T) = \emptyset$. Let $\widetilde{f}$ denote the extension of f to $\overline{D} \times T$.

There exist a non-empty subset D_1 of D and Nash functions $\phi_1, \ldots, \phi_k \in \mathcal{N}(D_1)$ with pairwise disjoint graphs such that

$$\{(z,w) \in D_1 \times U : Q(z,w)w = 0\} = \phi_1 \cup \ldots \cup \phi_k.$$

Comparing this equality with the definition of S we see that

$$S(\widetilde{f})(z) = \sum_{i=1}^{k} \frac{1}{N!} \frac{\partial^N}{\partial w^N} \left[(w - \phi_i(z))^{N+1} \frac{P(z,w)}{wQ(z,w)} \right] (z, \phi_i(z)) \quad \text{for } z \in D_1,$$

where N is a sufficiently large integer.

But a composition of Nash mappings is a Nash mapping and a partial derivative of a Nash function is a Nash function (see [7]), so $S(\widetilde{f})|D_1 \in \mathcal{N}(D_1)$ and consequently $S(\widetilde{f})|D \in \mathcal{N}(D)$. Hence $S(f) = S(\widetilde{f})|K \in \mathcal{N}(K)$ and the proof is complete.

The following example proves that $\mathcal{R}(K \times T)$ in Theorem 1 cannot be replaced by $\mathcal{N}(K \times T)$.

Example 1. Set $f(z,w) = (1 - z/(2w))^{-1/2}(1 - w/2)^{-1/2}$. Then obviously $f \in \mathcal{N}(U \times T)$. Simple computations show that $S(f)(z) = \sum_{n \in \mathbb{N}} \binom{2n}{n}^2 64^{-n} z^n$ is a transcendental function (cf. [4], [6]).

4. Diagonal operator. In this section we consider the diagonal operator

$$I : \mathcal{O}(T \times T) \mapsto \mathcal{O}(T)$$

defined by $I(f)(z)=\sum_{n\in\mathbb{Z}} a_{n,n}z^n$ where $f(x,y)=\sum_{p,q\in\mathbb{Z}} a_{p,q}x^py^q$ is the Laurent series of f. Simple computations show that

$$I(f)(z)=\frac{1}{2\pi i}\int_T f\left(\frac{z}{w},w\right)\frac{dw}{w}.$$

THEOREM 2. $I(\mathcal{R}(T\times T))=\mathcal{N}(T)$.

Proof. The mapping $\Phi:\mathcal{O}(T\times T)\to\mathcal{O}(T\times T)$ defined by $\Phi(f)(z,w)=f(zw,w)$ is a bijection and $\Phi(\mathcal{R}(T\times T))=\mathcal{R}(T\times T)$. Now, Theorem 2 is a direct consequence of Theorem 1 (in the case $K=T$) and the obvious formula $I\circ\Phi=S$.

In view of the inclusions $\mathcal{O}(\overline{U}\times\overline{U})\subset\mathcal{O}(T\times T)$ and $\mathcal{O}(\overline{U})\subset\mathcal{O}(T)$ we can consider the operator

$$I:\mathcal{O}(\overline{U}\times\overline{U})\to\mathcal{O}(\overline{U}).$$

We end this section with the following extension of Safonov's result ([6], Th. 1).

THEOREM 3. $I(\mathcal{R}(\overline{U}\times\overline{U}))=\mathcal{N}(\overline{U})$.

Proof. As the inclusion $I(\mathcal{R}(\overline{U}\times\overline{U}))\subset\mathcal{N}(\overline{U})$ is a direct consequence of Theorem 2 it is sufficient to prove the reverse one.

Let $g\in\mathcal{N}(\overline{U})$. There exist $\delta>0$ and $\widetilde{g}\in\mathcal{N}(B(0,1+3\delta))$ such that $\widetilde{g}|\overline{U}=g$, where $B(z_0,r)=\{z\in\mathbb{C}:|z-z_0|<r\}$ for $z_0\in\mathbb{C}$, $r>0$.

By Lemma 3 there exist $P\in\mathbb{C}[z]$, $R\in\mathcal{R}(B(0,1+3\delta))$ and $h\in\mathcal{N}(B(0,1+3\delta))$ such that:

(1) $h(0)=0$,
(2) $h(B(0,1+2\delta))\subset U$,
(3) $\overline{h}^Z\cap(B(0,1+2\delta)\times\overline{U})=h|B(0,1+2\delta)$,
(4) $\widetilde{g}=Ph+R$.

Let Q be an irreducible polynomial describing the graph of h. There exists $\varepsilon>0$ such that

$$Q^{-1}(0)\cap(B(0,1+\delta)\times B(0,1+\varepsilon))=h|B(0,1+\delta).$$

The function $h(z)/z$ is holomorphic in $B(0,1+\delta)$ and $|h(z)/z|\le 1/(1+\delta)$ for $z\in B(0,1+\delta)$.

Define

$$F(x,y)=y^2\frac{Q_w(xy,y)}{Q(xy,y)}\,.$$

It is obvious that $F\in\mathcal{R}(\overline{U}\times T)$ and $I(F)=h|\overline{U}$. From the construction we deduce that

$$Q(z,w)=(w-h(z))A(z,w),$$

where A is a non-vanishing holomorphic function on $B(0,1+\delta)\times B(0,1+\varepsilon)$. Therefore

$$F(x,y)=y\frac{Q_w(xy,y)}{\left(1-x\dfrac{h(xy)}{xy}\right)A(xy,y)},$$

and consequently $F\in\mathcal{R}(\overline{U}\times\overline{U})$.

Now define

$$f(x,y)=P(xy)F(x,y)+R(xy).$$

Then $f\in\mathcal{R}(\overline{U}\times\overline{U})$ and $I(f)=g$, so the proof is complete.

Finally, look at the following example which shows that $I(\mathcal{N}(\overline{U}\times\overline{U}))\not\subset\mathcal{N}(\overline{U})$.

Example 2. Set $f(x,y)=(1-x/2)^{-1/2}(1-y/2)^{-1/2}$. Then obviously $f\in\mathcal{N}(\overline{U}\times\overline{U})$. But the diagonal $I(f)(z)=\sum_{n=0}^{\infty}\binom{2n}{n}^2 64^{-n}z^n$ is the transcendental function from Example 1.

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