Automorphisms of Riemann surfaces
with two fixed points

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Abstract. We give an upper bound for the order of an automorphism of a Riemann surface with two fixed points. The main results are presented in Theorems 1.4 and 2.4.

0. Introduction. We assume the reader to be familiar with basic concepts in the theory of Riemann surfaces. Notions of hyperbolicity, genus, automorphism, quotient space etc. are used freely without further comments. Precise definitions may be found in [F-K]. As far as possible we also stick to the notation of that book.

The study of Riemann surfaces via their mappings is an old story. It is surprising how much information about the surface itself is contained in the properties of its mappings. The first important observation concerning the whole group of automorphisms of a hyperbolic, compact Riemann surface is contained in the Hurwitz theorem (see Theorem 1.1). The cyclic subgroups of that group were studied by Wiman, Kodaira and Lewittes. Their results were completed by Harvey (see Theorem 1.2).

It was Minda who noticed that all self-mappings of a hyperbolic Riemann surface with two fixed points must be automorphisms. This is not true in the non-hyperbolic case; fortunately there are only few such surfaces. Recently Jarnicki and Tworzewski [J-T] got a much sharper theorem (see Theorem 0.1).

Let X be a Riemann surface. The group of its automorphisms is denoted by $\mathrm{Aut}(X)$. If a, b are two distinct points of X, we denote by $G(X; a, b)$ the set of all self-mappings of X with a and b fixed.

THEOREM 0.1. *If X is a hyperbolic Riemann surface then $G(X; a, b)$ is a finite, cyclic subgroup of* $\mathrm{Aut}(X)$.

It seems natural to ask what the order of the above group can be. As

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different techniques are involved we deal separately with compact surfaces
and with domains in the complex plane.

1. Compact surfaces. Throughout this section M denotes a compact
Riemann surface of genus g. We assume that $g \geq 2$, that is, the surface is
hyperbolic. Further, T is an automorphism of M with $\nu(T)$ fixed points.
The subgroup of $\mathrm{Aut}(M)$ generated by T is denoted by $\langle T \rangle$. If G is any
group we denote the number of its elements by $\mathrm{ord}\, G$. Instead of $\mathrm{ord}\langle T \rangle$ we
simply write $\mathrm{ord}\, T$.

We look for a bound on the order of T in terms of the genus g of the
surface on which it operates. This seems very natural in view of the following
two statements.

THEOREM 1.1 (Hurwitz 1893). $\mathrm{ord}\,\mathrm{Aut}(M) \leq 84(g-1)$.

THEOREM 1.2 (Harvey, [H] 1966). *For any $T \in \mathrm{Aut}(M)$, $\mathrm{ord}\, T \leq 4g+2$.*

For automorphisms with one fixed point the above theorem is sharp:

EXAMPLE 1.3. Let M be the normalization of the projective closure of
a curve given by
$$w^2 = z(z^{2g+1} - 1)\,, \quad g \geq 2\,.$$
Thus M is a hyperelliptic Riemann surface of genus g.

Let $T(z, w) = (\varepsilon_{2g+1} z, \varepsilon_{4g+2} w)$, where ε_p denotes a p-primitive root of
1. It is easily seen that $T \in \mathrm{Aut}(M)$ and $(0, 0)$ is its fixed point. Of course
$\mathrm{ord}\, T = 4g + 2$.

In case of two fixed points Harvey's result can be sharpened:

THEOREM 1.4. *If $\nu(T) \geq 2$ then $\mathrm{ord}\, T \leq 4g$.*

P r o o f. Let M^* be the quotient Riemann surface $M/\langle T \rangle$, γ its genus
and $N = \mathrm{ord}\, T$. Let $\pi : M \to M^*$ be the quotient map. π is thus an
N-sheeted branch covering. We apply the Riemann–Hurwitz formula to π
to get

$(*)$ $\qquad\qquad\qquad 2g - 2 = N(2\gamma - 2) + B$

where B denotes the branch index of π.

C a s e 1: either $\nu(T) > 2$ or $\gamma > 0$. The branch index of the quotient
map at fixed points of T is $N - 1$, thus $B \geq \nu(T)(N - 1)$. Applying this in
$(*)$ we get $2g - 2 \geq N[2\gamma + \nu(T) - 2] + \nu(T)(N - 1)$. Since the expression
in square brackets is not zero we get immediately
$$N \leq \frac{2g + \nu(T) - 2}{2\gamma + \nu(T) - 2} \leq 2g + 1 \leq 4g\,.$$

C a s e 2: $\nu(T) = 2$ and $\gamma = 0$. From the Riemann–Hurwitz formula
it follows at once that T has at least one orbit of order p, where $1 < p <$

N (otherwise $g = 0$, which contradicts the hyperbolicity of M). From elementary group theory it follows that p must divide N and thus $p \leq N/2$. This time the branch index B satisfies

$$B \geq 2(N - 1) + p(N/p - 1) = 3N - 2 - p.$$

Once again we link this with $(*)$ to get $2g - 2 \geq -2N + 3N - 2 - p$ and consequently $N \leq 2g + p \leq 2g + N/2$, which yields the desired $N \leq 4g$. ∎

R e m a r k 1.5. Although our result is close to that of Harvey we got it using much simpler methods. In fact, only an elementary formula in the theory of compact Riemann surfaces (the Riemann–Hurwitz formula) was applied.

R e m a r k 1.6. The estimate of Theorem 1.4 is sharp, that is, for any $g \geq 2$ there exists a compact Riemann surface M_g of genus g and points $a, b \in M_g$ such that there exists an automorphism $T \in G(M; a, b)$ with $\operatorname{ord} T = 4g$.

EXAMPLE 1.7. Let M_g be the normalization of the projective closure of the curve given by

$$w^2 = z(z^{2g} - 1), \quad g \geq 2.$$

M_g is a hyperelliptic compact Riemann surface of genus g. Let $T(z, w) = (\varepsilon_{2g} z, \varepsilon_{4g} w)$, where ε_p is as in Example 1.3. It is easily seen that $T \in \operatorname{Aut}(M)$ and $(0, 0)$ and the point at infinity are its fixed points. Moreover, $\operatorname{ord} T = 4g$.

2. Domains in complex plane. The key concept in this part is that of "complex double". We sketch briefly the construction of this surface. The first step is to notice that any domain may be conformally mapped onto another one of a particularly nice type. Throughout the section Ω denotes an open domain in the complex plane, $\partial\Omega$ its boundary and $\overline{\Omega}$ its closure.

DEFINITION 2.1. Domain whose border consists of p circles, some of them possibly being points, is call *p-circular*. If all components of its boundary are circles with positive radius we call the domain *proper*.

In the sense of the above definition annuli are 2-circular proper domains. Circular domains are canonical as the following theorem shows.

THEOREM 2.2 (Goluzin [G]). *Let Ω be a p-connected bounded domain in the complex plane. Then there exists a bounded p-circular domain Δ and a conformal map $s : \Omega \to \Delta$.*

Of course s induces an isomorphism of automorphism groups,

$$\operatorname{Aut}(\Omega) \ni g \to s \circ g \circ s^{-1} \in \operatorname{Aut}(\Delta).$$

If Ω is a circular domain and $T \in \operatorname{Aut}(\Omega)$ then by the Riemann theorem on isolated singularities T may be extended through the isolated points of

$\partial\Omega$. So it suffices to consider automorphisms of proper circular domains. Moreover, the following holds.

THEOREM 2.3 (Behnke–Sommer [B-S]). *Let Ω be a proper p-circular domain and let $T \in \mathrm{Aut}(\Omega)$. Then T may be extended to a homeomorphism $\widetilde{T}$ of $\overline{\Omega}$.*

After these preparations we can construct the "complex double" (cf. [B-S]). Let Ω be a bounded proper p-circular domain. Let Ω^+ and Ω^- be two copies of Ω. We assume that Ω^+ has the usual complex structure and the other copy the conjugate one. If we glue $\overline{\Omega}^+$ and $\overline{\Omega}^-$ together by identifying the corresponding points of $\partial\Omega^+$ and $\partial\Omega^-$ the resulting compact manifold M clearly has genus $g = p - 1$. Let S denote the set of points from the glued borders. It is also easily seen that there is a unique complex structure on M compatible with those on Ω^+ and Ω^-. If now $T \in \mathrm{Aut}(\Omega)$ then it extends to an automorphism of M by the following formula which results from the Schwarz reflection principle:

$$T^*(z) = \begin{cases} T(z) & \text{if } z \in \Omega^+, \\ \widetilde{T}(z) & \text{if } z \in S, \\ \overline{T(\overline{z})} & \text{if } z \in \Omega^-. \end{cases}$$

where $\widetilde{T}$ is the homeomorphism from Theorem 2.3.

Of course $\mathrm{ord}\, T = \mathrm{ord}\, T^*$ but more important is the observation that $\nu(T^*) \geq 2\nu(T)$. The number of fixed points is thus at least doubled since there are now fixed points in both glued copies of Ω and perhaps some new ones on S.

Our main result in this part is now a simple consequence of the above observations.

THEOREM 2.4. *If Ω is a p-connected bounded domain in the complex plane and $T \in \mathrm{Aut}(\Omega)$ has at least 2 fixed points, then $\mathrm{ord}\, T \leq p$.*

Proof. First by Theorem 2.2 we can assume that Ω is a circular domain. Next, to estimate the order of T it is enough to assume that Ω is proper. Thus we can consider the complex double M and its automorphism T^* defined above. As M is a compact Riemann surface we can define M^* to be the quotient space $M/\langle T^* \rangle$ and apply the Riemann–Hurwitz formula to the quotient map $\pi : M \to M^*$. Let γ be as usual the genus of M^* and B the branch index of π. As T^* has at least 4 fixed points and the branch index of π at each of them is $N - 1$ where N is the order of T, we get $B \geq 4(N-1)$ and hence $2g - 2 \geq N(2\gamma - 2) + 4(N-1)$. Since $\gamma \geq 0$ it follows that $N \leq g + 1 = p$. ∎

R e m a r k 2.5. The assumption that Ω is bounded may be dropped; it suffices to assume that $\mathbb{C} \setminus \Omega$ has non-empty interior. In fact, if we consider

domains on the Riemann sphere we can map any such domain by a Möbius transformation ϕ in such a way that the point at infinity is in the interior of $\widehat{\mathbb{C}} \setminus \phi(\Omega)$.

R e m a r k 2.6. The result of Theorem 2.4 is sharp, that is, for every p there exist a p-connected domain Ω and $T \in \text{Aut}(\Omega)$ such that $\text{ord}\, T = p$.

If $p = 1$ then the domain is conformally equivalent to the unit disc and of course only its identity has two fixed points.

EXAMPLE 2.7. Let $p \geq 2$, and let ε_p be a p-primitive root of 1. Let δ be so small that the closed discs $D(\varepsilon_p^r, \delta)$, $r = 1, \ldots, p$, are pairwise disjoint. Then $\Omega = \widehat{\mathbb{C}} \setminus (D(\varepsilon_p^1, \delta) \cup \ldots \cup D(\varepsilon_p^p, \delta))$ is a p-connected domain on the Riemann sphere. $T : \Omega \ni z \to \varepsilon_p z \in \Omega$ is its automorphism fixing 0 and the point at infinity and of course $\text{ord}\, T = p$.

R e m a r k 2.8. Our paper does not deal with open Riemann surfaces at large but there are some straightforward conclusions. Namely, if we consider open Riemann surfaces of finite type (see Minda [M]) Theorem 1.4 still holds. For arbitrary open surfaces it seems that there is no good parameter to describe the order of an automorphism with two fixed points, and we must be satisfied with the result of Theorem 0.1.

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