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The classes of univalent functions connected with homographies

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Abstract. We define some new classes of univalent functions. The Schiffer differential equations are obtained for extremal functions from some of these classes.

1. Introduction. I would like to suggest studying some new classes of univalent functions. The idea of construction of these classes follows the definitions of the Bieberbach–Eilenberg and Gelfer functions.

Let D denote the unit disc $|z| < 1$, let $H_u(D)$ be the set of univalent holomorphic functions on D and let h be a homography. Define

$$T(h,a) = \{f \in H_u(D) : f(0) = a, w \in f(D) \Rightarrow h(w) \notin f(D)\}\,. \tag{1.1}$$

If $h(z) = -z$, $a = 1$ we get the class of Gelfer functions; for $h(z) = 1/z$, $a = 0$, we have the Bieberbach–Eilenberg functions.

From (1.1) it follows that $a \in f(D)$ while $h(a) \notin f(D)$ and $w_0 \notin f(D)$ where w_0 is a fixed point of h. Since either $h(a)$ or w_0 is not infinite, $T(h,a) \cup \{f(z) \equiv a\}$ is a compact family.

In this paper I study the form of Schiffer's differential equations for some $T(h,a)$ classes. The idea of writing these equations consists in writing them for some special classes and then translating information to the others.

2. Extremal functions and Schiffer's equations in $T(h,a)$. We start with two theorems:

THEOREM 1. *Let h, l, p be homographies. Suppose that $h(\infty) = \infty$, $l = p \circ h \circ p^{-1}$, $p(a) = b$. Then*

$$T(l,b) \subset p(T(h,a)) = \{p \circ f : f \in T(h,a)\}. \tag{2.1}$$

If $l(\infty) = \infty$ or $l(b) = \infty$ then

$$T(l,b) = p(T(h,a)). \tag{2.2}$$

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Proof. Let $g \in T(l,b)$. Define $f = p^{-1} \circ g$. Then f is univalent and $f(0) = a$. Suppose that $u \in g(D)$ and $p^{-1}(u) = \infty$. Then $l(u) = p \circ h \circ p^{-1}(u) = p(\infty) = u \in g(D)$. This is impossible, so that f is holomorphic in D. From $p^{-1} \circ g(z_1) = h \circ p^{-1} \circ g(z_2)$, $z_1, z_2 \in D$, it follows that $g(z_1) = l(g(z_2))$; this contradiction gives $f(z_1) \neq h(f(z_2))$ and $f \in T(h,a)$.

If $l(\infty) = \infty$ then $h \circ p^{-1}(\infty) = p^{-1} \circ l(\infty) = p^{-1}(\infty)$. If $l(b) = \infty$ then $p \circ h(a) = l \circ p(a) = \infty$. In both cases the pole of p is not in $f(D)$ for $f \in T(h,a)$. Hence $p \circ f$ is holomorphic and univalent in D. That $p \circ f \in T(l,b)$ is proved as above.

Theorem 1 implies

THEOREM 2. *For every $a \neq 0$, ∞ and every homography l there exist homographies h and p so that*:

(i) $l = p \circ h \circ p^{-1}$,
(ii) $h(z) = \lambda z$ *or* $h(z) = z + 1$,
(iii) $p(T(h,a)) = T(l,b)$.

Moreover, *if* $l(\infty) = \infty$ *then* b *is an arbitrary number*, *otherwise* b *is the pole of* l.

Proof. The proof of (i), (ii) can be found in [2]; (iii) follows from Theorem 1.

For a holomorphic function $f(z) = a + a_1 z + a_2 z^2 + \ldots$ let $\{f\}_s$ denote a_s. For $n \geq 2$, define

$$V_n = \{(x_1, y_1, \ldots, x_n, y_n) : x_s = \operatorname{Re}\{f\}_s, y_s = \operatorname{Im}\{f\}_s, \quad s = 1, \ldots, n,\ f \in T(h,a)\}. \tag{2.3}$$

Let $F(x_1, y_1, \ldots, x_n, y_n)$ be a real-valued function which satisfies the following conditions:

(2.4)
(a) F is defined in an open set $U \supset V_n \cup \{a\}$,
(b) F and its derivatives $F_s = \frac{1}{2}(\partial F/\partial x_s - i\partial F/\partial y_s)$, $s = 1 \ldots, n$, are continuous in U,
(c) $|\operatorname{grad} F| = (\sum_{s=1}^{n} |F_s|^2)^{1/2} > 0$ in U.

Then F defines a functional H by

$$H(f) = F(\operatorname{Re}\{f\}_1, \operatorname{Im}\{f\}_1, \ldots, \operatorname{Re}\{f\}_n, \operatorname{Im}\{f\}_n). \tag{2.5}$$

DEFINITION 1. A function $f^* \in T(h,a)$ is called *extremal* in $T(h,a)$ if $H(f^*) \geq H(f)$, $f \in T(h,a)$, for some F as above.

It is known [4], [6] that extremal functions in some classes of univalent functions satisfy the Schiffer differential equation

$$(zf'(z))^2 P(f(z)) = Q(z), \quad |z| < 1, \tag{2.6}$$

where $P(w)$, $Q(z)$ are rational functions, $Q(z)=\sum_{s=-n+1}^{n-1} B_s/z^s$, Q is real and nonnegative on $|z|=1$, B_0 is real and $B_{-s}=\overline{B}_s$.

These equations differ in $P(w)$ which depends on the class of univalent functions considered.

Now we may prove the following theorem:

THEOREM 3. *Let* $p(T(h,a))=T(l,b)$ *and* $l=p\circ h\circ p^{-1}$, *where* p, l, h *are homographies. Then*

(i) f *is an extremal function in* $T(h,a)$ *if and only if* $p\circ f$ *is extremal in* $T(l,b)$,

(ii) f *satisfies Schiffer's equation with* $P(w)$, $Q(z)$ *if and only if* $p\circ f$ *satisfies Schiffer's equation with* $[(p^{-1}(w))']^2 P(p^{-1}(w))$ *and* $Q(z)$.

Proof. (i) Let f^* be an extremal function in $T(h,a)$. There is a function $F(x_1,y_1,\ldots,x_n,y_n)$ so that $H(f^*)\ge H(f)$, $f\in T(h,a)$, where H is defined by (2.5). Let V_n and V_n^* denote the sets (2.3) for $T(h,a)$ and $T(l,b)$. We may define a mapping $m:V_n^*\to V_n$ by

$$V_n^*\ni(x_1^*,y_1^*,\ldots x_n^*,y_n^*)\to(\operatorname{Re}m_1,\operatorname{Im}m_1,\ldots,\operatorname{Re}m_n,\operatorname{Im}m_n)\in V_n$$

where $m_1=m_1(x_1^*,y_1^*)$, $m_2=m_2(x_1^*,y_1^*,x_2^*,y_2^*),\ldots$, $m_n=m_n(x_1^*,y_1^*,\ldots,x_n^*,y_n^*)$ are the polynomials appearing in the development

$$p^{-1}\circ g(z)=m_0+m_1(x_1^*,y_1^*)z+\ldots+m_n(x_1^*,y_1^*,\ldots,x_n^*,y_n^*)z^n+\ldots,$$
$$g(z)=b+(x_1^*+iy_1^*)z+\ldots+(x_n^*+iy_n^*)z^n+\ldots\in T(l,b).$$

It follows from $T(l,b)=p(T(h,a))$ that we may define a function F^* in an open set $U^*\supset V^*\cup\{b\}$ by

$$F^*(x_1^*,y_1^*,\ldots,x_n^*,y_n^*)=F(\operatorname{Re}m_1,\operatorname{Im}m_1,\ldots,\operatorname{Re}m_n,\operatorname{Im}m_n).$$

It is easy to see that for $p^{-1}(z)=\alpha z$ or $p^{-1}(z)=z+r$ or $p^{-1}(z)=1/z$ the Jacobian of m is not zero, so that F^* satisfies (2.4).We may define H^* as in (2.5). We have $H^*(g)=H(f)$ where $g=p\circ f$. Therefore $H^*(g^*)\ge H(g)$, $g\in T(l,b)$, where $g^*=p\circ f^*$, so that g^* is extremal in $T(l,b)$. The inverse implication is proved similarly.

(ii) The proof is obvious.

From Theorem 3 it follows that it is sufficient to investigate extremal problems in the classes $T(h,a)$ where $h(z)=\lambda z$ or $h(z)=z+1$.

3. Schiffer's equation in classical form for the Gelfer function. In [1] J. A. Hummel and M. M. Schiffer proved

THEOREM 4. *Suppose that* Ψ *is a functional on the class* E *of Bieberbach–Eilenberg functions. Suppose that for some* $f\in E$, $\operatorname{Re}\Psi(f)\ge\operatorname{Re}\Psi(f^*)$ *for* *every $f^* \in E$, and Ψ has a Gateaux derivative $L(f,\cdot)$ with respect to f. Then f satisfies the differential equation*

$$\left(\frac{zf'(z)}{f(z)}\right)^2 A(f(z)) = Q(z), \quad |z|<1, \tag{3.1}$$

where

$$\begin{aligned} A(w) &= D(w) + L(f,f) + D(1/w) \\ Q(\xi) &= E(\xi) + L(f, zf'(z)) + \overline{E(1/\overline{\xi})}, \\ D(w) &= L\left(f, \frac{wf(z)}{f(z)-w}\right), \qquad E(\xi) = L\left(f, \frac{zf'(z)}{z-\xi}\right). \end{aligned} \tag{3.2}$$

Further, $L(f, zf'(z))$ is real and Q is real and nonpositive on $|z|=1$. If $A(w) \not\equiv 0$ then $\overline{\mathbb{C}}\setminus(f(D)\cup h(D))$ has no interior points where $h(z) = 1/f(z)$, and $\{-1,1\} \subset \partial f(D)$.

This equation is written in functional form. We will write it in classical form.

Let f be an extremal function in E and let H be a functional of type (2.5). For every function g holomorphic in D and every "near" f, in the sense that $|\{f\}_\nu - \{g\}_\nu|$ is sufficiently small for $\nu = 1,\ldots,n$, we may define a functional Ψ by $\Psi(g) = H(g) - iH(f - i(f-g))$. If $g \in E$ is "near" f then $\operatorname{Re}\Psi(g) \le \operatorname{Re}\Psi(f)$. For ε sufficiently small it is easy to obtain

$$H(f+\varepsilon g) - H(f) = 2\operatorname{Re}\varepsilon\sum_{\nu=1}^{n} F_\nu\{g\}_\nu + o(\varepsilon).$$

Therefore

$$\begin{aligned} \Psi(f+\varepsilon g) - \Psi(f) &= 2\operatorname{Re}\varepsilon\sum_{\nu=1}^{n} F_\nu\{g\}_\nu - i2\varepsilon\operatorname{Re} i\sum_{\nu=1}^{n} F_\nu\{g\}_\nu + o(\varepsilon) \\ &= 2\varepsilon\sum_{\nu=1}^{n} F_\nu\{g\}_\nu + o(\varepsilon), \end{aligned}$$

so that Ψ has Gateaux derivative

$$L(f,g) = 2\sum_\nu F_\nu(\operatorname{Re}\{f\}_1, \operatorname{Im}\{f\}_1, \ldots, \operatorname{Re}\{f\}_n, \operatorname{Im}\{f\}_n)\{g\}_\nu.$$

By (3.2) we have

$$\begin{aligned} E(\xi) &= \sum_{\nu=1}^{n} F_\nu\left\{-zf'(z)\frac{1}{1-z/\xi}\right\}_\nu = B + \frac{B_{-1}}{\xi} + \ldots + \frac{B_{-n+1}}{\xi^{n-1}}, \\ \overline{E(1/\overline{\xi})} &= \overline{B} + \overline{B}_{-1}\xi + \ldots + \overline{B}_{-n+1}\xi^{n-1}. \end{aligned}$$

Hence

$$Q(\xi) = \sum_{\nu=-n+1}^{n-1} \frac{B_\nu}{\xi^\nu},$$

where $B_{-\nu} = \overline{B}_\nu$, $\nu = 1, \ldots, n-1$ and $B_0 = L(f, zf'(z)) + \overline{B} + B$ is real. Similarly

$$A(w) = \sum_{\nu=-n+1}^{n-1} \frac{A_\nu}{w^\nu} \quad \text{where } A_{-\nu} = A_\nu.$$

Without loss of generality we may multiply both sides by -1, and then Q is nonnegative on ∂D.

We have proved

THEOREM 5. *Let f be an extremal function in E. Then f satisfies the equation*

$$\left(\frac{zf'(z)}{f(z)}\right)^2 A(f(z)) = Q(z), \qquad z \in D,$$

where

$$A(w) = \sum_{\nu=-n+1}^{n-1} A_\nu / w^\nu, \qquad Q(z) = \sum_{\nu=-n+1}^{n-1} B_\nu / z^\nu. \tag{3.3}$$

The function Q is real and nonnegative on $|z| = 1$. B_0 is real and $B_{-\nu} = \overline{B}_\nu$, $A_{-\nu} = A_\nu$, $\nu = 1, \ldots, n-1$. The set $\overline{\mathbb{C}} \setminus (f(D) \cup h(D))$ has no interior points where $h(z) = 1/f(z)$, and $\{-1, 1\} \in \partial f(D)$.

We know that $E = T(h, 0)$ where $h(z) = 1/z$. Taking $p(z) = (1+z)/(1-z)$ we get $p(T(h, 0)) = T(l, 1)$ where $l(z) = -z$, the class of Gelfer functions. Using Theorems 3 and 5 we may prove:

THEOREM 6. *Let f be an extremal function in the class of Gelfer functions $T(l, 1)$ where $l(z) = -z$. Then f satisfies the equation*

$$(zf'(z))^2 P(f(z)) = Q(z) \tag{3.4}$$

where $P(w) = U(w)/(w^2 - 1)^{m+2}$, $m < n$, $U(w)$ is a polynomial, $U(-w) = U(w)$, $\deg U \le 2m$, and Q is as in Theorem 5. Moreover, $\{0, \infty\} \subset \partial f(D)$ and $\overline{\mathbb{C}} \setminus (f(D) \cup (-f(D))$ has no interior points.

Proof. By Theorem 3, f satisfies Schiffer's equations with $Q(z)$ as above and

$$P(w) = \frac{4}{(w+1)^4} \left(\frac{w+1}{w-1}\right)^2 A\left(\frac{w-1}{w+1}\right)$$

where $A(u)$ is defined by (3.3). Hence

$$P(w) = \frac{4}{(w^2-1)^2} A\left(\frac{w-1}{w+1}\right) = \frac{4}{(w^2-1)^2} \sum_{\nu=-n+1}^{n-1} \frac{A_\nu (w+1)^\nu}{(w-1)^\nu} .$$

We see that $P(-w) = P(w)$ because $A(u) = A(1/u)$. Let $A_{m+1} = A_{m+2} = \ldots = A_{n-1} = 0$. Then $P(w) = U(w)/(w^2-1)^{m+2}$ where U is a polynomial. Since $P(-w) = P(w)$ we have $U(-w) = U(w)$. The degree of U is not greater than $2m$.

The rest of the assertion follows from Theorem 5 and from the properties of the homography p.

Now we will obtain Schiffer's equation in the class $T(p \circ h \circ p^{-1}, p(1))$ where $h(z) = -z$ and p is a homography.

4. Schiffer's equations in some $T(h, a)$ classes. Let $h(z) = -z$ and $l = p \circ h \circ p^{-1}$ where p is a homography. Then l has two fixed points x, y. Suppose that $x \neq \infty$,$y \neq \infty$. Then p and p^{-1} have the form [3]

$$p^{-1}(z) = \lambda \frac{z-x}{z-y}, \qquad p(z) = \frac{yz - \lambda x}{z - \lambda}. \tag{4.1}$$

Hence

$$l(z) = \frac{\frac{1}{2}(y+x)z - xy}{z - \frac{1}{2}(x+y)}.$$

By Theorem 2, $P(T(h,1)) = T(l, \frac{1}{2}(x+y))$. Because $p(1) = \frac{1}{2}(x+y)$ the parameter λ is equal to -1. Now we may prove:

THEOREM 7. *Let l, p, h be homographies such that $h(z) = -z$, $l = p \circ h \circ p^{-1}$. Suppose that $x \neq \infty$, $y \neq \infty$ are fixed points of l. Let f be an extremal function in $T(l, \frac{1}{2}(x+y))$. Then f satisfies the equation*

$$\frac{(zf'(z))^2}{(f(z) - \frac{1}{2}(x+y))^2} A(f(z)) = Q(z), \qquad |z| < 1,$$

where $Q(z)$ is as in (3.3),

$$A(w) = \sum_{k=-n+1}^{n-1} \frac{C_k}{(w - \frac{1}{2}(x+y))^k} , \qquad C_{-k} = \left(\frac{4}{(x-y)^2}\right)^k C_k , \qquad k = 1, \ldots, n-1 .$$

The points x, y lie in $\partial f(D)$ and $\overline{\mathbb{C}} \setminus (f(D) \cup l(f(D)))$ has no interior points.

Proof. Let f be an extremal function in $T(l, \frac{1}{2}(x+y))$. Then $p^{-1} \circ f$ is an extremal function in $T(h,1)$ so that $p^{-1} \circ f$ satisfies (3.4):

$$[z(p^{-1} \circ f(z))']^2 U(p^{-1} \circ f(z)) \frac{1}{[1 - (p^{-1} \circ f(z))^2]^{m+2}} = Q(z).$$

By (4.1), $p^{-1}(w) = -(w-x)/(w-y)$. Therefore

$$\frac{(zf'(z))^2}{(f(z)-\frac{1}{2}(x+y))^2}A(f(z)) = Q(z)$$

where

$$A(w) = \frac{4U(p^{-1}(w))}{[1-(p-1(w))^2]^m} = \frac{KU\left(\dfrac{w-x}{w-y}\right)(w-y)^{2m}}{(w-\frac{1}{2}(x+y))^m},$$

K is a constant and U a polynomial of degree not greater than $2m$, $m<n$, $U(-w)=U(w)$. Using $U(-w)=U(w)$ it is easy to see that $A(l(w)) = A(w)$. The function $A(w)$ is rational and has one pole of degree m at $\frac{1}{2}(x+y)$. Therefore

$$A(w) = \sum_{k=-n+1}^{n-1} \frac{C_k}{(w-\frac{1}{2}(x+y))^k}, \qquad C_{-m-1} = C_{-m-2} = \ldots = C_{-n+1} = 0.$$

From $A(l(w)) = A(w)$ it follows that

$$C_{-k} = \left[\frac{4}{(x-y)^2}\right]^k C_k, \qquad k = 1,\ldots,n-1.$$

Because $0,\ \infty \in \partial(p^{-1}\circ f(D))$ the points x, y lie in $\partial f(D)$. The set $\overline{\mathbb{C}} \setminus (f(D)\cup l(f(D)))$ has no interior points because $\overline{\mathbb{C}}\setminus(p^{-1}\circ f(D)\cup h\circ p^{-1}\circ f(D))$ has no such points.

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