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New cases of equality between p-module and p-capacity

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Abstract. Let E_0, E_1 be two subsets of the closure $\overline{D}$ of a domain D of the Euclidean n-space $\mathbb{R}^n$ and $\Gamma(E_0, E_1, D)$ the family of arcs joining E_0 to E_1 in D. We establish new cases of equality $M_p\Gamma(E_0, E_1, D) = \operatorname{cap}_p(E_0, E_1, D)$, where $M_p\Gamma(E_0, E_1, D)$ is the p-module of the arc family $\Gamma(E_0, E_1, D)$, while $\operatorname{cap}_p(E_0, E_1, D)$ is the p-capacity of E_0, E_1 relative to D and $p > 1$. One of these cases is when $p = n$, $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, $E_i = E_i' \cup E_i'' \cup E_i''' \cup F_i$, E_i' is inaccessible from D by rectifiable arcs, E_i'' is open relative to $\overline{D}$ or to the boundary ∂D of D, E_i''' is at most countable, F_i is closed $(i = 0, 1)$ and D is bounded and m-smooth on $(F_0 \cup F_1) \cap \partial D$.

Let D be a domain of the Euclidean n-space $\mathbb{R}^n$, E_0, E_1 two sets contained in the closure $\overline{D}$ of D, $\Gamma = \Gamma(E_0, E_1, D)$ the family of arcs joining E_0 to E_1 in D, and let

$$F(\Gamma) = \{\varrho : \mathbb{R}^n \to \dot{\mathbb{R}}^+ ;\, \varrho \text{ Borel measurable and } \textstyle\int \varrho \, dH^1 \geq 1 \ \forall \gamma \in \Gamma\} ,$$

where $\dot{\mathbb{R}}^+ = [0, \infty]$ and H^1 is the linear Hausdorff measure. The *p-module* of Γ is

$$M_p\Gamma = \inf_{\varrho \in F(\Gamma)} \int \varrho^p \, dm \qquad (p > 1) ,$$

where dm is the n-dimensional Lebesgue measure.

Let $E_0, E_1 \subset \overline{D}$, $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, then the *$p$-capacity of* E_0, E_1 *relative to* D is

$$\operatorname{cap}_p(E_0, E_1, D) = \inf_{u \in \mathcal{U}} \int_D |\nabla u|^p \, dm ,$$

where

$$\mathcal{U} = \{u : D \cup \overline{E}_0 \cup \overline{E}_1 \to [0, 1];\ u \text{ continuous, } u_{|D} \text{ locally lipschitzian, } u_{|\overline{E}_0} = 0,\ u_{|\overline{E}_1} = 1\} ,$$

and $\nabla u = (\partial u / \partial x^1, \ldots, \partial u / \partial x^n)$ is the gradient of u.

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When the sets E_0, E_1 are closed, we denote them by F_0 and F_1, respectively.

In this paper, continuing my earlier research, I establish that

$$M_p\Gamma(E_0, E_1, D) = \operatorname{cap}_p(E_0, E_1, D) \tag{1}$$

in several new cases, for instance when $E_0, E_1 \subset \overline{D}$, $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, $E_i = F_i \cup E_i' \cup E_i''$, where F_i $(i = 0, 1)$ is compact, E_i' is not accessible from D by rectifiable arcs and E_i'' is open relative to $\overline{D}$ or to ∂D while D is m-smooth of order $p \geq n$ on $(F_0 \cup F_1) \cap \partial D$.

I begin by recalling several preliminary results and some concepts.

A domain D is said to be *m-connected at $\xi \in \partial D$* if m is the least integer for which there exist arbitrarily small neighbourhoods U_ξ of ξ such that $U_\xi \cap D$ consists of m components.

D is *m-smooth of order $p > 1$ at $\xi \in \partial D$* if:

1° D is m-connected at ξ;

2° there exist a constant $\lambda_p > 0$ and a neighbourhood U_ξ such that $U_\xi \cap D$ consists of m components $\Delta_1, \ldots, \Delta_m$ and if V_ξ is an arbitrary neighbourhood of ξ contained in U_ξ, there exists a neighbourhood $V_\xi' \subset V_\xi$ so that $M_p\Gamma(E_0, E_1, V_\xi \cap \Delta_k) \geq \lambda_p$ whenever $E_0, E_1 \subset \Delta_k$ $(k = 1, 2, \ldots)$ are connected and $E_i \cap \partial V_\xi, E_i \cap \partial V_\xi' \neq \emptyset$ $(i = 0, 1)$.

If D is m-smooth of order p at each point of a set $E \subset \partial D$, then D is called *m-smooth of order p on E*. In the particular case $p = n$, we obtain the definition of a domain m-smooth at ξ or on E (cf. J. Hesse [6]).

Proposition 1 (P. Caraman [4], Theorem 1). *If $F_0, F_1 \subset \overline{D}$ are compact, $F_0 \cap F_1 = \emptyset$ and D is m-smooth of order $p > 1$ on $(F_0 \cup F_1) \cap \partial D$, then*

$$M_p\Gamma(F_0, F_1, D) = \operatorname{cap}_p(F_0, F_1, D)\,.$$

Arguing as in Theorem 2.23 of J. Hesse's [6] Ph.D. thesis, we deduce

Proposition 2. *If $E_0, E_1 \subset \overline{D}$, $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ and either E_0 or E_1 is bounded, then $M_p\Gamma(E_0, E_1, D) < \infty$ $(p > 1)$.*

Let $\varrho \geq 0$ be a Borel measurable function on $\mathbb{R}^n$ and, for $r \in (0, 1)$, let $E_i(r) = \{x : d(x, E_i) < r\}$ $(i = 0, 1)$. Then, let $L(\varrho, r) = \inf_\gamma \int_\gamma \varrho\, dH^1$ and $L_1(\varrho, r) = \inf_\gamma \int_\gamma \varrho\, dH^1$, where the infimum is taken over all $\gamma \in \Gamma[E_0(r), E_1(r), D]$, and $\gamma \in \Gamma[E_0, E_1(r), D]$, respectively. If $r_1 > r_2 > \ldots > 0$ and $\lim_{k\to\infty} r_k = 0$, then

$$\Gamma[E_0(r_1), E_1(r_1), D] \supset \Gamma[E_0(r_2), E_1(r_2), D] \supset \ldots\,,$$
$$\Gamma[E_0, E_1(r_1), D] \supset \Gamma[E_0, E_1(r_2), D] \supset \ldots\,,$$

implying $L(\varrho, r_1) \leq L(\varrho, r_2) \leq \ldots$ and $L_1(\varrho, r_1) \leq L_1(\varrho, r_2) \leq \ldots$ Set $L(\varrho) = \lim_{k\to\infty} L(\varrho, r_k)$ and $L_1(\varrho) = \lim_{k\to\infty} L_1(\varrho, r_k)$.

PROPOSITION 3 (P. Caraman [4], corollary to Proposition 1). *If* $E_0, E_1 \subset \overline{D}$ *and* $\varrho \in F[\Gamma(E_0, E_1, D)]$, *then* $L(\varrho) \geq 1$ *iff* $\forall \varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) \in (0, 1)$ *such that* $\int_\gamma \varrho\, dH^1 \geq 1 - \varepsilon\ \forall \gamma \in \Gamma_r[E_0(r), E_1(r), D]\ \forall r \leq \delta$, *where* Γ_r *denotes the subfamily of the rectifiable arcs of* Γ.

Remark. We observe that each of the conditions $L(\varrho) \geq 1$ and $L(\varrho, r) \geq 1 - \varepsilon$ implies $E_0 \cap E_1 = \emptyset$, and that is why we did not mention this last condition explicitly.

PROPOSITION 4 (P. Caraman [4], Lemma 1). *If* $F_0, F_1 \subset \overline{D}$ *are compact and* D *is m-smooth of order* $p > 1$ *on* $(F_0 \cup F_1) \cap \partial D$, *then* $L(\varrho) \geq 1$ $\forall \varrho \in \mathcal{A}_p = \{\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^p;\ \varrho_{|\Delta}$ *continuous and* $\varrho(x) \geq \alpha_F^\varrho > 0$ $\forall x \in F\ \forall F$ *compact*$\}$, *where* $\Delta = D - (F_0 \cup F_1)$.

A direct consequence of the preceding two propositions is

COROLLARY. *Let* $F_0, F_1 \subset \overline{D}$ *be compact*, $F_0 \cap F_1 = \emptyset$, D *m-smooth of order* $p > 1$ *on* $(F_0 \cup F_1) \cap \partial D$ *and* $\varrho \in \mathcal{A}_p$. *Then* $\forall \varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) \in (0, 1)$ *such that* $\varrho/(1 - \varepsilon) \in F\{\Gamma[F_0(r), F_1(r), D]\}\ \forall r < \delta$.

PROPOSITION 5 (P. Caraman [3], Lemma 1). *If* D_S *is a superficial domain of the sphere* $S(x_0, r)$, $E_0, E_1 \subset \overline{D}_S$, $E_0 \cap E_1 = \emptyset$ *and there exists a spherical cap* $K \subset D_S$ *of* $S(x_0, r)$ *such that* $\overline{K} \cap E_i \neq \emptyset$ $(i = 0, 1)$ *and* $\varrho : \mathbb{R}^n \to \dot{\mathbb{R}}^+$ *is Borel measurable*, *then* $\forall \varepsilon > 0$ *there exists a circular arc* $\gamma \in \Gamma(E_0, E_1, K)$ *so that*

$$\int\limits_{S(x_0,r)} \varrho^p\, d\sigma \geq \frac{(1-\varepsilon)^p b_{n,p}}{r^{p-n+1}} \Big(\int\limits_\gamma \varrho\, ds \Big)^p, \tag{2}$$

where

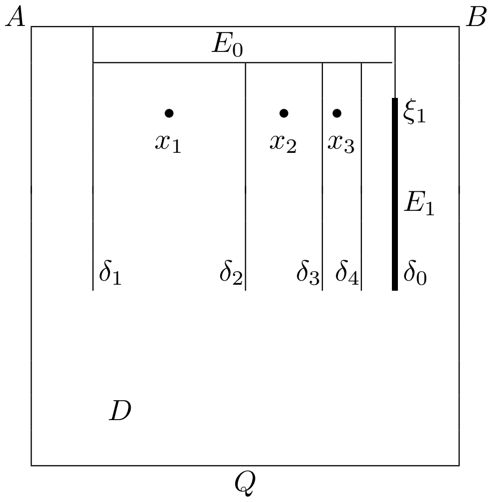
$$\begin{aligned} b_{n,p} &= \frac{\omega_{n-2}}{2^{2p-n+1}} \Bigg[\int\limits_0^\infty \frac{dt}{t^{\frac{n-2}{p-1}} (1+t)^{\frac{p-n+1}{p-1}}} \Bigg]^{1-p} \\ &\geq \frac{\omega_{n-2}}{2^{3p-n}} \left(\frac{p-n+2}{p-1} \right)^{p-n} \qquad (n > 2), \\ b_{2,p} &= 1/(2\pi)^{p-1}. \end{aligned} \tag{3}$$

A set E is said to be *open relative to another set* E' if there exists an open set G such that $E = G \cap E'$.

PROPOSITION 6 (P. Caraman [3], Lemma 2). *If* $E_0, E_1 \subset \overline{D}$ *are open relative to* $\overline{D}$ *or to* ∂D, $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ *and* $\varrho \in F[\Gamma(E_0, E_1, D)] \cap L^p$ $(p \geq n)$, *then* $\forall \varepsilon > 0$ *there exist* $b > 0$ *and two domains* $E_i^D(b)$ $(i = 0, 1)$ *such that if* $\gamma = \gamma(x_0, x_1) \subset D$ *has endpoints* $x_i \in E_i^D(b)$ $(i = 0, 1)$, *then* $\int_\gamma \varrho\, dH^1 \geq 1 - \varepsilon$.

Proposition 7 (P. Caraman [3], Theorem 1). *If* $E_0, E_1 \subset \overline{D}$ *are open relative to* $\overline{D}$ *or to* ∂D *and* $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, *then* (1) *holds for* $p \geq n$.

Remark. In the preceding proposition, it seems not to be enough to suppose that only one of the sets E_0, E_1 is open relative to $\overline{D}$ in order to have (1) $\forall p \geq n$, at least by the kind of proof used there. Indeed, in the case $n = 2$, consider a square Q (see the figure) with side length $l = 2$ and a sequence $\{\delta_k\}$ of parallel linear segments of length $1 + 2\varepsilon$ ($\varepsilon > 0$) with one endpoint belonging to the side $\overline{AB}$ of the square Q such that $d(\delta_1, \delta_2) = 2d(\delta_2, \delta_3) = 2^2 d(\delta_3, \delta_4) = \ldots$ and $\lim_{k\to\infty} \delta_k = \delta_0$.



Set $D = Q - \bigcup_{k=0}^{\infty} \delta_k$ and let E_0 be the rectangle open relative to $\overline{D}$, with one side on $\overline{AB}$ and the sides perpendicular to $\overline{AB}$ contained in δ_0 and δ_1 respectively, and having length ε. Next, let E_1 be the closed linear segment contained in δ_0 of length 1 and having its endpoints at distance 2ε and $1 + 2\varepsilon$, respectively, from $\overline{AB}$. Finally, let ϱ_0 be the characteristic function of D:

$$\varrho_0(x) = \begin{cases} 1 & \text{for } x \in D, \\ 0 & \text{for } x \in CD. \end{cases}$$

Clearly, $\varrho_0 \in F[\Gamma(E_0, E_1, D)]$. Now, let u_0 be the potential of ϱ_0, i.e. $u_0(x) = \inf_\gamma \int_\gamma \varrho_0 \, dH^1$, where the infimum is taken over all rectifiable $\gamma = \gamma(x, E_0)$ joining x to E_0 in D, and let $\{x_k\}$ be a sequence of points tending to ξ_1 in D, where ξ_1 is the endpoint of E_1 at distance 2ε of $\overline{AB}$, such that $d(x_k, E_0) = \varepsilon$. Then $u_0(x_k) = \int_{\lambda_k} \varrho_0 \, dt = \int_{\lambda_k} dt = \varepsilon$, where $\lambda_k \perp \overline{AB}$ is the linear segment joining x_k to E_0, hence $\lim_{k\to\infty} u_0(x_k) = \varepsilon$. On the other hand, $u_0(\xi_1) = \inf_\gamma \int_\gamma \varrho_0 \, dH^1 = \inf_\gamma \int_\gamma dH^1 = \inf_\gamma H^1(\gamma) > 1$, where the infimum is taken over all rectifiable arcs joining ξ_1 to E_0 in D, so that u_0 obtained in this way is not continuous in $D \cup E_0 \cup E_1$ and thus it is not admissible for $\operatorname{cap}_p(E_0, E_1, D)$.

A subfamily $\mathcal{A} \subset F[\Gamma(E_0, E_1, D)]$, where $E_0, E_1 \subset \overline{D}$, is called *p-complete* if $M_p\Gamma(E_0, E_1, D) = \inf_{\varrho\in\mathcal{A}} \int \varrho^p\,dm$.

PROPOSITION 8 (J. Hesse [7], Lemma 4.9). *If $F_0, F_1 \subset \overline{D} \subset \dot{\mathbb{R}}^n$ (where $\dot{\mathbb{R}}^n$ is the one-point compactification of $\mathbb{R}^n$) are compact, $F_0 \cap F_1 = \emptyset$ and there exists a p-complete family $\mathcal{A} \subset F[\Gamma(F_0, F_1, D)]$ such that $L(\varrho) \ge 1$, $\forall \varrho \in \mathcal{A}$, then the family $\mathcal{A}'_p = \{\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^p;\ \varrho$ lower semicontinuous and $\varrho_{|D}$ continuous$\}$ is p-complete.*

PROPOSITION 9 (P. Caraman [4], corollary to Proposition 4). *If $F_0, F_1 \subset \overline{D}$ are compact, $F_0 \cap F_1 = \emptyset$ and D is m-smooth of order $p > 1$ on $(F_0 \cup F_1) \cap \partial D$, then the family $\mathcal{A}''_p = \{\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^p; \varrho_{|D}$ continuous and $\varrho(x) \ge \alpha_F^\varrho > 0\ \forall x \in F\ \forall F$ compact$\}$ is p-complete.*

THEOREM 1. *If E is open relative to $\overline{D}$ or to ∂D, $F \subset \overline{D}$ is compact, $\overline{E} \cap F = \emptyset$ and D is m-smooth of order $p \ge n$ on $F \cap \partial D$, then*

$$M_p\Gamma(E, F, D) = \operatorname{cap}_p(E, F, D)\,. \tag{4}$$

Proof. We observe first that arguing as in W. Ziemer's [10] Lemma 3.1, we obtain

$$M_p\Gamma(E, F, D) \le \operatorname{cap}_p(E, F, D)\,, \tag{5}$$

so that we only have to prove that

$$\operatorname{cap}_p(E, F, D) \le M_p\Gamma(E, F, D)\,. \tag{6}$$

Proposition 2 yields that $M_p\Gamma(E, F, D) < \infty$ so that we may assume that $\forall \varepsilon > 0$ there exists $\varrho \in F[\Gamma(E, F, D)]$ such that

$$\int \varrho^p\,dm < M_p\Gamma(E, F, D) + \varepsilon\,. \tag{7}$$

By the same argument as in J. Hesse's [6] Lemma 4.40, it follows that the family

$$\mathcal{A}_p = \{\varrho \in F[\Gamma(E, F, D)] \cap L^p; \varrho_{|\Delta} \text{ continuous and } \varrho(x) \ge \alpha_K^\varrho\,,\ \forall x \in K\ \forall K \text{ compact}\}\,,$$

where $\Delta = D - (\overline{E} \cup F)$, is p-complete. Let us show that $L_1(\varrho) \ge 1\ \forall \varrho \in \mathcal{A}_p$.

Suppose first that $F = \{\xi\} \in \partial D$ and $\varrho \in \widetilde{\mathcal{A}}_p = \{\varrho \in F[\Gamma(E, \{\xi\}, D)] \cap L^p;\ \varrho(x) \ge \alpha_K^\varrho > 0\ \forall x \in K\ \forall K$ compact$\}$. Assume, by contradiction, that $L_1(\varrho) < 1$. Then, as in the proof of Proposition 4, let $\{\eta_k\}$ be a sequence of numbers $\eta_k \in (0, 1)$ $(k = 1, 2, \ldots)$ such that $\sum_{k=1}^\infty \eta_k < \infty$, $\{r_k\}$ a decreasing sequence such that $\lim_{k\to\infty} r_k = 0$ and $\{\gamma_k\}$ a sequence of arcs $\gamma_k \in \Gamma[E, B(\xi, r_k), D]$ so that $\int_{\gamma_k} \varrho\,dH^1 < L_1(\varrho, r_k) + \eta_k \le L_1(\varrho) + \eta_k$. Then all γ_k are rectifiable, so that they can be decomposed as $\gamma_k = \chi_k \circ \alpha'_k \circ \alpha_k$, where

$$\chi_k \in \Gamma[E, S(\xi, r_{k-2}), D]\,,$$
$$\alpha'_k \in \Gamma[S(\xi, r_{k-1}), S(\xi, r_{k-2}), B(\xi, r_{k-2})]\,,$$
$$\alpha_k \in \Gamma[B(\xi, r_k), S(\xi, r_{k-1}), B(\xi, r_{k-1})]\,.$$

Arguing as in Proposition 4 (with obvious changes), we obtain arcs $\widetilde{\gamma}_k \in \Gamma(E, F, D)$ $(k = 3, 4, \ldots)$ such that $1 \le \int_{\widetilde{\gamma}_k} \varrho\, dH^1 < 1$ for k sufficiently large. This contradiction yields $L_1(\varrho) \ge 1$ in this case.

Now, consider the general case of $\varrho \in \mathcal{A}_p$ and suppose that $L_1(\varrho) < 1$. Then $L_1(\varrho) < 1 - 2\varepsilon$ for $\varepsilon > 0$ sufficiently small. From the definition of $L_1(\varrho, r_k)$, with $\{r_k\}$ as above, there exists a sequence of arcs $\gamma_k \in \Gamma[E, F(r_k), D]$ such that

$$\int_{\gamma_k} \varrho\, dH^1 \le L_1(\varrho, r_k) + \varepsilon \le L_1(\varrho) + \varepsilon < 1 - \varepsilon \qquad (k = 1, 2, \ldots)\,. \tag{8}$$

Consider a sequence $\{\gamma'_k\}$, where $\gamma'_k \in \Gamma\{E, \overline{F(r_k)}, D - \overline{F(r_k)}\} \subset \Gamma[E, \overline{F(r_k)}, D]$ and $\gamma'_k \subset \gamma_k$. Then (8) yields

$$\int_{\gamma'_k} \varrho\, dH^1 \le \int_{\gamma_k} \varrho\, dH^1 < 1 - \varepsilon\,. \tag{9}$$

Let $\gamma'_k = \gamma(x_k, y_k)$ $(k = 1, 2, \ldots)$. Then we have several possibilities:

I. There exists a subsequence of $\{\gamma'_k\}$ (denoted again by $\{\gamma'_k\}$) such that $\lim y_k = \xi \in \partial D$. Since $\varrho \in \mathcal{A}_p \subset \widetilde{\mathcal{A}}_p$, the hypotheses of the preceding case $(F = \{\xi\} \subset \partial D)$ are fulfilled so that $\widetilde{L}_1(\varrho) = \lim_{k\to\infty} \widetilde{L}_1(\varrho, r_k) \ge 1$, where $\widetilde{L}_1(\varrho, r) = \inf_\gamma \int_\gamma \varrho\, dH^1$ and the infimum is taken over all $\gamma \in \Gamma(E, B(\xi, r), D)$. Hence, by the same argument as in Proposition 3, we deduce the existence of a $\delta = \delta(\varepsilon) \in (0, 1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[E, \overline{B(\xi, r)}, D]\}$ $\forall r < \delta$. On account of (9), it follows that, for k so large that $y_k \in B(\xi, \delta)$, we should have $1 - \varepsilon \le \int_{\gamma'_k} \varrho\, dH^1 < 1 - \varepsilon$. This contradiction implies $L_1(\varrho) \ge 1$ in this case too.

II. There exists a subsequence of $\{\gamma'_k\}$ (denoted again by $\{\gamma'_k\}$) such that $\lim_{k\to\infty} y_k = y_0 \in D$. Then, arguing as in the corresponding part of the proof of Proposition 6 (with obvious modifications), we infer that $L_1(\varrho) \ge 1$ also in this case.

Now, using the same notations as in Proposition 6, let

$$c = \begin{cases} b_n \left(\dfrac{\varepsilon}{2}\right)^n \log 2 & \text{for } p = n, \\[2mm] \dfrac{b_{n,p}}{2^p (p-n)(1-\varepsilon)^p} & \text{for } p > n, \end{cases}$$

where $b_n = b_{n,n}, b_{n,p} > 0$ are the constants appearing in Proposition 5. As in Proposition 6, we show there exists a constant $b > 0$ such that $2b < d(E, F)$

and $\int_{B(x,b)} \varrho^p\, dm \le c \ \forall x \in D$. Let $E = \bigcup_{k=1}^{\infty} E_k$, where E_k $(k = 1, 2, \dots)$ are the components of E, and let $E^D(b) = \{x \in D; d(x, E) < b$ and there exists $y \in E$ such that $d(x, y) = k < k' < b$, $S(x, k') \cap E_y \neq \emptyset$, $B(y, x') \cap [(\partial D - E) \cup F] = \emptyset\}$, where E_y is the component of E containing y and where $k' = 2k$ for $p = n$ and $1/k^{p-n} - 1/(k')^{p-n} = 1$ for $p > n$. It is easy to see that $E^D(b)$ is open.

In the first part of the proof, we have seen that $L_1(\varrho) \ge 1 \ \forall \varrho \in \mathcal{A}_p$, and arguing as in the preceding proposition, we conclude that the family

$$\mathcal{A}_p''' = \{\varrho \in F[\Gamma(E, F, D)] \cap L^p; \varrho_{|D-\overline{E}} \text{ continuous},\\ \varrho(x) \ge \alpha_K^{\varrho} > 0 \ \forall x \in K \ \forall K \text{ compact}\}$$

is p-complete. Next, from Proposition 3, we derive that there exists $\delta = \delta(\varepsilon) \in (0, 1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[E, \overline{F(r)}, D]\} \ \forall r < \delta$. Now, define, for $r < \delta$,

$$\varrho_1(x) = \begin{cases} \varrho/(1-\varepsilon) & \text{for } x \in D - [E^D(b) \cup F(r)], \\ 0 & \text{otherwise.} \end{cases}$$

Then, as in the proof of Proposition 7, $\forall \gamma \in \Gamma_r[E^D(b), F(r), D]$,

$$\int_\gamma \varrho_1\, dH^1 \ge \int_{\gamma'} \frac{\varrho}{1-\varepsilon}\, dH^1 \ge 1\,,$$

where $\gamma' \in \Gamma_r\{\overline{E^D(b)}, \overline{F(r)}, D - [\overline{E^D(b)} \cup \overline{F(r)}]\}$, hence, $\varrho_1 \in F\{\Gamma_r[E^D(b), F(r), D]\}$. Next, let $u(x) = \min(1, \inf_\gamma \int_\gamma \varrho_1\, dH^1)$, where the infimum is taken over all arcs γ joining x to E in D. By the same argument as in the corresponding part of the proof of Propositions 1 and 7, we find that u is locally lipschitzian in D and $\lim_{x \to x_0, x \in D} u(x) = 0 \ \forall x_0 \in E$, while $\lim_{x \to x_1, x \in D} u(x) = 1 \ \forall x_1 \in F$, implying the admissibility of u for $\operatorname{cap}_p(E, F, D)$. Finally, arguing as in Theorem 1 of [2], we deduce that u is differentiable a.e. in D and

$$|\nabla u(x)| \le \varrho_1(x) \tag{10}$$

a.e. in D. From the definition of ϱ_1 and (7), we obtain

$$\int \varrho_1^p\, dm \le \frac{1}{(1-\varepsilon)^p} \int \varrho^p\, dm < \frac{M_p \Gamma(E, F, D) + \varepsilon}{(1-\varepsilon)^p}\,.$$

Hence (10) yields

$$\operatorname{cap}_p(E, F, D) \le \int_D |\nabla u|^p\, dm \le \int \varrho_1^p < \frac{M_p \Gamma(E, F, D) + \varepsilon}{(1-\varepsilon)^p}\,,$$

and letting $\varepsilon \to 0$, we obtain (6), which, together with (5), implies (4), as desired.

Corollary. *If E is open, F is compact and $\overline{E}\cap F=\emptyset$, then*

$$M_p\mathit{\Gamma}(E,F)=\operatorname{cap}_p(E,F)\qquad(p\geq n)\,,$$

where $M_p\mathit{\Gamma}(E_0,E_1)=M_p\mathit{\Gamma}(E_0,E_1,\mathbb{R}^n)$ *and* $\operatorname{cap}_p(E_0,E_1)=\operatorname{cap}_p(E_0,E_1,\mathbb{R}^n)$.

Now, let $L_2(\varrho,r)=\inf_\gamma\int_\gamma\varrho\,dH^1$, where the infimum is taken over all $\gamma\in\mathit{\Gamma}[E_0\cup E_0'(r),E_1\cup E_1'(r),D]$. Hence, for a sequence $\{r_k\}$ as above, $L_2(\varrho,r_1)\leq L_2(\varrho,r_2)\leq\ldots\leq L_2(\varrho)$, where $L_2(\varrho)=\lim_{r\to0}L_2(\varrho,r)$.

Proposition 10 (P. Caraman [4], Proposition 2). *If $E_0,E_1\subset\overline{D}$, $\overline{E}_0\cap\overline{E}_1=\emptyset$ and $M_p\mathit{\Gamma}(E_0,E_1,D)<\infty$ $(p>1)$, then $\mathcal{A}_p$ (of Proposition 4) is p-complete.*

Theorem 2. *If $E_0\cap E_1=\emptyset$, $E_i=E_i''\cup F_i$, where E_i'' $(i=0,1)$ is open relative to $\overline{D}$ or to ∂D, while F_i is compact, and D is m-smooth of order $p\geq n$ on $(F_0\cup F_1)\cap\partial D$, then* (1) *holds.*

Proof. We observe first that, arguing as in Ziemer's [10] Lemma 3.1, we obtain the inequality

$$M_p\mathit{\Gamma}(E_0,E_1,D)\leq\operatorname{cap}_p(E_0,E_1,D)\,,\tag{11}$$

so that we only have to establish the opposite inequality

$$\operatorname{cap}_p(E_0,E_1,D)\leq M_p(E_0,E_1,D)\,.\tag{12}$$

If $M_p\mathit{\Gamma}(E_0,E_1,D)=\infty$, then (1) is a direct consequence of (11), so that we may assume that $M_p\mathit{\Gamma}(E_0,E_1,D)<\infty$. But then, from the preceding proposition, we deduce that the corresponding family $\mathcal{A}_p$ is p-complete so that $\forall\varepsilon>0$ there exists $\varrho\in\mathcal{A}_p$ such that

$$\int\varrho^p\,dm<\frac{M_p\mathit{\Gamma}(E_0,E_1,D)}{1-\varepsilon}\,.\tag{13}$$

Next, $L_2(\varrho)\geq1$ $\forall\varrho\in\mathcal{A}_p$. Indeed, $L_1(\varrho)\geq1$ corresponds to $\mathit{\Gamma}[F_0(r),E_1'',D]$ as well as to $\mathit{\Gamma}[E_0'',F_1(r),D]$, while $L(\varrho)\geq1$ to $\mathit{\Gamma}[F_0(r),F_1(r),D]$. If $\mathit{\Gamma}_0=\mathit{\Gamma}(E_0'',E_1'',D)$, $\mathit{\Gamma}'=\mathit{\Gamma}(F_0,E_1'',D)$, $\mathit{\Gamma}''=\mathit{\Gamma}(E_0'',F_1,D)$, $\mathit{\Gamma}'''=\mathit{\Gamma}(F_0,F_1,D)$ and $\widetilde{L}(\varrho)=\lim_{r\to0}\widetilde{L}(\varrho,r)$, $\widetilde{L}(\varrho,r)=\inf_\gamma\int_\gamma\varrho\,dH^1$, where the infimum is taken over all $\gamma\in\widetilde{\mathit{\Gamma}}=\mathit{\Gamma}'\cup\mathit{\Gamma}''\cup\mathit{\Gamma}'''$, then $\widetilde{L}(\varrho)\geq1$ since $\forall\varrho\in\widetilde{\mathcal{A}}_p=\{\varrho\in[F(\mathit{\Gamma}')\cap F(\mathit{\Gamma}'')\cap F(\mathit{\Gamma}''')]\cap L^p;\varrho_{|D-(\overline{E}_0\cup\overline{E}_1)}$ continuous, $\varrho(x)\geq\alpha_F>0$ $\forall x\in F$ $\forall F$ compact$\}$, we have

$$\begin{aligned}\widetilde{L}(\varrho,r)&=\inf_{\gamma\in\widetilde{\mathit{\Gamma}}}\int\varrho\,dH^1\\&=\min\Big(\inf_{\gamma\in\mathit{\Gamma}'}\int_\gamma\varrho\,dH^1,\inf_{\gamma\in\mathit{\Gamma}''}\int_\gamma\varrho\,dH^1,\inf_{\gamma\in\mathit{\Gamma}'''}\int_\gamma\varrho\,dH^1\Big)\\&=\min[L'(\varrho,r),L''(\varrho,r),L'''(\varrho,r)]\end{aligned}$$

$\forall r > 0$, so that

$$\widetilde{L}(\varrho) = \min[\lim_{r\to 0} L'(\varrho, r), \lim_{r\to 0} L''(\varrho, r), \lim_{r\to 0} L'''(\varrho, r)]$$
$$= \min[L'(\varrho), L''(\varrho), L'''(\varrho)]\,.$$

Hence, $L_2(\varrho) \geq 1$ because the family Γ_0 does not modify this result since $\int_\gamma \varrho\, dH^1 \geq 1\ \forall \gamma \in \Gamma(E_0'', E_1'', D)$, and, by the same argument as in Proposition 8, the family $\widetilde{\mathcal{A}}_p' = \{\varrho \in F[\Gamma(E_0, E_1, D)] \cap L^p; \varrho_{|D-(\overline{E}_0 \cup \overline{E}_1)}$ continuous, $\varrho(x) \geq \alpha_F > 0\ \forall x \in F\ \forall F$ compact$\} \subset \mathcal{A}_p$ is p-complete, so that, arguing as in Proposition 3, it follows that there is $\delta = \delta(\varepsilon) \in (0, 1)$ such that

$$\frac{\varrho}{1-\varepsilon} \in F\{\Gamma[E_0'' \cup F_0(r), E_1'' \cup F_1(r), D]\}$$

$\forall r < \delta$. Now, define

$$\varrho_1(x) = \begin{cases} \dfrac{\varrho(x)}{1-\varepsilon} & \text{if } x \in D - [E_0''^D(b) \cup F_0(r) \cup E_1''^D(b) \cup F_1(r)], \\ 0 & \text{otherwise.} \end{cases}$$

As in the corresponding part of the proof of Proposition 1, we deduce that $\varrho_1 \in F\{\Gamma_r[\overline{E_0''^D(b)} \cup \overline{F_0(r)}, \overline{E_1''^D(b)} \cup \overline{F_1(r)}, D]\}$ so that $u(x) = \min(1, \inf_\gamma \int_\gamma \varrho_1\, dH^1)$ (where the infimum is taken over all γ joining x to E_0 in D) is admissible for $\mathrm{cap}_p(E_0, E_1, D)$. Hence, as in the last part of the proof of the preceding theorem, we obtain (12), which, together with (11), yields (1), as desired.

Corollary. *If $E_i = E_i'' \cup F_i$, where E_i'' $(i = 0, 1)$ is open, F_i is compact and $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, then $M_p\Gamma(E_0, E_1) = \mathrm{cap}_p(E_0, E_1)$ $(p \geq n)$.*

Next, we give criteria for equality between p-module and p-capacity, where we only impose conditions on one of the sets E_0, E_1.

Proposition 11 (W. Ziemer [9], Theorem 2.5.1). *If $\Gamma_1 \subset \Gamma_2 \subset \ldots$ and $\Gamma = \bigcup_{k=1}^\infty \Gamma_k$, then $M_p\Gamma = \lim_{k\to\infty} M_p\Gamma_k$ $(p > 1)$.*

Proposition 12 (J. Väisälä [8], Theorem 2.3). *p-Almost every bounded curve $(p > 0)$ is rectifiable.*

We recall that an arc family Γ_2 is said to be *minorized* by an arc family Γ_1 (denoted by $\Gamma_1 \prec \Gamma_2$) if $\forall \gamma_2 \in \Gamma_2$ there exists a $\gamma_1 \in \Gamma_1$ so that $\gamma_1 \subset \gamma_2$.

Proposition 13 (B. Fuglede [5], Theorem 1). *If $\Gamma_1 \prec \Gamma_2$, then $M_p\Gamma_1 \geq M_p\Gamma_2$ $(p > 1)$.*

Theorem 3. *If $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ and E_0 is not accessible from D by rectifiable arcs, then*

$$M_p\Gamma(E_0, E_1, D) = \mathrm{cap}_p(E_0, E_1, D) = 0 \quad (p > 1)\,. \tag{14}$$

Proof. Clearly, $E_0 \subset \partial D$. Set $E(r,\infty) = \{x; d(E,x) > r\}$ and $E(r_1,r_2) = \{x; r_1 < d(E,x) < r_2\}$, where $d(E,x)$ is the distance between the set E and the point x. Since $\Gamma[E_0, E_0(r_1,r_2), D\cap E_0(r_2)] \prec \Gamma(E_0,E_1,D)$, it follows that if E_0 is bounded and $r_1 < d(E_0,E_1)$, then, by the preceding two propositions,

$$M_p\Gamma(E_0,E_1,D) \le M_p\Gamma[E_0,E_0(r_1,r_2),D\cap E_0(r_2)] = 0\,. \tag{15}$$

If E_0 is unbounded, set $E_R = E_0 \cap B(R)$. Then

$$M_p\Gamma(E_R,E_1,D) \le M_p\Gamma[E_R,E_R(r_1,r_2),D\cap E_R(r_2)] = 0\,.$$

Hence, letting $R\to\infty$ and taking into account Proposition 11,

$$\begin{aligned} M_p\Gamma(E_0,E_1,D) &= \lim_{R\to\infty} M_p\Gamma(E_R,E_1,D) \\ &\le \lim_{R\to\infty} M_p\Gamma[E_R,E_R(r_1,r_2),D\cap E_R(r_2)] = 0\,. \end{aligned}$$

Next, let us show that

$$\operatorname{cap}_p[E_0,E_0(r_1,r_2),D\cap E_0(r_2)] = M_p\Gamma[E_0,E_0(r_1,r_2),D\cap E_0(r_2)]\,,$$

where $0<r_1<r_2<d(E_0,E_1)$.

Suppose first that E_0 is bounded. Then $\forall\varepsilon>0$ there exists $R=R(\varepsilon)$ such that if ϱ is the characteristic function of $E_0(R)\cap D$, then

$$\int \varrho^p\,dm = \int_{E_0(R)} dm = mE_0(R) < \varepsilon\,.$$

If E_0 is unbounded, we may consider its intersection with the annuli $A(0,k,k+1) = \{x; k\le |x| < k+1\}$ $(k=0,1,\ldots)$ and define

$$\varrho(x) = \begin{cases} 1 & \text{if } x\in E_0(R_k)\cap D\cap A(0,k,k+1)\ (k=0,1,\ldots),\\ 0 & \text{otherwise}\end{cases}$$

where $\{R_k\}$ is a non-increasing sequence such that $R_1 < r_1$, $R_k\to 0$ as $k\to\infty$ and

$$\int \varrho^p\,dm = \sum_{k=0}^{\infty}\ \int_{A(0,k,k+1)} \varrho^p\,dm = \sum_{k=0}^{\infty} m[E_0(R_k)\cap D\cap A(0,k,k+1)] < \varepsilon\,.$$

Next, let $u(x) = \inf_\gamma \int_{\gamma[x,E_0(r_1,r_2)]} \varrho\,dH^1$, where the infimum is taken over all arcs γ joining x to $E_0(r_1,r_2)\cap D$. Clearly, $u(x)\to 0$ as $x\to E_0(r_1,r_2)\cap D$. Indeed, $E_0(r_1,r_2)$ is open and $\forall x_0\in E_0(r_1,r_2)\cap D$ each x sufficiently close to x_0 belongs to $E_0(r_1,r_2)\cap D$ so that it may be joined to $E_0(r_1,r_2)$ by an arc of length 0 (joining x to x), hence $u(x)=0$ for any x in a sufficiently small neighbourhood of x_0. Set $v(x)=\min[1,u(x)]$. Then $v(x)\to 0$ as $x\to E_0(r_1,r_2)\cap D$ in D and we may extend v by setting $v=0$ on $E_0(r_1,r_2)\cap CD$, so that $v_{|E_0(r_1,r_2)} = 0$. Next, since E_0 is not accessible by rectifiable arcs, and $\varrho(x) = 1$ in a sufficiently small neighbourhood of E_0, it follows that

$$u(x) = \inf_{\gamma} \int_{\gamma[x,E_0(r_1,r_2)]} \varrho \, dH^1$$
$$= \inf H^1 \Big\{ \gamma[x, E_0(r_1, r_2)] \cap \Big[\bigcup_{k=0}^{\infty} E_0(R_k) \cap D \cap A(0, k, k+1) \Big] \Big\}$$

becomes as large as one wishes as $x \to E_0$ in D. Hence $u(x) \to \infty$ and $v(x) \to 1$ as $x \to E_0$, so that, if $w(x) = 1 - v(x)$, then $w(x) \to 0$ as $x \to E_0$ in D and $w(x) \to 1$ as $x \to E_0(r_1, r_2)$ in D. But, since ϱ is bounded in $\mathbb{R}^n$, it follows that u, and hence also w, is locally lipschitzian in $D \cap E_0(r_2)$. Now, arguing as in Theorem 1 of [2], we obtain $|\nabla w(x)| \le \varrho(x)$ in $D \cap E_0(r_2)$, hence w is admissible for $\mathrm{cap}_p[E_0, E_0(r_1, r_2), D \cap E_0(r_2)]$, so that

$$\mathrm{cap}_p[E_0, E_0(r_1, r_2), D \cap E_0(r_2)] \le \int_{D \cap E_0(r_2)} |\nabla w|^p \, dm \le \int \varrho^p \, dm < \varepsilon \, .$$

Letting $\varepsilon \to 0$ yields $\mathrm{cap}_p[E_0, E_0(r_1, r_2), D \cap E_0(r_2)] = 0$. Finally, letting $r_2 \to \infty$ and taking into account the monotonicity of the p-capacity (cf. Lemma 6 of [2]), we get

$$(16) \quad \mathrm{cap}_p(E_0, E_1, D) \le \mathrm{cap}_p[E_0, E_0(r_1, \infty), D] = \inf_{u \in \mathcal{U}_1} \int_D |\nabla w|^p \, dm$$
$$= \inf_{u \in \mathcal{U}_1} \int_{D \cap E_0(r_2)} |\nabla w|^p \, dm = \inf_{u \in \mathcal{U}_2} \int_{D \cap E_0(r_2)} |\nabla w|^p \, dm$$
$$= \mathrm{cap}_p[E_0, E_0(r_1, r_2), D \cap E_0(r_2)] = 0 \, ,$$

where

$$\mathcal{U}_1 = \{ w : D \cup E_0 \cup E_0(r_1, \infty) \to [0,1]; w \text{ continuous}, \; w_{|D} \text{ locally lipschitzian}, \; w_{|E_0} = 0 \, , \; w_{|E_1} = 1 \} \, ,$$
$$\mathcal{U}_2 = \{ w : [D \cup E_0(r_2)] \cup E_0 \cup E_0(r_1, r_2) \to [0,1]; w \text{ continuous}, \; w_{|D \cap E_0(r_2)} \text{ locally lipschitzian}, \; w_{|E_0} = 0 \, , \; w_{|E_0(r_1, r_2)} = 1 \} \, .$$

Now, (15) and (16) imply (14), as desired.

PROPOSITION 14 (P. Caraman [2], Lemma 6). *If* $E_0 \subset \bigcup_{k=1}^{\infty} E_0^k$, $E_1 \cap \bigcup_{k=1}^{\infty} E_0^k = \emptyset$ *and* $E_0, E_1 \subset \overline{D}$, *then*

$$\mathrm{cap}_p(E_0, E_1, D) \le \sum_{k=1}^{\infty} \mathrm{cap}_p(E_0^k, E_1, D) \quad (p > 1) \, .$$

COROLLARY. *If* $E_0 \subset E_0^*$ *and* $\overline{E}_1 \cap \overline{E_0^*} = \emptyset$, *then* $\mathrm{cap}_p(E_0, E_1, D) \le \mathrm{cap}_p(E_0^*, E_1, D)$ $(p > 1)$.

THEOREM 4. *If* $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ *and* $E_i = E_i' \cup F_i$, *where* E_i' $(i = 0, 1)$ *is* *not accessible by rectifiable arcs*, F_i *is compact*, *and* D *is* m*-smooth of order* $p > 1$ *on* $(F_0 \cup F_1) \cap \partial D$, *then* (1) *holds.*

P r o o f. Indeed, by the preceding theorem and Theorem 1 of B. Fuglede [5],

$$\begin{aligned} M_p\Gamma(F_0, F_1, D) &\le M_p\Gamma(E_0, E_1, D) \\ &\le M_p\Gamma(E_0', E_1, D) + M_p\Gamma(E_0, E_1', D) + M_p\Gamma(F_0, F_1, D) \\ &= M_p\Gamma(F_0, F_1, D) \,. \end{aligned}$$

Hence, taking into account Proposition 1 and the corollary of the preceding proposition, we obtain

$$\begin{aligned} M_p\Gamma(E_0, E_1, D) &= M_p\Gamma(F_0, F_1, D) = \operatorname{cap}_p(F_0, F_1, D) \le \operatorname{cap}_p(E_0, E_1, D) \\ &\le \operatorname{cap}_p(E_0', E_1, D) + \operatorname{cap}_p(E_0, E_1', D) + \operatorname{cap}_p(F_0, F_1, D) \\ &= \operatorname{cap}_p(F_0, F_1, D) \,, \end{aligned}$$

hence,

$$M_p\Gamma(E_0, E_1, D) = \operatorname{cap}_p(F_0, F_1, D) = \operatorname{cap}_p(E_0, E_1, D) \,,$$

as desired.

Arguing as in the preceding theorem, on account of Propositions 1, 7 and of the preceding theorem, we deduce

COROLLARY 1. *If* $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ *and* $E_i = E_i' \cup E_i'' \cup F_i$, *where* E_i' *is inaccessible from* D *by rectifiable arcs*, E_i'' *is open relative to* $\overline{D}$ *or to* ∂D, F_i *is compact* $(i = 0, 1)$, *and* D *is* m*-smooth of order* $p \ge n$ *on* $(F_0 \cup F_1) \cap \partial D$, *then* (1) *holds.*

COROLLARY 2. *With the notations of the preceding corollary*, *if* $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ *and* $E_i = E_i' \cup E_i''$ $(i = 0, 1)$, *then* (1) *holds* $\forall p \ge n$.

THEOREM 5. *If* $\operatorname{cap}_p(E_0, E_1, D) = 0$, *then* $M_p\Gamma(E_0, E_1, D) = 0$ $(p > 1)$.

P r o o f. From the definition of the p-capacity, it follows that $E_0 \cap \overline{E}_1 = \overline{E}_0 \cap E_1 = \overline{E}_0 \cap \overline{E}_1 \cap D = \emptyset$. Thus the theorem is a direct consequence of (11).

LEMMA 2. $\overline{E}_0 \cap \overline{E}_1 = \emptyset \Rightarrow \operatorname{cap}_p(E_0, E_1, D) \le \operatorname{cap}_p(E_0, E_1)$ $(p > 1)$.

P r o o f. Define

$$\begin{aligned} \mathcal{U}_D = \{&u : D \cup E_0 \cup E_1 \to [0,1]; u \text{ continuous}, \\ &u_{|D} \text{ locally lipshitzian}, \ u_{|E_0} = 0 \,,\ u_{|E_1} = 1\} \,, \\ \mathcal{U} = \{&u : \mathbb{R}^n \to [0,1]; u \text{ continuous and locally lipschitzian}, \\ &u_{|E_0} = 0 \,,\ u_{|E_1} = 1\} \,. \end{aligned}$$

Then $\mathcal{U}_{|D} \subset \mathcal{U}_D$, where $\mathcal{U}_{|D} = \{u_{|D}; u \in \mathcal{U}\}$. Hence,

$$\operatorname{cap}_p(E_0, E_1, D) = \inf_{u \in \mathcal{U}_D} \int_D |\nabla u|^p \, dm \le \inf_{u \in \mathcal{U}_{|D}} \int_D |\nabla u|^p \, dm$$

$$\le \inf_{u \in \mathcal{U}} \int_D |\nabla u|^p \, dm = \operatorname{cap}_p(E_0, E_1) \,,$$

as desired.

PROPOSITION 15 (P. Caraman [1], Lemma 13). *If D is bounded, $F_0 \subset D$ and $F_1 \subset \overline{D}$ are closed, $F_0 \cap F_1 = \emptyset$, $\varrho \in F[\Gamma(F_0, F_1, D - F_1)] \cap L^n$, then $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[F_0(r), F_1, D - F_1]\}$ $\forall r < \delta$.*

Hence and on account of the corollary of Propositions 3 and 4, we have

COROLLARY 1. *(D bounded, $F_0 \subset D$ and $F_1 \subset \overline{D}$ closed, $F_0 \cap F_1 = \emptyset$, $\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^p$ $(p > 1)$) $\Rightarrow \forall \varepsilon > 0$ there is $\delta = \delta(\varepsilon) \in (0,1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[F_0(r), F_1, D]\}$ $\forall r < \delta$.*

Arguing as in the preceding proposition, we also obtain

COROLLARY 2. *(D bounded, $E \subset \overline{D}$, $F \subset D$ closed, $\overline{E} \cap F = \emptyset$ and $\varrho \in F[\Gamma(E, F, D)] \cap L^p$ $(p > 1)$) $\Rightarrow \forall \varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) \in (0,1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[E, F(r), D]\}$ $\forall r < \delta$.*

By the same argument as in the preceding corollary, we get

COROLLARY 3. *(F_0, F_1 compact, $F_0 \cap F_1 = \emptyset$ and $\varrho \in F[\Gamma(F_0, F_1, D] \cap L^p$ $(p > 1)$) $\Rightarrow \forall \varepsilon > 0$ there is $\delta = \delta(\varepsilon) \in (0,1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[F_0, F_1(r), D]\}$ $\forall r < \delta$.*

LEMMA 3. *(D bounded, $E \subset \overline{D}$, $F \subset D$ closed and $\overline{E} \cap F = \emptyset$) $\Rightarrow$*

$$M_p\Gamma(E, F, D) = \lim_{r \to 0} M_p\Gamma[E, F(r), D] \quad (p > 1)\,. \tag{17}$$

P r o o f. Clearly,

$$M_p\Gamma(E, F, D) \le \lim_{r \to 0} M_p\Gamma[E, F(r), D]\,, \tag{18}$$

so that we only have to prove the opposite inquality. By Proposition 2, $M_p\Gamma(E, F, D) < \infty$ so that $\forall \varepsilon > 0$ there exists $\varrho \in F[\Gamma(E, F, D)]$ satisfying (7). Now, by Corollary 2 of the preceding proposition, there is $\delta = \delta(\varepsilon) \in (0,1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma[E, F(r), D]\}$ $\forall r < \delta$. Therefore, on account of (7),

$$M_p\Gamma[E, F(r), D] < \int \frac{\varrho^p \, dm}{(1-\varepsilon)^p} < \frac{M_p\Gamma(E, F, D) + \varepsilon}{(1-\varepsilon)^p} \quad \forall r < \delta\,.$$

Hence, letting $r \to 0$,

$$\lim_{r\to 0} M_p\Gamma[E, F(r), D] \le \frac{M_p\Gamma(E,F,D)+\varepsilon}{(1-\varepsilon)^p},$$

and letting $\varepsilon \to 0$,

$$\lim_{r\to 0} M_p\Gamma[E, F(r), D] \le M_p\Gamma(E,F,D),$$

which, together with (18), yields (17), as desired.

Arguing as in the preceding lemma and taking into account the preceding corollary (instead of Corollary 2 of the preceding proposition), we obtain

COROLLARY 1. *(F_0, F_1 compact and $F_0 \cap F_1 = \emptyset$) $\Rightarrow$ $M_p\Gamma(F_0,F_1) = \lim_{r\to 0} M_p\Gamma[F_0, F_1(r)]$ ($p > 1$).*

COROLLARY 2. *Under the hypotheses of the preceding corollary, $M_p\Gamma(F_0,F_1) = \lim_{r\to 0} M_p\Gamma[F_0, \overline{F_1(r)}]$ ($p > 1$).*

LEMMA 4. *(F_0, F_1 compact, D m-smooth of order $p > 1$ on $(F_0\cup F_1)\cap\partial D$ and $F_0 \cap F_1 = \emptyset$) $\Rightarrow$*

$$M_p\Gamma(F_0,F_1,D) = \lim_{r\to 0} M_p\Gamma[F_0(r), F_1(r), D] \quad (p>1). \tag{19}$$

Proof. Clearly,

$$M_p\Gamma(F_0,F_1,D) \le \lim_{r\to 0} M_p\Gamma[F_0(r), F_1(r), D] \quad (p>1), \tag{20}$$

so that we only have to prove the opposite inequality. On account of Proposition 2, $M_p\Gamma(F_0,F_1,D) < \infty$, so that we may assume that $\varrho \in L^p$. Hence, by Proposition 4, $L(\varrho) \ge 1$ $\forall\varrho \in \mathcal{A}_p$, and so, by Proposition 3, $\forall\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that $\varrho/(1-\varepsilon) \in F\{\Gamma_r[F_0(r), F_1(r), D]\}$ $\forall r < \delta$. Consequently, we may choose a ϱ satisfying (7) and

$$M_p\Gamma[F_0(r), F_1(r), D] \le \frac{1}{(1-\varepsilon)^p}\int \varrho^p\,dm < \frac{M_p\Gamma(F_0,F_1,D)+\varepsilon}{(1-\varepsilon)^p} \quad \forall r > 0.$$

Letting $r \to 0$ and then $\varepsilon \to 0$ shows that

$$\lim_{r\to 0} M_p\Gamma[F_0(r), F_1(r), D] \le M_p\Gamma(F_0,F_1,D),$$

which, together with (20), gives (19), as desired.

Arguing as in Lemma 8 of [2], we obtain

PROPOSITION 16. *(F_0, F_1 compact, $F_0 \cap F_1 = \emptyset$ and D m-smooth on $(F_0 \cup F_1) \cap \partial D$) $\Rightarrow$ $L(\varrho) \ge 1$ $\forall\varrho \in \mathcal{A}_0' = \{\varrho \in F[\Gamma(F_0,F_1,D)] \cap L^n;$ $\varrho(x) \ge \alpha_F^\varrho > 0$ $\forall x \in F$ $\forall F$ compact$\}$.*

Hence, we deduce

LEMMA 5. *(F_0, F_1 closed, $F_0 \cap F_1 = \emptyset$, D bounded and m-smooth on $(F_0 \cup F_1) \cap \partial D) \Rightarrow L(\varrho) \geq 1\ \forall \varrho \in \widetilde{\mathcal{A}}'_p = \{\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^p;\ \varrho_{|CD} = 0$, $\varrho(x) \geq \alpha_F^\varrho > 0\ \forall x \in F\ \forall F$ compact$\}$ ($p \geq n$).*

P r o o f. It is enough to show that the hypotheses of the preceding proposition are satisfied, especially the condition $\varrho \in L^n$. Indeed,

$$\int \varrho^n\,dm = \int_D \varrho^n\,dm = \int_{E_1} \varrho^n\,dm + \int_{E_2} \varrho^n\,dm$$

$$\leq \int_{E_1} dm + \int_{E_2} \varrho^p\,dm \leq mE_1 + \int \varrho^p\,dm \leq mD + \int \varrho^p\,dm < \infty\,,$$

where $E_1 = \{x \in D; \varrho(x) \leq 1\}$, $E_2 = \{x \in D; \varrho(x) > 1\}$.

By the same argument, we also obtain

COROLLARY 1. *(F closed, $\overline{E} \cap F = \emptyset$, D bounded and m-smooth on $F \cap \partial D) \Rightarrow L_1(\varrho) \geq 1\ \forall \varrho \in \mathcal{A}_p$ ($p \geq n$).*

By the same argument as in Lemma 4 and using the preceding lemma (instead of Proposition 4), we get

COROLLARY 2. *(F_0, F_1 closed, $F_0 \cap F_1 = \emptyset$, D bounded and m-smooth on $(F_0 \cup F_1) \cap \partial D) \Rightarrow M_p\Gamma(F_0, F_1, D) = \lim_{r \to 0} M_p\Gamma[F_0(r), F_1(r), D]$ ($p \geq n$).*

A similar argument to the one used in Theorem 4 yields

COROLLARY 3. *($E_i = E'_i \cup F_i$, F_i ($i = 0, 1$) compact, $F_0 \cap F_1 = \emptyset$ and D m-smooth of order $p > 1$ on $(F_0 \cup F_1) \cap \partial D) \Rightarrow$*

$$M_p\Gamma(E_0, E_1, D) = \lim_{r \to 0} M_p\Gamma[E'_0 \cup F_0(r), E'_1 \cup F_1(r), D]\,. \tag{21}$$

COROLLARY 4. *($E_i = E'_i \cup F_i$, F_i ($i = 0, 1$) closed, $F_0 \cap F_1 = \emptyset$, D bounded and m-smooth on $(F_0 \cup F_1) \cap \partial D) \Rightarrow$ (21) holds for $p \geq n$.*

In the particular case $p = n$, Proposition 4 yields

COROLLARY 5. *(F_0, F_1 compact, $F_0 \cap F_1 = \emptyset$ and D m-smooth on $(F_0 \cup F_1) \cap \partial D) \Rightarrow L(\varrho) \geq 1\ \forall \varrho \in \mathcal{A}_n = \{\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^n;\ \varrho_{|\Delta}$ continuous, $\varrho(x) \geq \alpha_F^\varrho > 0\ \forall x \in F\ \forall F$ compact$\}$.*

By the same argument as in the preceding lemma, we obtain

COROLLARY 6. *(F_0, F_1 closed, $F_0 \cap F_1 = \emptyset$, D bounded and m-smooth on $(F_0 \cup F_1) \cap \partial D) \Rightarrow L(\varrho) \geq 1\ \forall \varrho \in \mathcal{A}^*_p = \{\varrho \in F[\Gamma(F_0, F_1, D)] \cap L^p; \varrho_{|\Delta}$ continuous, $\varrho_{|CD} = 0$ and $\varrho(x) \geq \alpha_F > 0\ \forall x \in F\ \forall F$ compact$\}$ ($p \geq n$).*

Arguing as in Proposition 1 and using the preceding corollary, we deduce

COROLLARY 7. *(F_0, F_1 closed, $F_0 \cap F_1 = \emptyset$, D bounded and m-smooth on $(F_0 \cup F_1) \cap \partial D) \Rightarrow M_p\Gamma(F_0, F_1, D) = \mathrm{cap}_p(F_0, F_1, D)$ ($p \geq n$).*

Proposition 17 (J. Hesse [6], Theorem 5.21). *If* $\{F'_k\}$, $\{F''_k\}$ *are two decreasing sequences of compact sets*, $F' = \bigcap_k F'_k$, $F'' = \bigcap_k F''_k$ *and* $F'_1 \cap F''_1 = \emptyset$, *then* $\lim_{k\to\infty} M_p\Gamma(F'_k, F''_k) = M_p\Gamma(F', F'')$.

Proposition 18 (B. Fuglede [5]). *If* $\Gamma_0 = \{\gamma; x_0 \in \gamma\}$, *then* $M_p\Gamma_0 = 0$ $(p \le n)$.

Theorem 6. ($\overline{E}_0 \cap \overline{E}_1 = \emptyset$ *and* E_0 *at most countable*) $\Rightarrow$

$$\operatorname{cap}_p(E_0, E_1, D) = M_p\Gamma(E_0, E_1, D) = 0 \quad (p \le n). \tag{22}$$

Proof. Suppose that $E_0 = \{x_0\}$ and that E_1 is bounded. Let $\{r_k\}$ be a strictly decreasing sequence such that $\lim_{k\to\infty} r_k = 0$ and let $r_0, r_1 < d(x_0, E_1)$. By the corollary of Proposition 14, Lemma 2, Proposition 1 and the preceding two propositions, we obtain

$$\begin{aligned}\operatorname{cap}_p(x_0, E_1, D) &\le \operatorname{cap}_p[x_0, \overline{E_1(r_0)}, D] \le \operatorname{cap}_p[x_0, \overline{E_1(r_0)}]\\ &\le \lim_{k\to\infty} \operatorname{cap}_p[\overline{B(x_0, r_k)}, \overline{E_1(r_0)}]\\ &= \lim_{k\to\infty} M_p\Gamma[\overline{B(x_0, r_k)}, \overline{E_1(r_0)}] = M_p\Gamma[x_0, E_1(r_0)] = 0\end{aligned}$$

since $\overline{E_1(r_0)}$ is closed and bounded, hence compact. On the other hand, by the preceding proposition,

$$M_p\Gamma(x_0, E_1, D) \le M_p\Gamma(x_0, \mathbb{R}^n - x_0) = 0, \tag{23}$$

hence

$$\operatorname{cap}_p(x_0, E_1, D) = M_p\Gamma(x_0, E_1, D) = 0$$

when E_1 is bounded.

Now, let us get rid of this restrictive condition. We have $E_1 = \bigcup_{k=0}^{\infty} E_1^k$, where $E_1^k = E_1 \cap A(0, k, k+1)$, and we may assume without loss of generality that $0 \in E_1$. By Proposition 14 and the first part of the proof,

$$\operatorname{cap}_p(x_0, E_1, D) \le \sum_{k=0}^{\infty} \operatorname{cap}_p(x_0, E_1^k, D) = 0 \quad (p \le n).$$

Since (23) is valid in the general case, we have

$$\operatorname{cap}_p(x_0, E_1, D) = M_p\Gamma(x_0, E_1, D) = 0 \quad (p \le n).$$

Finally, write $E_0 = \{x_k\}$. Then, by Proposition 14 and the first part of the proof,

$$\begin{aligned}\operatorname{cap}_p(E_0, E_1, D) = \operatorname{cap}_p(\{x_k\}, E_1, D) &\le \sum_{k=1}^{\infty} \operatorname{cap}_p(x_k, E_1, D)\\ &= \sum_{k=1}^{\infty} M_p\Gamma(x_k, E_1, D) = 0\end{aligned}$$

and since

$$M_p\Gamma(E_0, E_1, D) \le \sum_{k=1}^{\infty} M_p\Gamma(x_k, E_1, D) = 0\,,$$

we obtain (22), as desired.

COROLLARY. *Under the hypotheses of the preceding theorem*, $\operatorname{cap}_p(E_0, E_1) = M_p\Gamma(E_0, E_1) = 0$ $(p \le n)$.

THEOREM 7. $(\overline{E}_0 \cap \overline{E}_1 = \emptyset$, $\overline{E}_i - E_i$ $(i = 0, 1)$ *at most countable*, D *bounded and m-smooth of order* $p \le n$ *on* $(\overline{E}_0 \cup \overline{E}_1) \cap \partial D) \Rightarrow (1)$ *holds.*

P r o o f. From the preceding theorem and Proposition 1, we deduce that

$$\begin{aligned}
&M_p\Gamma(E_0, E_1, D)\\
&\quad = M_p\Gamma(\overline{E}_0 - E_0, \overline{E}_1, D) + M_p\Gamma(\overline{E}_0, \overline{E}_1 - E_1, D) + M_p\Gamma(E_0, E_1, D)\\
&\quad = M_p\Gamma(\overline{E}_0, \overline{E}_1, D) = \operatorname{cap}_p(\overline{E}_0, \overline{E}_1, D)\\
&\quad = \operatorname{cap}_p(E_0, E_1, D) + \operatorname{cap}_p(\overline{E}_0 - E_0, \overline{E}_1, D) + \operatorname{cap}_p(\overline{E}_0, \overline{E}_1 - E_1, D)\\
&\quad = \operatorname{cap}_p(E_0, E_1, D)\,.
\end{aligned}$$

COROLLARY. $(E_i$ *bounded*, $\overline{E}_i - E_i$ $(i = 0, 1)$ *at most countable and* $\overline{E}_0 \cap \overline{E}_1 = \emptyset) \Rightarrow M_p\Gamma(E_0, E_1) = \operatorname{cap}_p(E_0, E_1)$ $(p \le n)$.

LEMMA 6. *If* $\overline{E}_0 - E_0$ *is at most countable*, *then* $M_p\Gamma(E_0, E_1, D) = M_p\Gamma(\overline{E}_0, E_1, D)$ $(p \le n)$.

P r o o f. By Theorem 6, since $E_0 \subset E_0^*$ implies $M_p\Gamma(E_0, E_1, D) \le M_p\Gamma(E_0^*, E_1, D)$, we have

$$\begin{aligned}
M_p\Gamma(E_0, E_1, D) &\le M_p\Gamma(\overline{E}_0, E_1, D)\\
&\le M_p\Gamma(E_0, E_1, D) + M_p\Gamma(\overline{E}_0 - E_0, E_1, D)\\
&= M_p\Gamma(E_0, E_1, D)\,.
\end{aligned}$$

As a consequence of Lemmas 3 and 6, we deduce

THEOREM 8. *If* D *is bounded*, $\overline{E}_0 \subset D$, $E_1 \subset \overline{D}$, $\overline{E}_0 \cap \overline{E}_1 = \emptyset$ *and* $\overline{E}_0 - E_0$ *is at most countable*, *then* $M_p\Gamma(E_0, E_1, D) = \lim_{r\to 0} M_p\Gamma[E_0(r), E_1, D]$ $(p \le n)$.

P r o o f. Lemmas 3 and 6 yield

$$\begin{aligned}
M_p\Gamma(E_0, E_1, D) &= M_p\Gamma(\overline{E}_0, E_1, D)\\
&= \lim_{r\to 0} M_p\Gamma[\overline{E}_0(r), E_1, D] = \lim_{r\to 0} M_p\Gamma[E_0(r), E_1, D]\,.
\end{aligned}$$

PROPOSITION 19 (P. Caraman [1], Lemma 14). *If* D *is bounded*, $F_0, F_1 \subset \overline{D}$ *are closed*, $F_0 \subset D$ *and* $F_0 \cap F_1 = \emptyset$, *then* $\mathcal{A} = \{\varrho \in F[\Gamma(F_0, F_1, D)];\, \varrho$ *continuous in* $D - F_1\}$ *is p-complete.*

By the preceding theorem, arguing as in the preceding proposition we obtain

COROLLARY. *(F compact, $E \subset D$, $\overline{E} - E \subset D$ at most countable and $\overline{E} \cap F = \emptyset$) $\Rightarrow \mathcal{A}' = \{\varrho \in F[\Gamma(E,F,D)]; \varrho$ continuous in $D - F\}$ is p-complete.*

THEOREM 9. *If $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, $E_i = E_i' \cup E_i'' \cup E_i''' \cup F_i$ $(i = 0,1)$, E_i' is inaccessible by rectifiable arcs from D, E_i'' is open relative to $\overline{D}$ or to ∂D, E_i''' is at most countable, F_i is compact and D is m-smooth on $(F_0 \cup F_1) \cap \partial D$, then*

$$M\Gamma(E_0, E_1, D) = \operatorname{cap}(E_0, E_1, D)\,.$$

Proof. Corollary 1 of Theorem 4 and Theorem 6 yield

$$\begin{aligned} M\Gamma(E_0, E_1, D) &= M\Gamma(E_0' \cup E_0'' \cup F_0, E_1' \cup E_1'' \cup F_1, D) \\ &= \operatorname{cap}(E_o' \cup E_0'' \cup F_0, E_1' \cup E_1'' \cup F_1, D) = \operatorname{cap}(E_0, E_1, D)\,. \end{aligned}$$

COROLLARY 1. *With the notations of the preceding theorem, if $\overline{E}_0 \cap \overline{E}_1 = \emptyset$, and $E_i = E_i' \cup E_i'''$ $(i = 0,1)$, then (22) holds.*

Now, let us recall the following definitions of a topological cylinder (with respect to the euclidean metric).

A triple (B_0, B_1, Z), where Z is a domain and $B_0, B_1 \subset \partial Z$, is called a *topological cylinder with closed bases* if there exists a homeomorphism $\varphi : Z_0 \cup B_0^0 \cup B_1^0 \to Z \cup B_0 \cup B_1$ such that $\varphi(B_i^0) = B_i$, $Z_0 = \{x; (x^1)^2 + \ldots + (x^{n-1})^2 < 1\,,\ 0 < x^n < 1\}$ is the unit cylinder and $B_i^0 = \{x; (x^1)^2 + \ldots + (x^{n-1})^2 \le 1\,,\ x^n = i\}$ $(i = 0,1)$ are its bases. The B_i are the *bases* of the topological cylinder.

A triple (B_0, B_1, Z) is called a *topological cylinder with open bases* if the unit cylinder corresponding to φ has the bases $B_i^0 = \{x; (x^1)^2 + \ldots + (x^{n-1})^2 < 1\,,\ x^n = i\}$ $(i = 0,1)$.

As a direct consequence of Proposition 1, we have

COROLLARY 1. *If $Z = (B_0, B_1, Z)$ is a topological cylinder with closed bases and Z is smooth of order $p > 1$ on $B_0 \cup B_1$, then $M_p Z = \operatorname{cap}_p Z$.*

As a direct consequence of Corollary 7 of Lemma 5, we obtain

COROLLARY 2. *If a topological cylinder with closed bases is smooth on $B_0 \cup B_1$, then $M_p Z = \operatorname{cap}_p Z$ $(p \ge n)$.*

Remarks. 1. The condition for Z to be smooth (i.e. 1-smooth) on $B_0 \cup B_1$ is not more restrictive than to be m-smooth because a topological cylinder is locally connected on its bases (i.e. 1-connected), hence, if it is m-smooth, it has to be smooth.

2. Observe that we cannot have $B_i = F_i \cup E_i' \cup E_i'''$ $(i = 0,1)$, where F_i is closed, E_i' is inaccessible by rectifiable arcs, E_i''' is at most countable

and $F_i \neq B_i$. Indeed, assume otherwise. Since $B_i - F_i$ is then open in the topology induced on B_i, each $\xi_i \in B_i - F_i$ is an interior point (for the induced topology), i.e. there exists a superficial neighbourhood of ξ_i obtained as the intersection of a spatial neighbourhood of ξ_i with B_i and which is disjoint from F_i, e.g. $V_{\xi_i} = B(\xi_i, r_i) \cap B_i$, where $r_i < d(\xi_i, F_i)$; hence, $V_{\xi_i} \subset B_i - F_i \subset E_i' \cup E_i'''$, so that $E_i' \cup E_i'''$ may not be countable. Define $\dot{B}_i = B_i - \partial B_i$ (where ∂B_i is the relative boundary of B_i). Clearly, $V_{\xi_i} \cap \dot{B}_i \neq \emptyset$. Indeed, let $U_{\xi_i} = B(\xi_i, r_i) \cap (Z \cup B_i)$ and $U_{\xi_i^0} = \varphi^{-1}(U_{\xi_i})$. Since φ is a homeomorphism, $U_{\xi_i^0}$ is open in the topology induced on $Z_0 \cup B_i^0$, where $\xi_i^0 = \varphi^{-1}(\xi_i)$, while $V_{\xi_i^0} = \varphi^{-1}(V_{\xi_i})$ is open in the topology induced on B_i^0. Hence, $V_{\xi_i^0} \cap \dot{B}_i^0 \neq \emptyset$, where $\dot{B}_i^0 = B_i^0 - \partial B_i^0$ is an $(n-1)$-dimensional ball. Let $\eta_i^0 \in V_{\xi_i^0} \cap \dot{B}_i^0$ and $\eta_i = \varphi(\eta_i^0)$. Since $V_{\xi_i^0} \cap \dot{B}_i^0$ is open in the relative topology induced in $B_i^0, \varphi(V_{\xi_i^0} \cap \dot{B}_i^0) \subset \dot{B}_i$ is open in the relative topology induced in B_i and $\eta_i \in \dot{B}_i$ is an interior point of $E_i' \cup E_i''$.

Now, consider the ball $B(\eta_i, r_i')$, where $r_i' < d(\eta_i, F_i \cup \partial B_i)$, a point $x_i \in B(\eta_i, r_i') \cap Z$ and the relative neighbourhood $U_{\eta_i} = B_i \cap B(\eta_i, r_i')$. The family $\{\lambda\}$ of all linear segments joining x_i to U_{η_i} is uncountable, while the subfamily of linear segments containing points of E_i''' is at most countable. Let $\lambda = (x_i, \eta_i) \subset B(\eta_i, r_i')$ be a linear segment in $\{\lambda\}$ such that $\lambda \cap E_i''' = \emptyset$ and ξ_i' is the first point of B_i on $\overline{x_i \eta_i}$ from x_i toward η_i. Then the segment $\lambda' = (x_i, \xi_i') \subset Z$ is a rectifiable arc joining x_i to E_i' in Z, contradicting the hypotheses.

However, we want to point out that the bases B_i may contain points inaccessible from Z by rectifiable arcs.

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