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A distortion theorem for quasiconformal automorphisms of the unit disk

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Abstract. We give a distortion theorem for quasiconformal automorphisms of the unit disk and its application to improving some results due to Douady and Earle.

1. Introduction. In various estimates connected with the conformally natural quasiconformal extension of a quasisymmetric automorphism f of the unit circle T, as given by Douady and Earle [3], the following distortion problem plays an important role. If f is a K-quasiconformal self-mapping of the unit disk normalized by the condition $\int_T f(z)|dz| = 0$, find an estimate $r(K)$ of $|f(0)|$. We were able to find an estimate $r(K)$ such that $r(K) \to 0$ as K tends to 1. Consequently, in a simple way, some results due to Douady and Earle [3] could be improved for K near 1.

2. Let T be the unit circle. We start with the following

Lemma. *If f is a K-quasiconformal self-mapping of the unit disk D and*

$$\int_T f(z)|dz| = 0 \tag{1}$$

then

$$|f(0)| \le 2p(K) = \frac{8}{\pi(K+1)}(K2^{7(1-1/K)/2} - 2^{7(1-K)/2}) \tag{2}$$

for $1 \le K \le 1.044$ *and*

$$|f(0)| \le 1 - 2(1 + \sqrt{3(4^{3K-2} - 1)})^{-1} \tag{3}$$

for other K.

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Proof. Using the fact that for every arc $I \subset T$ of length $|I| = \frac{2}{3}\pi$ we have, by (1), $|f(I)| \le \frac{4}{3}\pi$ (cf. [5]), and from the quasi-invariance of the harmonic measure [4], we get the estimate

$$|f(0)| \le \frac{1}{2} + \frac{\sqrt{3}}{2}\cot\left(\frac{\pi}{3} + \arccos \Phi_K\left(\frac{\sqrt{3}}{2}\right)\right), \tag{4}$$

where $\Phi_K = \mu^{-1}(K^{-1}\mu)$ and $\mu(r)$, $0 < r < 1$, is the module of D slit from 0 to r along the positive real axis [6]. For details cf. [5]. Since $\Phi_K^2(r) + \Phi_{1/K}^2(\sqrt{1-r^2}) = 1$ for $K > 0$ and $0 \le r \le 1$, as shown in [1], from the inequality [6]

$$\Phi_K(r) \le 4^{1-1/K} r^{1/K}, \quad K \ge 1,\ 0 \le r \le 1,$$

we derive (3) after suitable transformations of (4) for every $K \ge 1$. But (3) is not sharp because its r.h.s. tends to $1/2$ as $K \to 1$. In what follows we are going to replace the r.h.s. in (3) so as to obtain an asymptotically sharp estimate.

Let

$$h_a(z) = \frac{z-a}{1-\overline{a}z}, \quad z, a \in D,$$

be a Möbius transformation of D. Assume $f(0) = a$. By the Darboux property there exist z_1, $z_2 \in T$ such that $h_a \circ f(z_1) = z_2$ and $h_a \circ f(-z_1) = -z_2$. Then the function $\overline{z}_2 h_a \circ f(z_1 z)$ is a K-quasiconformal self-mapping of D which keeps the points -1, 0, 1 fixed, and due to its Hölder continuity [6], we have

$$16^{1-K}|z-1|^K \le |\overline{z}_2 h_a \circ f(z_1 z) - 1| \le 16^{1-1/K}|z-1|^{1/K},$$
$$16^{1-K}|z+1|^K \le |\overline{z}_2 h_a \circ f(z_1 z) + 1| \le 16^{1-1/K}|z+1|^{1/K},$$

for every $z \in T$. Hence

$$\frac{1}{2\pi}\int_T |h_a \circ f(z) - z_2\overline{z}_1 z|\,|dz| \tag{5}$$
$$\le \frac{4}{2\pi}\int_0^{\pi/2}\left(2^{1/K}16^{1-1/K}\left(\sin\frac{t}{2}\right)^{1/K}\cos\frac{t}{2} - 2^K 16^{1-K}\left(\sin\frac{t}{2}\right)^K\cos\frac{t}{2}\right)dt$$
$$= \frac{4}{\pi(K+1)}(K2^{7(1-1/K)/2} - 2^{7(1-K)/2}) = p(K).$$

From this and (1) we get

$$|a| = \frac{1}{2\pi}\left|\int_T h_{-a}(z_2\overline{z}_1 z)\,|dz| - \int_T h_{-a}(h_a \circ f(z))\,|dz|\right|$$

$$= \frac{1}{2\pi}\left| \int_T \frac{(1-|a|^2)(z_2\overline{z}_1 z - h_a\circ f(z))}{(1+\overline{a}z_2\overline{z}_1 z)(1+\overline{a}h_a\circ f(z))}\,|dz|\right| \le \frac{1+|a|}{1-|a|}p(K)\,,$$

which implies, in view of (3),

$$|a| \le \frac{1-p(K)-\sqrt{(1-p(K))^2-4p(K)}}{2} \le 2p(K) < \frac{1}{3}\,,$$

since $1 \le K \le 1.044$, and this ends the proof.

3. For any automorphism γ of T, as shown by Choquet [2], the mapping

$$D \ni w \mapsto \frac{1}{2\pi}\int_T h_z\circ\gamma(u)\,\mathrm{Re}\,\frac{u+w}{u-w}\,|du| \in D$$

is an automorphism of D for any fixed $z\in D$. Therefore the equation

$$\frac{1}{2\pi}\int_T h_z\circ\gamma(u)\,\mathrm{Re}\,\frac{u+w}{u-w}\,|du| = 0 \tag{6}$$

implicitly defines a function $w = F_\gamma(z)$. It is quite easy to show that F_γ is a real-analytic diffeomorphic self-mapping of D which has a continuous extension to an automorphism γ^{-1} of T, and for any Möbius transformations η_1, η_2

$$F_{\eta_1\circ\gamma\circ\eta_2} = \eta_2^{-1}\circ F_\gamma\circ\eta_1^{-1}\,. \tag{7}$$

For details see [5]. As a matter of fact, F_γ^{-1} coincides with the mapping $E(\gamma)$ found by Douady and Earle [3; Theorem 1], but the construction of F_γ is much simpler as compared with that of $E(\gamma)$.

Assume that γ admits a K-quasiconformal extension f to D. We shall apply the lemma to estimate the complex dilatation of F_γ in D for small K. In view of (7) it is sufficient to do this at the point 0 in the case when $F_\gamma(0)=0$. Then by (6)

$$\int_T f(u)\,|du| = 0\,,$$

and differentiating both sides of (6) at 0 with respect to z and $\overline{z}$ we obtain

$$\partial_z F_\gamma(0) = \frac{\overline{A}-\overline{C}B}{|A|^2-|B|^2}\,,\qquad \partial_{\overline{z}} F_\gamma(0) = \frac{\overline{A}C-B}{|A|^2-|B|^2}\,,$$

where

$$A = \frac{1}{2\pi}\int_T \overline{u}f(u)\,|du|\,,\quad B = \frac{1}{2\pi}\int_T uf(u)\,|du|\,,\quad C = -\frac{1}{2\pi}\int_T f^2(u)\,|du|\,.$$

Let $a = f(0)$ and let z_1, z_2 be as in the proof of the lemma. Since $|h_a(u) - u| \le 2|a|$ for every $u\in T$, by the lemma and (5) we obtain for $K \le 1.044$

the following estimates:

$$
\begin{aligned}
1-|A| &\le \frac{1}{2\pi}\Big|\int_T z_2\overline{z}_1 u\overline{u}\,|du| - \int_T \overline{u}f(u)\,|du|\Big| \\
&\le \frac{1}{2\pi}\int_T |z_2\overline{z}_1 u - h_a\circ f(u)|\,|du| + \frac{1}{2\pi}\int_T |h_a\circ f(u) - f(u)|\,|du| \\
&\le p(K) + 2|a| = 5p(K)\,, \\
|B| &= \frac{1}{2\pi}\Big|\int_T uf(u)\,|du| - \int_T z_2\overline{z}_1 u^2\,|du|\Big| \\
&\le \frac{1}{2\pi}\int_T |f(u) - h_a\circ f(u)|\,|du| + \frac{1}{2\pi}\int_T |h_a\circ f(u) - z_2\overline{z}_1 u|\,|du| \\
&\le p(K) + 2|a| = 5p(K)\,, \\
|C| &= \frac{1}{2\pi}\Big|\int_T (z_2\overline{z}_1)^2 u^2\,|du| - \int_T f^2(u)\,|du|\Big| \\
&\le \frac{2}{2\pi}\int_T |z_2\overline{z}_1 u - h_a\circ f(u)|\,|du| + \frac{2}{2\pi}\int_T |h_a\circ f(u) - f(u)||du| \\
&\le 2p(K) + 4|a| \le 10p(K)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|\frac{\partial_{\overline{z}}F_\gamma(0)}{\partial_z F_\gamma(0)}\right| = \left|\frac{\overline{A}C - B}{\overline{A} - \overline{C}B}\right| &\le \frac{|C| + |B|}{1 - |1-A| - |C||B|} \\
&\le \frac{15p(K)}{1 - 5p(K) - 50p^2(K)} < 1
\end{aligned}
$$

if $K \le 1.01$.

Thus, in a simple and short way we have obtained the

THEOREM. *If an automorphism γ of T admits a K-quasiconformal extension to D, where $1 \le K \le 1.01$, then the mappings F_γ and $E(\gamma)$ are K^*-quasiconformal, where*

$$K^* \le 1 + 30p(K)(1 + 116p(K)).$$

What is important, we have obtained an explicit estimate which tends to 1 as $K \to 1$. In this sense the above theorem improves the results found by Douady and Earle [3; Corollary 2, Proposition 7]. Using the theory of Teichmüller mappings they have proved that, given $\epsilon > 0$, there exists $\delta > 0$ such that $K^* \le K^{3+\epsilon}$ if $K \le 1 + \delta$ [3; Corollary 2]. Their explicit estimate [3; Proposition 7] starts from $4 \cdot 10^8 e^{35}$ for K near 1.

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