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On some majorization of derivatives in the class $S^*(\gamma)$

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Abstract. We give a generalization of some result of J. Janowski and J. Stankiewicz [2].

Introduction. Let $S^*(\gamma)$ ($|\gamma| < \pi/2$) denote the class of γ-spiral-starlike functions, i.e., functions $F(z)$ holomorphic in the disc $K_1 = \{z; |z| < 1\}$ and satisfying the conditions

$$F(0) = 0\,, \quad F'(0) = 1\,, \quad \operatorname{Re}\left\{e^{-i\gamma}\frac{zF'(z)}{F(z)}\right\} > 0\,, \quad z \in K_1\,.$$

Among the many results concerning this class we have the sharp estimation [1]

$$m_1(r,\gamma) \le \left|\frac{zF'(z)}{F(z)}\right| \le m_2(r,\gamma)\,, \tag{1}$$

where

$$m_1(r,\gamma) = \frac{\sqrt{1+2r^2\cos 2\gamma + r^4} - 2r\cos\gamma}{1-r^2}\,, \tag{2}$$

$$m_2(r,\gamma) = \frac{\sqrt{1+2r^2\cos 2\gamma + r^4} + 2r\cos\gamma}{1-r^2}\,, \tag{3}$$

$$r = |z| < 1\,, \quad F \in S^*(\gamma)\,.$$

Let H denote the class of holomorphic functions in K_1. Define $\Omega = \{w \in H; |w(x)| \le 1 \text{ for } z \in K_1\}$.

Let f_1, f_2 be two holomorphic functions in K_1. We say that f_1 is *majorized* by f_2 in $K_R = \{z; |z| < R\}$ and write $f_1 \ll f_2$ if there exists a holomorphic function ϕ such that $|\phi(z)| \le 1$ and $f_1(z) = \phi(z)f_2(z)$, $z \in K_R$.

J. Janowski and J. Stankiewicz [2] considered the following problem. Let A, B be two fixed classes of holomorphic functions in K_1. Find the smallest

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function $T(r) = T(r, A, B)$, $r \in [0, 1)$, such that for every pair of functions $f \in A$, $F \in B$ we have the implication

$$f \ll F \text{ in } K_1 \Rightarrow |f'(z)| \leq T(r, A, B)|F'(z)| \text{ for } |z| = r < 1\,.$$

Main result. Let $f \in H$ and $F \in S^*(\gamma)$.

THEOREM. *If $f \ll F$ in K_1 and $|z| = r < 1$, then*

$$|f'(z)| \leq T(r, H, S^*(\gamma))|F'(z)|\,, \text{ where} \tag{4}$$

$$T(r, H, S^*(\gamma)) = \begin{cases} 1 & \text{for } r \in [0, r^*],\\ \dfrac{r^4 + 8r^2\cos^2\gamma + 2r^2 - 4rP(r,\gamma)\cos\gamma + 1}{4r(P(r,\gamma) - 2r\cos\gamma)} & \text{for } r \in (r^*, 1), \end{cases} \tag{5}$$

$$r^* = \sqrt{3 + 4\cos\gamma - 2\sqrt{2 + 6\cos\gamma + 4\cos^2\gamma}}\,, \tag{6}$$

$$P(r,\gamma) = \sqrt{1 + 2r^2\cos 2\gamma + r^4}\,. \tag{7}$$

The result is sharp.

Proof. If $f \ll F$ in K_1 then there exists $\phi \in \Omega$ such that

$$f(z) = \phi(z)F(z) \quad \text{for } z \in K_1\,. \tag{8}$$

Differentiating (8) and dividing by $F'(z)$ we obtain

$$\frac{f'(z)}{F'(z)} = \phi'(z)\frac{F(z)}{F'(z)} + \phi(z)\,. \tag{9}$$

Applying the well-known estimation for $\phi \in \Omega$,

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad \text{for } z \in K_1\,, \tag{10}$$

we have from (9), (1) and (10)

$$\left|\frac{f'(z)}{F'(z)}\right| \leq \frac{1 - |\phi(z)|^2}{1 - r^2} \cdot \frac{r(1 - r^2)}{\sqrt{1 + 2r^2\cos 2\gamma + r^4} - 2r\cos\gamma} + |\phi(z)|\,. \tag{11}$$

Define $u = |\phi(z)|$ and denote the right hand side of the above inequality by $G(u)$. Thus

$$G(u) = -\frac{r}{P(r,\gamma) - 2r\cos\gamma}u^2 + u + \frac{r}{P(r,\gamma) - 2r\cos\gamma}$$

where $P(r,\gamma) = \sqrt{1 + 2r^2\cos 2\gamma + r^4}$.

The function $G(u)$ attains its maximum at $u = 1$ when $r \in [0, r^*]$ and at

$$u = \frac{P(r,\gamma) - 2r\cos\gamma}{2r} \quad \text{when } r \in (r^*, 1)\,,$$

where

$$r^* = \sqrt{3 + 4\cos\gamma - 2\sqrt{2 + 6\cos\gamma + 4\cos^2\gamma}}\,.$$

Therefore

$$
(12)\qquad \max_{u\in[0,1]} G(u) = \begin{cases} 1 & \text{for } r\in[0,r^*],\\ \dfrac{r^4+8r^2\cos^2\gamma+2r^2-4rP(r,\gamma)\cos\gamma+1}{4r(P(r,\gamma)-2r\cos\gamma)} & \text{for } r\in(r^*,1). \end{cases}
$$

Thus by (12) and (11) we have (4).

The result is sharp. For $r\in(0,r^*]$ and for every pair of functions $\{f(z)=e^{i\theta}F(z),F(z)\}$, $F\in S^*(\gamma)$, θ an arbitrary real number, we have $T(r,H,S^*(\gamma))=1$.

If $r\in(r^*,1)$ equality is achieved at $z_0=re^{i\theta_0}$ for the functions $\widehat{f}(z)=\widehat{\phi}(z)\widehat{F}(z)$ and $\widehat{F}(z)$, where

$$
\widehat{\phi}(z)=\frac{z+\alpha}{1+\alpha z}\,,\qquad \alpha=\frac{-2r^2-2r\cos\gamma+P(r,\gamma)}{r(2+2r\cos\gamma-P(r,\gamma))}\,,
$$

$$
\widehat{F}(z)=\frac{z}{(1+ze^{-i\theta_0})^{2e^{i\gamma}\cos\gamma}}\,,\qquad \widehat{F}\in S^*(\gamma)\,.
$$

By simple calculation we can check that

$$
|\widehat{f}'(z_0)/\widehat{F}'(z_0)|=T(r,H,S^*(\gamma))
$$

and this completes the proof.

Remark. For $\gamma=0$ we obtain a result of [2] (Theorem 2, p. 54).

References

[1] C. Bucka and K. Ciozda, *Some estimations and problems of the majorization in the classes of functions* $S^k_{(\alpha,\beta)}$, Ann. Univ. Mariae Curie-Skłodowska 29 (4) (1975), 29–41.

[2] J. Janowski and J. Stankiewicz, *A relative growth of modulus of derivatives for majorized functions*, ibid. 32 (4) (1978), 51–61.

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