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The growth of regular functions on algebraic sets

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Abstract. We are concerned with the set of all growth exponents of regular functions on an algebraic subset V of $\mathbb{C}^n$. We show that its elements form an increasing sequence of rational numbers and we study the dependence of its structure on the geometric properties of V.

1. Introduction. Let $V \subset \mathbb{C}^N$ be an algebraic set of positive dimension and let $f \in \mathbb{C}[V]$. Then, by the definition of $\mathbb{C}[V]$, f is the restriction to V of a polynomial $F \in \mathbb{C}[X_1, \ldots, X_N]$ and so there exist two non-negative constants A, B such that

$$|f(z)| \leq A(1 + |z|)^B \quad \text{for } z \in V\,,$$

where $|\cdot|$ is a norm in $\mathbb{C}^N$ and $B \leq \deg F$. For simplicity, throughout the paper we will use the norm

$$|(z_1, \ldots, z_N)| = \max_{1 \leq i \leq N} |z_i|\,.$$

Define

$$\begin{aligned} M(V,f) &:= \{B \geq 0 : \exists A \geq O \text{ such that } |f(z)| \leq A(1+|z|)^B \text{ for } z \in V\}\,,\\ B(V,f) &:= \inf M(V,f)\,,\\ B_V &:= \{B(V,f) : f \in \mathbb{C}[V]\}\,. \end{aligned}$$

We will call $B(V,f)$ the *growth exponent* of f. The aim of this paper is to study the dependence of the structure of the set B_V of growth exponents on the geometric properties of V.

In Section 2 it is shown (Theorem 2.1) that $M(V,f) \neq \emptyset$, $B(V,f) \in M(V,f)$ and

$$\mathbb{N} \subset B_V \subset \{p/q : p, q \in \mathbb{N}\,,\ (p,q) = 1\,,\ 1 \leq q \leq d\}\,,$$

where d is the maximum degree of the irreducible components of V and (p,q) denotes the greatest common divisor of p and q.

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Examples 3.1–3.4 show that all combinations of equalities and strict inclusions are possible in this theorem. The fact that $na \in B_V$ whenever $a \in B_V$ and $n \in \mathbb{N}$ allows us to call $G \subset B_V$ a *generating set* of B_V if for each $a \in B_V$ there exist $g \in G$, $n \in \mathbb{N}$ such that $a = ng$. The set B_V defined in Example 3.5 is not closed under addition and has no one-element generating set. In Examples 3.6, 3.7 the sets B_V are not generated by any finite set.

In Section 4 we show that neither the smallest number of generators nor the number of denominators of irreducible ratios belonging to B_V are invariants of biregular mappings of $\mathbb{C}^N$. However, if V has a one-dimensional polynomial parametrization we can calculate B_V. In particular, if V is biregular with $\mathbb{C}$ then B_V is generated by one element (Proposition 4.2).

The theorems of Section 5 give us a better characterization of B_V. If V is a curve in $\mathbb{C}^N$ then B_V is contained in the set of ratios with denominators equal to the intersection multiplicities of the irreducible components of the analytic germs of the projective closure of V at points at infinity with the hyperplane at infinity. If $V \subset \mathbb{C}^N$ is an algebraic set of pure dimension then there exist natural numbers $q_1, \ldots, q_r$ such that $q_1 + \ldots + q_r \le \deg V$, $B_V \subset \{m/q_i : i = 1, \ldots r, m \in \mathbb{N}\}$. The proof of this fact bases on the following property: if $f \in \mathbb{C}[V]$ then $B(C, f|C) = B(V, f)$ for "almost all" algebraic curves $C \subset V$.

Section 6 deals with the case of a hypersurface V. We study the dependence of B_V on the multiplicities of the irreducible factors of the leading form of the polynomial describing V. In particular, we show that if B_V contains an irreducible ratio with denominator k, then the leading form of the polynomial describing V is divisible by the kth power of a homogeneous non-constant polynomial.

2. The basic theorem

THEOREM 2.1. *If $V \subset \mathbb{C}^N$ is an algebraic set of positive dimension and if $f \in \mathbb{C}[V]$ then $M(V, f) \neq \emptyset$ and $B(V, f) \in M(V, f)$. Moreover, if d denotes the maximum degree of the irreducible components of V, then*

$$\mathbb{N} \subset B_V \subset D_d := \{p/q : p, q \in \mathbb{N}, \ (p, q) = 1, \ 1 \le q \le d\}.$$

The theorem will be proved by means of a sequence of lemmas.

LEMMA 2.2. *If $V \subset \mathbb{C}^{n+k}$ is an algebraic set of pure dimension $n > 0$ and of degree d and if $f \in \mathbb{C}[V]$, then $B(V, f) \in M(V, f)$ and $B_V \subset D_d$.*

Proof. Let $f \in \mathbb{C}[V]$. By Sadullaev's theorem (see [2], VII, 7.1) we may assume that for some $C > 0$

$$V \subset \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^k : |y| \le C(1 + |x|)\}. \tag{1}$$

Let $\Phi : V \ni (x,y) \to (x, f(x,y)) \in \mathbb{C}^n \times \mathbb{C}$. Since Φ is a proper holomorphic mapping, $W := \Phi(V)$ is an analytic set of pure dimension n, by Remmert's theorem.

The projection $\pi|W : W \ni (x,t) \to x \in \mathbb{C}^n$ is proper, hence there exist a proper analytic subset S of $\mathbb{C}^n$ and a natural number s such that $\#(\pi|W)^{-1}(x) = s$ for all $x \in \mathbb{C}^n \setminus S$, and there exist holomorphic functions $\sigma_1, \ldots, \sigma_s : \mathbb{C}^n \to \mathbb{C}$ such that

$$W = \{(x,t) \in \mathbb{C}^n \times \mathbb{C} : t^s + \sigma_1(x)t^{s-1} + \ldots + \sigma_s(x) = 0\}$$

(see e.g. [4], p. 71, Lemma 1).

Clearly, $\#(\pi|W)^{-1}(x) \le \deg V = d$ for $x \in \mathbb{C}^n$, and so $s \le d$. By the Vieta formulae we obtain

$$|\sigma_i(x)| \le \binom{s}{i} \max\{|t|^i : (x,t) \in W\} = \binom{s}{i} \max\{|f(x,y)|^i : (x,y) \in V\}.$$

Moreover, if $f = F|V$, $F \in \mathbb{C}[X_1, \ldots, X_{n+k}]$ then for some $A > 0$

$$\begin{aligned}|f(x,y)| &\le A(1 + |(x,y)|)^{\deg F} \le A(1 + |x| + C(1+|x|))^{\deg F}\\ &= (C+1)^{\deg F} A(1+|x|)^{\deg F}.\end{aligned}$$

Hence, by the Liouville theorem, $\sigma_i(x)$ is a polynomial (for $i = 1, \ldots, s$). Thus W is algebraic.

Now, it suffices to prove

Lemma 2.3. *Let* $W = \{(x,t) \in \mathbb{C}^n \times \mathbb{C} : t^s + \sigma_1(x)t^{s-1} + \ldots + \sigma_s(x) = 0\}$, *where* σ_i *is a polynomial of degree* p_i *for* $i = 1, \ldots, s$. *If* $B := \max_{1 \le i \le s}\{p_i/i\}$, $M(W) := \{D \ge 0 : \exists A \ge 0$ *such that* $W \subset \{(x,t) : |t| \le A(1+|x|)^D\}\}$, *then* $B \in M(W)$ *and* $B = \min M(W)$.

Indeed, from (1) and the definition of W it follows that $M(V,f) = M(W)$. Thus, since $s \le d$, Lemma 2.3 implies Lemma 2.2. To prove Lemma 2.3, choose A_0 such that $|\sigma_i(x)| \le A_0(1+|x|)^{p_i}$ for $i = 1, \ldots, s$. Set $A := \max\{sA_0, 1\}$. Suppose that there exist $(x,t) \in W$ such that $|t| > A(1+|x|)^B$. Since $t^s = -\sigma_1(x)t^{s-1} + \ldots - \sigma_s(x)$ for $(x,t) \in W$, we have

$$\begin{aligned}|t| &= \left|\sigma_i(x) + \frac{\sigma_2(x)}{t} + \ldots + \frac{\sigma_s(x)}{t^{s-1}}\right| \le \sum_{i=1}^{s} \frac{|\sigma_i(x)|}{|t|^{i-1}}\\ &\le \frac{\sum_{i=1}^{s} A_0(1+|x|)^{p_i}}{A^{i-1}(1+|x|)^{B(i-1)}} = \sum_{i=1}^{s} \frac{A_0}{A^{i-1}}(1+|x|)^{p_i - B(i-1)}.\end{aligned}$$

Since $p_i - B(i-1) \le p_i - (p_i/i)(i-1) = p_i/i \le B$, we obtain

$$|t| \le \Big(\sum_{i=1}^{s} A_0/A^{i-1}\Big)(1+|x|)^B,$$

which contradicts our assumption. Hence $W \subset \{(x,t) : |t| \le A(1+|x|)^B\}$, and so $B \in M(W)$.

Suppose now that there exists $D \in M(W)$ such that $D < B$. Then there exist $A > 0$ and $1 \le i \le s$ such that $D < p_i/i$ and $|t| \le A(1+|x|)^D$ for $(x,t) \in W$. By the Vieta formulae

$$|\sigma_i(x)| \le \binom{s}{i} A^i (1+|x|)^{iD},$$

and so $\deg \sigma_i \le Di < p_i$. This contradiction completes the proof of Lemma 2.3. ■

As a simple consequence of Lemma 2.2 we obtain

LEMMA 2.4. *If* $V = \bigcup_{j=1}^s V_j$ *is the decomposition of an algebraic set* $V \subset \mathbb{C}^N$ *into irreducible components and if* $f \in \mathbb{C}[V]$, *then*

$$B(V,f) = \max_{1\le j\le s} B(V_j, f|V_j) \in M(V,f).$$

To prove Theorem 2.1 it remains to show $\mathbb{N} \subset B_V$. By Sadullaev's theorem we can assume that, for some $C > 0$,

$$V \subset \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^{N-n} : |y| < C(1+|x|)\},$$

where $n = \dim V$. Since the projection $\pi|V : V \ni (x,y) \to x \in \mathbb{C}^n$ is proper, by the Remmert theorem, $\pi(V) = \mathbb{C}^n$. Now, it is easy to check that $B(V, X_1^k) = k$ for $k \in \mathbb{N}$, where $x = (x_1, \ldots, x_n)$. ■

3. Examples. Under the assumptions of Theorem 2.1 we have $\mathbb{N} \subset B_V \subset D_d$. The following examples show that all combinations of equalities and strict inclusions are possible here.

EXAMPLE 3.1. If V is a linear subspace of $\mathbb{C}^N$, then $\mathbb{N} = B_V = D_d$.

EXAMPLE 3.2. Let $V = \{(x,y) \in \mathbb{C}^2 : xy = 1\}$. Then $\mathbb{N} = B_V \subsetneq D_d = \{n/2 : n \in \mathbb{N}\}$.

EXAMPLE 3.3. Let $V = \{(x,y) \in \mathbb{C}^2 : y - x^2 = 0\}$. Then $\mathbb{N} \subsetneq B_V = D_d = \{n/2 : n \in \mathbb{N}\}$.

EXAMPLE 3.4. Let $V = \{(x,y) \in \mathbb{C}^2 : y - x^n = 0\}$, $n > 2$. If $|(x,y)| \ge 1$, $(x,y) \in V$ then $|(x,y)| = |y|$. For $f \in \mathbb{C}[V]$,

$$f(x,y) = \sum_{\text{finite}} a_{i,j} x^i y^j = \sum a_{i,j} \varepsilon_n^i y^{j+i/n},$$

where $a_{i,j} \in \mathbb{C}$, $\varepsilon_n^n = 1$. Hence $B_V = \{k/n : k \in \mathbb{N}\}$. As $d = \deg V = n$ we have $\mathbb{N} \subsetneq B_V \subsetneq D_d$.

EXAMPLE 3.5. Let $k, n \in \mathbb{N}_+$, $V = \{(x,y) \in \mathbb{C}^2 : x^n y^k = 1\}$. Then $B_V = \{is/k, js/n : i,j \in \mathbb{N},\ s := (k,n)\}$. Choosing for instance $k = 2$,

$n = 3$ we obtain the set B_V not closed under addition and having no one-element generating set.

EXAMPLE 3.6. Let $k, n \in \mathbb{N}_+$ and let $n \le k$. For $V = \{(x,y) \in \mathbb{C}^2 : x^n - y^k = 0\}$ we have $B_V = \{in/k + j : i, j \in \mathbb{N}\}$. In particular, for $n = 2$ and $k = 3$ $B_V = \{0, 2/3, 1, 4/3, 5/3, 2, \ldots\} = \{n/3 : n \in \mathbb{N} \setminus \{1\}\}$.

EXAMPLE 3.7. If $V = \{(x,y) \in \mathbb{C}^2 : (y^3 - 1)x^2 - 1 = 0\}$, then $B_V = \{0, 1, 3/2, 2, 5/2, \ldots\} = \{n/2 : n \in \mathbb{N} \setminus \{1\}\}$.

4. The growth exponents on biregular sets

EXAMPLE 4.1. Neither the smallest number of generators of B_V nor the number of denominators of irreducible ratios belonging to B_V are invariants of biregular mappings of $\mathbb{C}^N$. For instance the biregular mapping $\Phi : \mathbb{C}^2 \ni (x,y) \to (x - y^3, y) \in \mathbb{C}^2$ maps $V = \{(x,y) \in \mathbb{C}^2 : xy^2 - 1 = 0\}$ onto $W = \{(w,z) \in \mathbb{C}^2 : (w + z^3)z^2 - 1 = 0\}$ and $B_V = \{i/2 : i \in \mathbb{N}\}$, $B_W = \{i/2, j/3 : i, j \in \mathbb{N}\}$.

PROPOSITION 4.2. *Let $f = (f_1, \ldots, f_n) : \mathbb{C} \to \mathbb{C}^n$ be a polynomial mapping and let $V = f(\mathbb{C})$. If $p := \max_{1 \le i \le n} \deg f_i$ then*

$$B_V = \{i/p : \exists h \in \mathbb{C}[V] \text{ such that } \deg(h \circ f) = i\}\ .$$

In particular, if V is biregular with $\mathbb{C}$ then $B_V = \{i/p : i \in \mathbb{N}\}$.

The proof of this proposition bases on the following

LEMMA 4.3. *If $a, b \in \mathbb{C}[t]$, $\deg a = r$, $\deg b = s$ then there exists $A > 0$ such that $|a(t)| \le A(1 + |b(t)|)^{r/s}$, $t \in \mathbb{C}$.*

Since $\lim_{t\to\infty} |a(t)^s/b(t)^r| = M \in (0, \infty)$, there exists R such that $|a(t)^s/b(t)^r| < 2M$ for $|t| > R$, and so

$$A := \max\{(2M)^{1/s}, \sup_{|t| \le R} |a(t)|\}$$

is the required constant. ■

To prove Proposition 4.2 choose i such that $\deg f_i = p$ and let $h \in \mathbb{C}[V]$. If $r := \deg(h \circ f)$ then, by Lemma 4.3, there exists $A > 0$ such that

$$|h(x)| = |h \circ f(t)| \le A(1 + |f_i(t)|)^{r/p} \le A(1 + |x|)^{r/p}\,,$$

where $x = f(t)$, $t \in \mathbb{C}$. Again by Lemma 4.3, there exists A' such that

$$|f_j(t)| \le A'(1 + |h \circ f(t)|)^{p/r} \quad \text{for } j = 1, \ldots, n\,.$$

Since $|x| = \max_{1 \le j \le n} |x_j|$, putting $A'' := A'^{-r/p}$ we obtain

$$A''|x|^{r/p} - 1 \le |h(x)| \le A(1 + |x|)^{r/p}\,.$$

Hence $B(V, h) = r/p$. ■

5. A sharpened version of the basic theorem. Let $V \subset \mathbb{C}^n \subset \mathbb{P}^n$ be an affine algebraic set, let $\overline{V}$ denote its projective closure and let $H_\infty := \mathbb{P}^n \setminus \mathbb{C}^n$. For analytic subsets W, Z of an open neighbourhood of $a \in \mathbb{P}^n$ let $i(W \cdot Z, a)$ denote the intersection multiplicity of W and Z at a (in the sense of [1], see also [5]).

THEOREM 5.1. *Let $V \subset \mathbb{C}^n$ be an algebraic set of pure dimension* 1. *Let $\overline{V} \cap H_\infty = \{a_1, \ldots, a_r\}$ and let, for $i = 1, \ldots, r$, $\overline{V}_{a_i} = A_{i,1} \cup \ldots \cup A_{i,s_i}$ be the decomposition of the germ $\overline{V}_{a_i}$ into irreducible analytic germs. If $q_{i,j} := i(A_{i,j} \cdot H_\infty, a_i)$ for $i = 1, \ldots, r$, $j = 1, \ldots, s_i$, then*

$$B_V \subset \{m/q_{i,j} : m \in \mathbb{N}\,,\ i = 1, \ldots, r\,,\ j = 1, \ldots, s_i\}\,.$$

Proof. For any $Z \subset \mathbb{C}^n$ and $f \in \mathbb{C}[X_1, \ldots, X_n]$ we set

$$B(Z, f) := \inf\{B \geq 0 : \exists A > 0 \text{ such that } |f(x)| \leq A(1 + |x|)^B \text{ for } x \in Z\}\,.$$

Now, since $B(Z_1 \cup \ldots \cup Z_t, f) = \max_{1 \leq i \leq t} B(Z_i, f)$ and $B(K, f) = 0$ for any compact set K, it suffices to prove the following:

(L) *Each germ $A_{i,j}$, $i = 1, \ldots, r$, $j = 1, \ldots, s_i$, has a representative $\mathcal{A}_{i,j}$ such that for any $f \in \mathbb{C}[X_1, \ldots, X_n]$ and any open neighbourhood U_i of a_i we have*

(1) $B_{i,j}(f) := B(\mathcal{A}_{i,j} \cap U_i \cap \mathbb{C}^n, f)$ *does not depend on the choice of U_i,*

$$B_{i,j}(f) \in \{m/q_{i,j} : m \in \mathbb{N}\}. \tag{2}$$

Fix $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s_i\}$. In suitable projective coordinates $\langle x_0, \ldots, x_n \rangle$ in $\mathbb{P}^n$ we have $H_\infty = \{x_0 = 0\}$, $a := a_i = \langle 0, \ldots, 0, 1 \rangle$. Then $A := A_{i,j}$ is an irreducible analytic germ at a, $\dim A = 1$, $q := i(A \cdot H_\infty, a) = q_{i,j}$ and $A \cap H_\infty = \{a\}$. By the Puiseux theorem (see e.g. [2], II, 6.2, III, 4.4), there exist $p \in \mathbb{N}_+$, a holomorphic mapping $h : \{t \in \mathbb{C} : |t| < \delta\} \to \mathbb{C}^{n-1}$ with $h(0) = 0$ and a representative W of the germ A such that $W = \{\langle t^p, h(t), 1 \rangle : |t| < \delta\}$ and the mapping $\{|t| < \delta\} \ni t \to \langle t^p, h(t), 1 \rangle \in W$ is homeomorphic. The mapping

$$\pi|W : W \ni \langle t^p, h(t), 1 \rangle \to t^p \in \{\langle x_0, 0, \ldots, 0, 1 \rangle : |x_0| < \delta^p\}$$

is a p-sheeted branched covering, and so $p = i(A \cdot H_\infty, a) = q$. If we set $\mathcal{A}_{i,j} := W$ then

$$\mathcal{A}_{i,j} \cap \mathbb{C}^n = \{\langle 1, h(t)t^{-q}, t^{-q} \rangle : |t| < \delta\}\,.$$

As $h(0) = 0$ we may assume that $|h(t)| < 1$. Since $|x| = \max_{1 \leq j \leq n} |x_j|$, we have $|x| = |t|^{-q}$ for $x = x(t) = (h(t)t^{-q}, t^{-q}) \in \mathcal{A}_{i,j} \cap \mathbb{C}^n$.

Fix $f \in \mathbb{C}[X_1, \ldots, X_n]$. Then, for $x = x(t) \in \mathcal{A}_{i,j} \cap \mathbb{C}^n$, $f(x) = f(x(t)) = \sum_{m=d}^{\infty} f_m t^m$, where $f_m \in \mathbb{C}$, $d \in \mathbb{Z}$, $f_d \neq 0$. Since

$$|f(x(t))/t^d| = \Big|\sum_{m=0}^{\infty} f_{m+d} t^m\Big| \xrightarrow[t \to 0]{} |f_d| \in (0, \infty)\,,$$

we have

$$\tfrac{1}{2}|f_d|\,|t|^d \le |f(x(t))| \le 2|f_d|\,|t|^d\,, \qquad \text{for } |t| < \delta'\,.$$

As $|t| = |x|^{-1/q}$ we obtain

$$\tfrac{1}{2}|f_d|\,|x|^{-d/q} \le |f(x)| \le 2|f_d|\,|x(t)|^{-d/q}\,.$$

Now, since $B(K,f) = 0$ for any bounded set K, and $B(W\cup Z,f) = \max\{B(W,f), B(Z,f)\}$, it is easily seen that for any open neighbourhood U_i of a_i

$$B(\mathcal{A}_{i,j}\cap U_i\cap \mathbb{C}^n, f) = \max\{-d/q, 0\}\,,$$

which completes the proof of (L).

The following theorem is a generalization of Theorem 5.1 for algebraic sets of any pure dimension. Since irreducible components of any algebraic set have pure dimension, we may apply this theorem to any algebraic set, by Lemma 2.4.

THEOREM 5.2. *If $V\subset\mathbb{C}^N$ is an algebraic set of pure dimension, then there exist natural numbers $q_1,\dots,q_r$ such that $q_1+\ldots+q_r\le\deg V$ and $B_V\subset\{m/q_i : i=1,\dots,r,\ m\in\mathbb{N}\}$.*

LEMMA 5.3. *Let $V\subset\{(x,y)\in\mathbb{C}^n\times\mathbb{C}^k : |y|\le C(1+|x|)\}$ be an algebraic set of pure dimension n, and let $f\in\mathbb{C}[V]$. For $x\in\mathbb{C}^n$ we put*

$$l_x := \{ax : a\in\mathbb{C}\}\subset\mathbb{C}^n\,,$$

$$V_x := \pi^{-1}(l_x)\cap V\,, \quad \textit{where} \quad \pi:\mathbb{C}^n\times\mathbb{C}^k\ni(u,w)\to u\in\mathbb{C}^n\,,$$

$$f_x := f|V_x\,.$$

Then there exists an algebraic cone $Z\subsetneq\mathbb{C}^n$ such that $B(V_x,f_x) = B(V,f)$ for $x\in\mathbb{C}^n\setminus Z$.

P r o o f. As in the proof of Lemma 2.2, we put

$$\Phi : V\ni(x,y)\to(x,f(x,y))\in\mathbb{C}^n\times\mathbb{C}\,, \qquad W := \Phi(V)\,.$$

Then there exist $\sigma_1,\dots,\sigma_s\in\mathbb{C}[X_1,\dots,X_n]$ such that

$$W = \{(x,t)\in\mathbb{C}^n\times\mathbb{C} : t^s+\sigma_1(x)t^{s-1}+\ldots+\sigma_s(x) = 0\}$$

and $B(V,f) = \max_{1\le i\le s}\{p_i/i\}$, where $p_i = \deg\sigma_i$. If $\sigma_i = \sigma_{i,0}+\ldots+\sigma_{i,p_i}$ is the decomposition of σ_i into homogeneous forms, then

$$\sigma_i^x(a) := \sigma_i(ax) = \sigma_{i,0}+\ldots+\sigma_{i,p_i}(x)a^{p_i} \qquad \text{for } a\in\mathbb{C}\,.$$

Applying Lemma 2.3 to the set

$$W_x = \Phi(V_x) = \{(ax,t) : t^s+\sigma_1^x(a)t^{s-1}+\ldots+\sigma_s^x(a) = 0\}$$

we obtain

$$B(V_x,f_x) = \max_{1\le i\le s}\{(\deg\sigma_i^x/i\}\,.$$

Then the set $Z \subsetneq \mathbb{C}^n$ defined by

$$Z := \Big\{x \in \mathbb{C}^n : \prod_{i=1}^{s} \sigma_{i,p_i}(x) = 0\Big\}$$

is an algebraic cone and $B(V_x, f_x) = B(V, f)$ for $x \in \mathbb{C}^n \setminus Z$. ■

P r o o f o f T h e o r e m 5.2. Let $p_1, \ldots, p_s$ be all the different denominators of the irreducible ratios belonging to B_V (see Theorem 2.1), and let $f_1, \ldots, f_s \in \mathbb{C}[V]$ be such that $B(V, f_i) = n_i/p_i$, where $(n_i, p_i) = 1$ for $i = 1, \ldots, s$. By Sadullaev's theorem, we can assume that $V \subset \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^{N-n} : |y| < C(1 + |x|)\}$, where $n = \dim V$. By Lemma 5.3 there exist algebraic cones $Z_i \subsetneq \mathbb{C}^n$ such that

$$B(V_{x_i}, (f_i)_{x_i}) = n_i/p_i \quad \text{ for } i = 1, \ldots, s\,,\ x_i \in \mathbb{C} \setminus Z_i\,.$$

Let $x \in \mathbb{C} \setminus (Z_1 \cup \ldots \cup Z_s)$. The V_x has pure dimension 1. By the Bézout theorem and Theorem 5.1,

$$B_{V_x} \subset \{m/q_i : i = 1, \ldots, r\,,\ m \in \mathbb{N}\}\,,$$

where $q_1 + \ldots + q_r \le \deg V_x \le \deg V$. As $B(V_x, (f_i)_x) = n_i/p_i$, $i = 1, \ldots, s$, we have

$$B_V \subset \{m/q_i : i = 1, \ldots, r\,,\ m \in \mathbb{N}\}\,. \ ■$$

Corollary 5.4. *For any algebraic set $V \subset \mathbb{C}^N$ of pure positive dimension there exists a curve $\Gamma \subset V$ such that $B_V \subset B_\Gamma$.*

To prove the corollary we apply Lemma 5.3 and notice that a countable sum of proper algebraic cones is nowhere dense, by the Baire theorem.

6. The case of a hypersurface. Let $F \in \mathbb{C}[X_1, \ldots, X_n]$, $n > 0$. By a *reduced decomposition* of F we mean a decomposition $F = p_1^{d_1} \ldots p_l^{d_l}$ such that

(1) $d_1 < \ldots < d_l$,
(2) $(p_i, p_j) = 1$ for $i \ne j$,
(3) p_i has no multiple factors for $i = 1, \ldots, l$.

It is easily seen that each $F \in \mathbb{C}[X_1, \ldots, X_n]$ has a unique reduced decomposition.

Theorem 6.1. *If $V = \{x \in \mathbb{C}^n : F(x) = 0\}$, where $F \in \mathbb{C}[X_1, \ldots, X_n]$, and if $F_d = p_1^{d_1} \ldots p_l^{d_l}$ is the reduced decomposition of the leading form of F, where $\deg p_i = r_i$, then there exist natural numbers $m_{i,j,k}$ for $i = 1, \ldots, l$, $j = 1, \ldots, r_i$, $k = 1, \ldots, s_{i,j}$, such that $d_i = \sum_{k=1}^{s_{i,j}} m_{i,j,k}$ for $j = 1, \ldots, r_i$ and*

$$B_V \subset \{t/m_{i,j,k} : t \in \mathbb{N}\,,\ i = 1, \ldots, l\,, j = 1, \ldots, r_i\,,\ k = 1, \ldots, s_{i,j}\}\,.$$

In particular, if k is the maximum denominator of the irreducible ratios in B_V then F_d is divisible by the k-th power of a non-constant polynomial.

LEMMA 6.2. *If $V = \{(x, y) \in \mathbb{C}^2 : F(x, y) = 0\}$, $F \in \mathbb{C}[X, Y]$, then there exist $m_1, \ldots, m_s \in \mathbb{N}_+$ and homogeneous linear polynomials $p_1, \ldots, p_s \in \mathbb{C}[X, Y]$ such that $F_d = p_1^{m_1} \ldots p_s^{m_s}$ is the leading form of F and $B_V \subset \{n/m_i : n \in \mathbb{N}, i = 1, \ldots, s\}$ ($p_1, \ldots, p_s$ need not be different).*

This lemma is a consequence of Theorem 5.1 for in the notation of that theorem

$$\sum_{j=1}^{s_i} i(A_{i,j} \cdot H_\infty, a_i) = i(\overline{V}_{a_i} \cdot H_\infty, a_i)$$
$$= \{\text{multiplicity of the factor } y_i X_i - x_i Y_i \text{ in } F_d\},$$

where $a_i = \langle 0, x_i, y_i \rangle$.

Proof of Theorem 6.1. We can assume, by changing linear coordinates if needed, that the coefficient at X_i^d in F_d is different from 0 for $i = 1, \ldots, n$. Let $Y := X_n$, $X' := (X_1, \ldots, X_{n-1})$. Then, dividing F by a non-zero constant if needed, we obtain

$$F(X) = F(X', Y) = Y^d + a_1(X')Y^{d-1} + \ldots + a_d(X'),$$

where $a_1, \ldots, a_d \in \mathbb{C}[X']$, $\deg a_i \le i$, $\deg a_d = d$. From Lemma 2.3 we have

$$V \subset \{(x', y) \in \mathbb{C}^{n-1} \times \mathbb{C} : |y| \le C(1 + |x'|)\},$$

where C is a suitable constant.

Let $B_V = \{B(V, f_m) : m \in \mathbb{N}\}$. Using the notation of Lemma 5.3, by that lemma, we obtain, for each $m \in \mathbb{N}$, an algebraic cone $Z_m \subsetneq \mathbb{C}^{n-1}$ such that

$$B(V, f_m) = B(V_{x'}, (f_m)_{x'}), \quad \text{for } x' \in \mathbb{C}^{n-1} \setminus Z_m.$$

The set

$$U := \mathbb{C}^{n-1} \setminus \bigcup_{m=0}^{\infty} Z_m$$

is of the second Baire category and $B_V \subset B_{V_{x'}}$ for $x' \in U$.

Let $x' \in U$. Then $V_{x'} = \{(ax', y) \in \mathbb{C}^n : F(ax', y) = 0\}$. Write

$$F(x')(A, Y) := F(Ax', Y), \quad F(x') \in \mathbb{C}[A, Y],$$

$$W(x') := \{(a, y) \in \mathbb{C}^2 : (ax', y) \in V_{x'}\} = \{(a, y) \in \mathbb{C}^2 : F(x')(a, y) = 0\},$$

$$f(x')(a, y) := f(ax', y) \quad \text{for } f \in \mathbb{C}[V],\ f(x') \in \mathbb{C}[W(x')].$$

Since $|ax'| = |x'|\,|a|$, $B(V_{x'}, f_{x'}) = B(W(x'), f(x'))$ for $f \in \mathbb{C}[V]$. Hence $B_V \subset B_{W(x')}$.

If $F_d(x')(A,Y) := F_d(Ax',Y)$, then $F_d(x')$ is the leading form of $F(x')$. Since $F_d(x')$ is a homogeneous polynomial in two variables with the coefficient at Y^d equal to 1, it can be decomposed into linear factors

$$F_d(x')(A,Y) = (Y - \xi_1(x')A)\ldots(Y - \xi_d(x')A)\,.$$

By Lemma 6.2, there are $m_1(x'),\ldots,m_{s(x')}(x') \in \mathbb{N}$ such that

$$B_V \subset B_{W(x')} \subset \{k/m_i(x') : i = 1,\ldots,s(x')\,,\ k \in \mathbb{N}\}$$

and

$$\begin{aligned} F_d(x',Y) &= (Y - \xi_1(x'))^{m_1(x')}\ldots(Y - \xi_{s(x')}(x'))^{m_{s(x')}(x')} \\ &= (Y - \eta_1(x'))^{t_1(x')}\ldots(Y - \eta_{r(x')}(x'))^{t_{r(x')}(x')}\,, \end{aligned}$$

where $\eta_i(x') \neq \eta_j(x')$ for $i \neq j$, $t_1(x') \leq \ldots \leq t_{r(x')}(x')$, $t_j(x') = m_{j,1}(x') + \ldots + m_{j,r_j(x')}(x')$ for $j = 1,\ldots,r(x')$ and $\{m_{j,k}(x') : j = 1,\ldots,r(x')\,,\ k = 1,\ldots,r_j(x')\} = \{m_i(x') : i = 1,\ldots,s(x')\}$.

Since $\sum_{j=1}^{r(x')} t_j(x') = d < \infty$, there exists a set $U' \subset U$ of the second Baire category such that for all $x' \in U'$ we have $s(x') = s$, $r(x') = r$, $t_j(x') = t_j$ for $j = 1,\ldots,r$, $m_{j,k}(x') = m_{j,k}$ for $j = 1,\ldots,r$, $k = 1,\ldots,r_j$.

Let $d_1 < \ldots < d_l$ be natural numbers such that

$$\begin{gathered} d_1 = t_1 = \ldots = t_{l_1}\,, \qquad d_2 = t_{l_1+1} = \ldots = t_{l_1+l_2}\,, \ldots\,, \\ d_l = t_{l_1+\ldots+l_{l-1}+1} = \ldots = t_r\,. \end{gathered}$$

Then for $x' \in U'$

$$F_d(x',Y) = [q_1(x')(Y)]^{d_1} \cdot \ldots \cdot [q_l(x')(Y)]^{d_l}\,,$$

where $q_i(x') \in \mathbb{C}[Y]$ is a monic polynomial of degree l_i. Moreover, $q_i(x')$ and $q_j(x')$ have no common roots for $i \neq j$, and $q_i(x')$ has no multiple roots.

To complete our proof it suffices to prove the following

Lemma 6.3. *Let $U \subset \mathbb{C}^n$ be of the second Baire category and let $g \in \mathbb{C}[X,Y]$ be monic on Y, where $X = (X_1,\ldots,X_n)$. If, for each $x \in U$, $g(x,Y)$ has a root of multiplicity $\geq k$ then $g = p^k q$, where $p \in \mathbb{C}[X,Y] \setminus \mathbb{C}$, $q \in \mathbb{C}[X,Y]$ are monic on Y.*

Indeed, applying Lemma 6.3 to the polynomial $F_d(X',Y)$ and the set U' we obtain $F_d = p^{d_l} q$, with $p \in \mathbb{C}[X',Y] \setminus \mathbb{C}$ and $q \in \mathbb{C}[X',Y]$. If $p(x')$ denotes the polynomial $\{y \to p(x',y)\} \in \mathbb{C}[Y]$ then we have $p(x') | q_l(x')$ for $x' \in U'$. Thus

$$q(x',Y) = [q_1(x')(Y)]^{d_1}\ldots[q_{l-1}(x')(Y)]^{d_{l-1}} \cdot [q_l(x')(Y)/p(x')(Y)]^{d_l}\,.$$

Repeated application of Lemma 6.3 to the subsequently obtained remainders q gives us finally $F_d = p_1^{d_1}\ldots p_l^{d_l}$, where $p_i \in \mathbb{C}[X',Y] \setminus \mathbb{C}$, $p_i(x') = q_i(x')$ for $x' \in U'$, $i = 1,\ldots,l$. Hence $(p_i,p_j) = 1$ for $i \neq j$, and p_i has no multiple factors for $i = 1,\ldots,l$, which completes the proof of Theorem 6.1.

P r o o f o f L e m m a 6.3. Let $g = p_1 \ldots p_s$ be the decomposition of g in $\mathbb{C}[X, Y]$ into irreducible polynomials monic in Y. For $i = 1, \ldots, s$ we can treat p_i as a monic irreducible polynomial of Y with coefficients in $\mathbb{C}[X]$, so its discriminant $\Delta(p_i) \in \mathbb{C}[X]$ is different from 0. Hence

$$W_0 := \{x \in \mathbb{C}^n : \Delta(p_1)(x) \ldots \Delta(p_s)(x) = 0\}$$

is a proper algebraic subset of $\mathbb{C}^n$. For fixed x, let $p_i(x)$ denote the polynomial $\{y \to p_i(x, y)\} \in \mathbb{C}[Y]$. For $x \in U \setminus W_0$ there exists a k-tuple $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subset \{1, \ldots, s\}$ such that $p_{\alpha_1}(x), \ldots, p_{\alpha_k}(x)$ have a common root, because none of $p_1(x), \ldots, p_s(x)$ has a multiple root and $q(x) = p_1(x) \ldots p_s(x)$ has a root of multiplicity $\geq k$. In particular, for any $i, j = 1, \ldots, k$, the resultant $R(p_{\alpha_i}(x), p_{\alpha_j}(x)) = 0$. We have

$$R(p_{\alpha_i}(x), p_{\alpha_j}(x)) = R(p_{\alpha_i}, p_{\alpha_j})(x)\,,$$

where $R(p_{\alpha_i}, p_{\alpha_j}) \in \mathbb{C}[X]$ is the resultant of $p_{\alpha_i}, p_{\alpha_j}$ as polynomials of Y. Hence the sets

$$W_\alpha := \{x \in \mathbb{C}^n : R(p_{\alpha_i}(x), p_{\alpha_j}(x)) = 0 \text{ for } i, j = 1, \ldots, k\}\,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \subset \{1, \ldots, s\}$, are algebraic subsets of $\mathbb{C}^n$. Since $\bigcup_{\alpha \subset \{1,\ldots,s\}} W_\alpha \supset U \setminus W_0$ is of the second Baire category, there exists α such that $W_\alpha = \mathbb{C}^n$. Thus $R(p_{\alpha_i}(x), p_{\alpha_j}(x)) = 0$ for $i, j = 1, \ldots, k$ and $x \in \mathbb{C}^n$. Hence $R(p_{\alpha_i}, p_{\alpha_j}) = 0$ in $\mathbb{C}[X]$. Therefore $p_{\alpha_i}, p_{\alpha_j}$ have a common root ξ in the algebraic closure of $\mathbb{C}(X)$, and so are both divisible by the minimal polynomial of ξ in $\mathbb{C}(X)[Y]$. However, being monic and irreducible in $\mathbb{C}[X][Y]$, they are irreducible in $\mathbb{C}(X)[Y]$. Hence $p_{\alpha_i} = p_{\alpha_j} =: p$ for $i, j = 1, \ldots, k$. ∎

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