The classes of univalent functions connected with homographies

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Abstract. We define some new classes of univalent functions. The Schiffer differential equations are obtained for extremal functions from some of these classes.

1. Introduction. I would like to suggest studying some new classes of univalent functions. The idea of construction of these classes follows the definitions of the Bieberbach–Eilenberg and Gelfer functions.

Let \( D \) denote the unit disc \(|z| < 1\), let \( H_u(D) \) be the set of univalent holomorphic functions on \( D \) and let \( h \) be a homography. Define

\[
T(h,a) = \{ f \in H_u(D) : f(0) = a, w \in f(D) \Rightarrow h(w) \notin f(D) \}.
\]

If \( h(z) = -z, a = 1 \) we get the class of Gelfer functions; for \( h(z) = 1/z, a = 0 \), we have the Bieberbach–Eilenberg functions.

From (1.1) it follows that \( a \in f(D) \) while \( h(a) \notin f(D) \) and \( w_0 \notin f(D) \) where \( w_0 \) is a fixed point of \( h \). Since either \( h(a) \) or \( w_0 \) is not infinite, \( T(h,a) \cup \{ f(z) = a \} \) is a compact family.

In this paper I study the form of Schiffer’s differential equations for some \( T(h,a) \) classes. The idea of writing these equations consists in writing them for some special classes and then translating information to the others.

2. Extremal functions and Schiffer’s equations in \( T(h,a) \). We start with two theorems:

Theorem 1. Let \( h, l, p \) be homographies. Suppose that \( h(\infty) = \infty, \)
\( l = p \circ h \circ p^{-1}, p(a) = b \). Then

\[
T(l,b) \subset p(T(h,a)) = \{ p \circ f : f \in T(h,a) \}.
\]

If \( l(\infty) = \infty \) or \( l(b) = \infty \) then

\[
T(l,b) = p(T(h,a)).
\]

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functions satisfy the Schiffer differential equation.

Theorem 1. Let \( g \in T(l,b) \). Define \( f = p^{-1} \circ g \). Then \( f \) is univalent and \( f(0) = a \). Suppose that \( u \in g(D) \) and \( p^{-1}(u) = 0 \). Then \( l(u) = p \circ h \circ p^{-1}(u) = p(\infty) = u \in g(D) \). This is impossible, so that \( f \) is holomorphic in \( D \). From \( p^{-1} \circ g(z_1) = h \circ p^{-1} \circ g(z_2) \), \( z_1, z_2 \in D \), it follows that \( g(z_1) = l(g(z_2)) \); this contradiction gives \( f(z_1) \neq h(f(z_2)) \) and \( f \in T(h,a) \).

If \( l(\infty) = \infty \) then \( h \circ p^{-1}(\infty) = p^{-1} \circ l(\infty) = p^{-1}(\infty) \). If \( l(b) = \infty \) then \( p \circ h(a) = l \circ p(a) = \infty \). In both cases the pole of \( p \) is not in \( f(D) \) for \( f \in T(h,a) \). Hence \( p \circ f \) is holomorphic and univalent in \( D \). That \( p \circ f \in T(l,b) \) is proved as above.

Theorem 1 implies

**Theorem 2.** For every \( a \neq 0, \infty \) and every homography \( l \) there exist homographies \( h \) and \( p \) so that:

(i) \( l = p \circ h \circ p^{-1} \),

(ii) \( h(z) = \lambda z \) or \( h(z) = z + 1 \),

(iii) \( p(T(h,a)) = T(l,b) \).

Moreover, if \( l(\infty) = \infty \) then \( b \) is an arbitrary number, otherwise \( b \) is the pole of \( l \).

**Proof.** The proof of (i), (ii) can be found in [2]; (iii) follows from Theorem 1.

For a holomorphic function \( f(z) = a + a_1 z + a_2 z^2 + \ldots \) let \( \{f\}_s \) denote \( a_s \). For \( n \geq 2 \), define

\[
V_n = \{(x_1,y_1,\ldots,x_n,y_n) : x_s = \text{Re}\{f\}_s, y_s = \text{Im}\{f\}_s, \quad s = 1,\ldots,n, \ f \in T(h,a)\}.
\]

Let \( F(x_1,y_1,\ldots,x_n,y_n) \) be a real-valued function which satisfies the following conditions:

(a) \( F \) is defined in an open set \( U \supset V_n \cup \{a\} \),

(b) \( F \) and its derivatives \( F_s = \frac{1}{2}(\partial F/\partial x_s - i\partial F/\partial y_s) \), \( s = 1,\ldots,n \), are continuous in \( U \),

(c) \( \text{grad } F = (\sum_{s=1}^n |F_s|^2)^{1/2} > 0 \) in \( U \).

Then \( F \) defines a functional \( H \) by

\[
H(f) = F(\text{Re}\{f\}_1,\text{Im}\{f\}_1,\ldots,\text{Re}\{f\}_n,\text{Im}\{f\}_n).
\]

**Definition 1.** A function \( f^* \in T(h,a) \) is called extremal in \( T(h,a) \) if \( H(f^*) \geq H(f) \), \( f \in T(h,a) \), for some \( F \) as above.

It is known [4], [6] that extremal functions in some classes of univalent functions satisfy the Schiffer differential equation

\[
(zf'(z))^2 P(f(z)) = Q(z), \quad |z| < 1,
\]
where \( P(w), Q(z) \) are rational functions, \( Q(z) = \sum_{s=-n+1}^{n-1} B_s/z^s \), \( Q \) is real and nonnegative on \(|z| = 1\), \( B_0 \) is real and \( B_{-s} = B_s \).

These equations differ in \( P(w) \) which depends on the class of univalent functions considered.

Now we may prove the following theorem:

**Theorem 3.** Let \( p(T(h, a)) = T(l, b) \) and \( l = p \circ h \circ p^{-1} \), where \( p, l, h \) are homographies. Then

(i) \( f \) is an extremal function in \( T(h, a) \) if and only if \( p \circ f \) is extremal in \( T(l, b) \).

(ii) \( f \) satisfies Schiffer’s equation with \( P(w), Q(z) \) if and only if \( p \circ f \) satisfies Schiffer’s equation with \([p^{-1}(w)]^2 P(p^{-1}(w)) \) and \( Q(z) \).

**Proof.** (i) Let \( f^* \) be an extremal function in \( T(h, a) \). There is a function \( F(x_1, y_1, \ldots, x_n, y_n) \) so that \( H(f^*) \geq H(f) \), \( f \in T(h, a) \), where \( H \) is defined by (2.5). Let \( V_n \) and \( V_n^* \) denote the sets (2.3) for \( T(h, a) \) and \( T(l, b) \). We may define a mapping \( m : V_n^* \rightarrow V_n \) by

\[
V_n^* \ni (x_1^*, y_1^*, \ldots, x_n^*, y_n^*) \rightarrow (Re m_1, Im m_1, \ldots, Re m_n, Im m_n) \in V_n
\]

where \( m_1 = m_1(x_1^*, y_1^*), m_2 = m_2(x_1^*, y_1^*, x_2^*, y_2^*), \ldots, m_n = m_n(x_1^*, y_1^*, \ldots, x_n^*, y_n^*) \) are the polynomials appearing in the development

\[
p^{-1} \circ g(z) = m_0 + m_1(x_1^*, y_1^*)z + \ldots + m_n(x_1^*, y_1^*, \ldots, x_n^*, y_n^*)z^n + \ldots,
\]

\[
g(z) = b + (x_1^* + iy_1^*)z + \ldots + (x_n^* + iy_n^*)z^n + \ldots \in T(l, b).
\]

It follows from \( T(l, b) = p(T(h, a)) \) that we may define a function \( F^* \) in an open set \( U^* \supset V^* \cup \{b\} \) by

\[
F^*(x_1^*, y_1^*, \ldots, x_n^*, y_n^*) = F(Re m_1, Im m_1, \ldots, Re m_n, Im m_n).
\]

It is easy to see that for \( p^{-1}(z) = az \) or \( p^{-1}(z) = z + r \) or \( p^{-1}(z) = 1/z \) the Jacobian of \( m \) is not zero, so that \( F^* \) satisfies (2.4). We may define \( H^* \) as in (2.5). We have \( H^*(g) = H(f) \) where \( g = p \circ f \). Therefore \( H^*(g^*) \geq H(g), \) \( g \in T(l, b) \), where \( g^* = p \circ f^* \), so that \( g^* \) is extremal in \( T(l, b) \). The inverse implication is proved similarly.

(ii) The proof is obvious.

From Theorem 3 it follows that it is sufficient to investigate extremal problems in the classes \( T(h, a) \) where \( h(z) = \lambda z \) or \( h(z) = z + 1 \).

3. **Schiffer’s equation in classical form for the Gelfer function.**


**Theorem 4.** Suppose that \( \Psi \) is a functional on the class \( E \) of Bieberbach–Eilenberg functions. Suppose that for some \( f \in E \), \( Re \Psi(f) \geq Re \Psi(f^*) \) for
every $f^* \in E$, and $\Psi$ has a Gateaux derivative $L(f, \cdot)$ with respect to $f$. Then $f$ satisfies the differential equation

$$
\left(\frac{zf'(z)}{f(z)}\right)^2 A(f(z)) = Q(z), \quad |z| < 1,
$$

where

$$
A(w) = D(w) + L(f, f) + D(1/w)
$$

and

$$
Q(\xi) = E(\xi) + L(f, zf'(z)) + E(1/\xi),
$$

$$
D(w) = L\left(f, \frac{wf(z)}{f(z) - w}\right), \quad E(\xi) = L\left(f, \frac{zf'(z)}{z - \xi}\right).
$$

Further, $L(f, zf'(z))$ is real and $Q$ is real and nonpositive on $|z| = 1$. If $A(w) \not\equiv 0$ then $\mathbb{C}\setminus((f(D) \cup h(D))$ has no interior points where $h(z) = 1/f(z)$, and $\{-1, 1\} \subset \partial f(D)$.

This equation is written in functional form. We will write it in classical form.

Let $f$ be an extremal function in $E$ and let $H$ be a functional of type (2.5). For every function $g$ holomorphic in $D$ and every “near” $f$, in the sense that $|\{f\}_\nu - \{g\}_\nu|$ is sufficiently small for $\nu = 1, \ldots, n$, we may define a functional $\Psi$ by $\Psi(g) = H(g) - iH(f - i(f - g))$. If $g \in E$ is “near” $f$ then $\text{Re} \Psi(g) \leq \text{Re} \Psi(f)$. For $\varepsilon$ sufficiently small it is easy to obtain

$$
H(f + \varepsilon g) - H(f) = 2 \text{Re} \varepsilon \sum_{\nu=1}^n F_{\nu} \{g\}_\nu + o(\varepsilon).
$$

Therefore

$$
\Psi(f + \varepsilon g) - \Psi(f) = 2 \text{Re} \varepsilon \sum_{\nu=1}^n F_{\nu} \{g\}_\nu - i2\varepsilon \text{Re} i \sum_{\nu=1}^n F_{\nu} \{g\}_\nu + o(\varepsilon)
$$

$$
= 2\varepsilon \sum_{\nu=1}^n F_{\nu} \{g\}_\nu + o(\varepsilon),
$$

so that $\Psi$ has Gateaux derivative

$$
L(f, g) = 2 \sum_{\nu} F_{\nu} \{\text{Re}\{f\}_1, \text{Im}\{f\}_1, \ldots, \text{Re}\{f\}_n, \text{Im}\{f\}_n\} \{g\}_\nu.
$$

By (3.2) we have

$$
E(\xi) = \sum_{\nu=1}^n F_{\nu} \left\{-zf'(z) \frac{1}{1-z/\xi}\right\}_\nu = B + \frac{B_{-1}}{\xi} + \ldots + \frac{B_{-n+1}}{\xi^{n-1}},
$$

$$
E(1/\xi) = B + B_{-1} \xi + \ldots + B_{-n+1} \xi^{n-1}.
$$
Hence
\[ Q(\xi) = \sum_{\nu=-n+1}^{n-1} \frac{B_\nu}{\xi^\nu}, \]
where \( B_{-\nu} = \overline{B}_\nu \), \( \nu = 1, \ldots, n - 1 \) and \( B_0 = L(f, zf'(z)) + \overline{B} + B \) is real.

Similarly
\[ A(w) = \sum_{\nu=-n+1}^{n-1} \frac{A_\nu}{w^\nu}, \quad A_{-\nu} = A_\nu. \]

Without loss of generality we may multiply both sides by \(-1\), and then \( Q \) is nonnegative on \( \partial D \).

We have proved

**Theorem 5.** Let \( f \) be an extremal function in \( E \). Then \( f \) satisfies the equation
\[
\left( \frac{zf'(z)}{f(z)} \right)^2 A(f(z)) = Q(z), \quad z \in D,
\]
where
\[
(3.3) \quad A(w) = \sum_{\nu=-n+1}^{n-1} \frac{A_\nu}{w^\nu}, \quad Q(z) = \sum_{\nu=-n+1}^{n-1} \frac{B_\nu}{z^\nu}.
\]

The function \( Q \) is real and nonnegative on \( |z| = 1 \). \( B_0 \) is real and \( B_{-\nu} = \overline{B}_\nu \), \( A_{-\nu} = A_\nu \), \( \nu = 1, \ldots, n - 1 \). The set \( \mathbb{C} \setminus (f(D) \cup h(D)) \) has no interior points where \( h(z) = 1/f(z) \), and \( \{-1, 1\} \in \partial f(D) \).

We know that \( E = T(h, 0) \) where \( h(z) = 1/z \). Taking \( p(z) = (1+z)/(1-z) \) we get \( p(T(h, 0)) = T(l, 1) \) where \( l(z) = -z \), the class of Gelfer functions. Using Theorems 3 and 5 we may prove:

**Theorem 6.** Let \( f \) be an extremal function in the class of Gelfer functions \( T(l, 1) \) where \( l(z) = -z \). Then \( f \) satisfies the equation
\[
(3.4) \quad (zf'(z))^2 P(f(z)) = Q(z)
\]
where \( P(w) = U(w)/(w^2 - 1)^{m+2}, m < n \), \( U(w) \) is a polynomial, \( U(-w) = U(w) \), \( \deg U \leq 2m \), and \( Q \) is as in Theorem 5. Moreover, \( \{0, \infty\} \subset \partial f(D) \) and \( \mathbb{C} \setminus (f(D) \cup (-f(D)) \) has no interior points.

**Proof.** By Theorem 3, \( f \) satisfies Schiffer’s equations with \( Q(z) \) as above and
\[
P(w) = \frac{4}{(w+1)^2} \left( \frac{w+1}{w-1} \right)^2 A \left( \frac{w-1}{w+1} \right)
\]
where $A(u)$ is defined by (3.3). Hence
\[
P(w) = \frac{4}{(w^2 - 1)^2} A \left( \frac{w - 1}{w + 1} \right) = \frac{4}{(w^2 - 1)^2} \sum_{\nu=-n+1}^{n-1} A_\nu (w + 1)^\nu \frac{1}{(w - 1)^\nu}.
\]
We see that $P(-w) = P(w)$ because $A(u) = A(1/u)$. Let $A_{m+1} = A_{m+2} = \ldots = A_{n-1} = 0$. Then $P(w) = U(w)/(w^2 - 1)^{m+2}$ where $U$ is a polynomial. Since $P(-w) = P(w)$ we have $U(-w) = U(w)$. The degree of $U$ is not greater than $2m$.

The rest of the assertion follows from Theorem 5 and from the properties of the homography $p$.

Now we will obtain Schiffer’s equation in the class $T(p \circ h \circ p^{-1}, p(1))$ where $h(z) = -z$ and $p$ is a homography.

4. Schiffer’s equations in some $T(h, a)$ classes. Let $h(z) = -z$ and $l = p \circ h \circ p^{-1}$ where $p$ is a homography. Then $l$ has two fixed points $x$, $y$. Suppose that $x \neq \infty, y \neq \infty$. Then $p$ and $p^{-1}$ have the form [3]
\[
(4.1) \quad p^{-1}(z) = \frac{\lambda z - x}{z - y}, \quad p(z) = \frac{y z - \lambda x}{z - \lambda}.
\]
Hence
\[
l(z) = \frac{1}{2}(yz + x)z - xy = \frac{1}{2}(x + y).
\]
By Theorem 2, $P(T(h, 1)) = T(l, \frac{x}{2}(x + y))$. Because $p(1) = \frac{1}{2}(x + y)$ the parameter $\lambda$ is equal to $-1$. Now we may prove:

**Theorem 7.** Let $l$, $p$, $h$ be homeomorphisms such that $h(z) = -z$, $l = p \circ h \circ p^{-1}$. Suppose that $x \neq \infty, y \neq \infty$ are fixed points of $l$. Let $f$ be an extremal function in $T(l, \frac{1}{2}(x + y))$. Then $f$ satisfies the equation
\[
\frac{(zf'(z))^2}{(f(z) - \frac{1}{2}(x + y))^2} A(f(z)) = Q(z), \quad |z| < 1,
\]
where $Q(z)$ is as in (3.3),
\[
A(u) = \sum_{k=-n+1}^{n-1} \left( \frac{C_k}{w - \frac{1}{2}(x + y)} \right)^k, \quad C_k = \left( \frac{4}{(x - y)^2} \right)^k C_k,
\]
k = 1, \ldots, n - 1.

The points $x, y$ lie in $\partial f(D)$ and $\overline{C \setminus (f(D) \cup l(f(D)))}$ has no interior points.

**Proof.** Let $f$ be an extremal function in $T(l, \frac{1}{2}(x + y))$. Then $p^{-1} \circ f$ is an extremal function in $T(h, 1)$ so that $p^{-1} \circ f$ satisfies (3.4):
\[
[z(p^{-1} \circ f(z))]^2 U(p^{-1} \circ f(z)) \frac{1}{[1 - (p^{-1} \circ f(z))^2]^{m+2}} = Q(z).
\]
By (4.1), \( p^{-1}(w) = -(w - x)/(w - y) \). Therefore
\[
\frac{(zf'(z))^2}{(f(z) - \frac{1}{2}(x + y))^2} A(f(z)) = Q(z)
\]
where
\[
A(w) = \frac{4U(p^{-1}(w))}{1 - (p^{-1}(w))^2} = \frac{KU\left(\frac{w - x}{w - y}\right)(w - y)^{2m}}{(w - \frac{1}{2}(x + y))^{m}},
\]
\( K \) is a constant and \( U \) a polynomial of degree not greater than \( 2m \), \( m < n \), \( U(-w) = U(w) \). Using \( U(-w) = U(w) \) it is easy to see that \( A(l(w)) = A(w) \). The function \( A(w) \) is rational and has one pole of degree \( m \) at \( \frac{1}{2}(x + y) \). Therefore
\[
A(w) = \sum_{k=-n+1}^{-1} \frac{C_k}{(w - \frac{1}{2}(x + y))^k}, \quad C_{-m-1} = C_{-m-2} = \ldots = C_{-n+1} = 0.
\]
From \( A(l(w)) = A(w) \) it follows that
\[
C_{-k} = \left[\frac{4}{(x - y)^2}\right]^k C_k, \quad k = 1, \ldots, n-1.
\]
Because \( 0, \infty \in \partial(p^{-1} \circ f(D)) \) the points \( x, y \) lie in \( \partial f(D) \). The set \( \mathbf{T} \setminus (f(D) \cup l(f(D))) \) has no interior points because \( \mathbf{T} \setminus (p^{-1} \circ f(D) \cup h \circ p^{-1} \circ f(D)) \) has no such points.

References