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A new division formula for complete intersections

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Abstract. We provide a new division formula for holomorphic mappings. It is given in terms of residue currents and has the advantage of being more explicit and simpler to prove than the previously known formulas.

1. Let X be a connected complex manifold with $\dim_{\mathbb{C}} X = n$, and $f : X \to \mathbb{C}^p$ a holomorphic map. Given also a holomorphic function h on X, we consider the problem of determining whether or not h is "divisible" by f in the sense that h belongs to $\mathcal{I}_f$, the ideal (in some ring of holomorphic functions on X) generated by $f_1, \ldots, f_p$. And in case this holds we wish to find a "quotient", that is, a new holomorphic map $g : X \to \mathbb{C}^p$ such that $h = g \cdot f = \sum g_j f_j$.

These two questions lie at the heart of the so-called fundamental principle for systems of linear partial differential equations with constant coefficients (see for instance [3] and [1]). And the more explicit the solution of the division problem, the more explicit the fundamental principle.

We shall restrict our attention to the case where f is a complete intersection, that is, $\dim_{\mathbb{C}} f^{-1}(0) = n - p$, and we shall prove a representation formula of the following type:

$$h(z) = \sum_{j=1}^{p} g_j(z) f_j(z) + \langle h \cdot \overline{\partial}[1/f], \varphi(\cdot, z) \rangle ,$$

where $\overline{\partial}[1/f]$ is a residue current (see definition below), φ a test form, and the functions g_j are also given in terms of currents. Such a formula was obtained in [4], but our new formula expresses the g_j more explicitly and is also easier to prove.

2. Let us briefly recall the definition of the residue currents that we need.

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First let $\chi : (0,\infty) \to [0,1]$ be a smooth approximation of the characteristic function for the interval $[1,\infty)$, that is, a smooth increasing function χ such that $\chi - \chi_{[1,\infty)}$ has compact support, and put $\chi_j^\varepsilon(z) = \chi(|f_j(z)|/\varepsilon_j)$, for $\varepsilon \in (0,1]^p$. Then take any two disjoint subsets $I = \{i_1,\ldots,i_q\}$ and $J = \{j_1,\ldots,j_r\}$ of $\{1,\ldots,p\}$, and consider the differential form of bidegree $(0,r)$

$$[\chi^\varepsilon/f]_I \overline{\partial}[\chi^\varepsilon/f]_J = \frac{\chi_{i_1}^\varepsilon \ldots \chi_{i_q}^\varepsilon \overline{\partial}\chi_{j_1}^\varepsilon \wedge \ldots \wedge \overline{\partial}\chi_{j_r}^\varepsilon}{f_{i_1}\ldots f_{i_q} \cdot f_{j_1}\ldots f_{j_r}}\,.$$

It is then known (see [2] and [5]) that if $\varepsilon \to 0$ in such a way that

$$\varepsilon_1^{k_1}\ldots\varepsilon_p^{k_p} \to 0 \text{ or } \infty, \tag{1}$$

for every $k \in \mathbb{Z}^p$, then

$$[1/f]_I\overline{\partial}[1/f]_J = \lim_{\varepsilon\to 0}\,[\chi^\varepsilon/f]_I\overline{\partial}[\chi^\varepsilon/f]_J$$

exists as a current of bidegree $(0,r)$. The condition (1) seems in fact to be of a technical nature and we conjecture that continuity holds at the origin—*in the case of a complete intersection*. When $J = \{1,\ldots,p\}$ we denote the corresponding current simply by $\overline{\partial}[1/f]$.

Moreover, the following identities hold (see [2] and [5]):

$$\begin{aligned} &f_k \cdot [1/f]_I\overline{\partial}[1/f]_J = [1/f]_{I\setminus\{k\}}\overline{\partial}[1/f]_J \qquad \text{if } k \in I\,,\\ &f_k \cdot [1/f]_I\overline{\partial}[1/f]_J = 0 \qquad\qquad\qquad\qquad \text{if } k \in J\,. \end{aligned} \tag{2}$$

In particular,

$$h\cdot\overline{\partial}[1/f] = 0\,, \qquad \text{for } h \in \mathcal{I}_f\,. \tag{3}$$

3. Now we specialize somewhat and let $X = D \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary. As in [4, p. 119] we introduce particular test forms $A^{N,k}(\zeta,z)$ on $\mathbb{C}^n \times D$ with support in $\overline{D}\times D$ and holomorphic in z.

We may then formulate our result as follows.

Theorem. *Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary, $f : D \to \mathbb{C}^p$ a complete intersection extending holomorphically to a neighborhood of D, and h a holomorphic function on D which is smooth up to the boundary. Then*

$$h(z) = \sum_{k=1}^{p}\sum_{\substack{|I|=p-k\\|J|=k}} c_{n,k}[f(z)]_I\langle h\cdot[1/f]_I\overline{\partial}[1/f]_J, [b(\cdot,z)]_J \wedge A^{N,n-k}(\cdot,z)\rangle\,,$$

where

$$[f(z)]_I = f_{i_1}(z)\ldots f_{i_{p-k}}(z)\,, \qquad [b(\zeta,z)]_J = b_{j_1}(\zeta,z)\wedge\ldots\wedge b_{j_k}(\zeta,z)$$

with

$$b_j(\zeta, z) = \sum_{\ell=1}^{n} B_{j\ell}(\zeta, z)\, d\zeta_\ell$$

the $B_{j\ell}$ *being any holomorphic functions satisfying* $f_j(z) - f_j(\zeta) = \sum B_{j\ell}(\zeta, z)(z_\ell - \zeta_\ell)$.

Remark. In case D is convex and ϱ a convex defining function for $D = \{\varrho < 0\}$ we can make the explicit choices

$$A^{N,k}(\zeta, z) = \frac{\varrho(\zeta)^{N+k}}{(\langle \partial\varrho(\zeta), z-\zeta\rangle + \varrho(\zeta))^{N+k}} \left(\overline{\partial}\frac{\partial\varrho(\zeta)}{\varrho(\zeta)}\right)^k, \qquad \zeta \in D\,,$$

$$B_{j\ell}(\zeta, z) = \int_0^1 \frac{\partial f_j}{\partial z_\ell}(\zeta + \lambda(z-\zeta))\, d\lambda\,.$$

COROLLARY. *The function* h *belongs to the ideal* $\mathcal{I}_f$ *if and only if* $h \cdot \overline{\partial}[1/f] = 0$.

4. Proof of the Theorem. We start from the representation formula given in Proposition 5.1.6 of [4]:

$$h(z) = \sum_{|\alpha|=k} c_{n,\alpha} \int_D h(\zeta) A^{N,n-k}(\zeta, z) \wedge G_1^{(\alpha_1)} \ldots$$
$$\ldots G_M^{(\alpha_M)} (\overline{\partial} q_1)^{\alpha_1} \wedge \ldots \wedge (\overline{\partial} q_M)^{\alpha_M}\,,$$

where the G_j are any holomorphic functions of one variable satisfying $G_j(0) = 1$, $G_j^{(k)}$ means the derivative $(\partial/\partial t)^k G_j$ evaluated at $t = \langle Q_j(\zeta, z), z-\zeta\rangle$, the Q_j are smooth mappings $D \times D \to \mathbb{C}^n$ and $q_j = \sum Q_{j\ell}(\zeta, z)\, d\zeta_\ell$.

We then make the following choices:

$$Q_j(\zeta, z) = Q_j^\varepsilon(\zeta, z) = \frac{\chi_j^\varepsilon(\zeta) B_j(\zeta, z)}{f_j(\zeta)}\,, \qquad j = 1, \ldots, p,$$
$$G_j(t) = 1 + t\,, \qquad j = 1, \ldots, p.$$

This gives $G^{(1)} = 1$ and

$$G^{(0)} = f_j(z)[\chi_j^\varepsilon / f_j] + (1 - \chi_j^\varepsilon)\,.$$

Now, $\chi_j^\varepsilon = f_j[\chi_j^\varepsilon / f_j]$ and *since no products of the type* $\chi_j^\varepsilon \overline{\partial} \chi_j^\varepsilon$ *will occur* it follows by (2) that when $\varepsilon \to 0$ all terms containing the factors $1 - \chi_j^\varepsilon$ will vanish. What remains is precisely the desired formula.

Proof of the Corollary. Suppose first that $h \cdot \overline{\partial}[1/f] = 0$. Then, in the formula of the Theorem, the term corresponding to $k = p$ vanishes, and the other terms are all contained in the ideal. The converse follows immediately from (3).

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