ANNALES
POLONICI MATHEMATICI
55 (1991)

On roots of the automorphism group of a circular domain in $\mathbb{C}^n$

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Abstract. We study the properties of the group $\mathrm{Aut}(D)$ of all biholomorphic transformations of a bounded circular domain D in $\mathbb{C}^n$ containing the origin. We characterize the set of all possible roots for the Lie algebra of $\mathrm{Aut}(D)$. There exists an n-element set P such that any root is of the form α or $-\alpha$ or $\alpha - \beta$ for suitable $\alpha, \beta \in P$.

1. Introduction. A bounded domain in $\mathbb{C}^n$ is said to be *circular* if for all $z \in D$ and all $t \in \mathbb{R}$, $e^{it}z \in D$ where $i^2 = -1$.

Given any bounded domain $D \subset \mathbb{C}^n$, denote by $\mathrm{Aut}(D)$ the set of all biholomorphic transformations of D onto itself. $\mathrm{Aut}(D)$, when equipped with the compact-open topology, is a locally compact Lie group. A proof of this theorem due to H. Cartan can be found in [4].

In the present paper we study the properties of the Lie algebra of $\mathrm{Aut}(D)$. In Section 2 we review some facts on maximal tori of $\mathrm{Aut}(D)$. Every finite-dimensional complex linear representation of a compact abelian group can be decomposed into a direct sum of one-dimensional subrepresentations (see [1]). To any complex one-dimensional representation of a maximal torus T there corresponds a $\mathbb{C}$-linear functional on the complexification of the Lie algebra of T. This functional is called a root of $\mathrm{Aut}(D)$ and is a generalization of that defined for semisimple groups. In Section 3 we characterize the set of all possible roots of $\mathrm{Aut}(D)$. This result generalizes an analogous one obtained by Sunada [5] for n-circular domains in $\mathbb{C}^n$.

2. Properties of maximal tori of $\mathrm{Aut}(D)$**.** Assume that $D \subset \mathbb{C}^n$ is a bounded circular domain. Let G be the identity component of $\mathrm{Aut}(D)$, denote by 0 the origin of $\mathbb{C}^n$ and assume that $0 \in D$. By a theorem of H. Cartan the set $K := \{f \in G : f(0) = 0\}$ is a compact subgroup in G

1991 *Mathematics Subject Classification*: Primary 17B05, 32A07.
Key words and phrases: circular domain, automorphism group, maximal torus, Lie algebra, adjoint representation, root, root subspace.

and any $f \in K$ is the restriction to D of a $\mathbb{C}$-linear transformation of $\mathbb{C}^n$ (see [4]).

A group T is called a *torus* if it is abelian, connected and compact. For any two maximal tori in K there exists an inner automorphism of K transforming one of them onto the other (see [1], p. 71). The (real) dimension of a maximal torus is called the *rank* of K. We will denote by T any fixed maximal torus in K and by r the rank of K.

Let X be any real vector field on D. In the standard frame $\partial/\partial z_1, \ldots, \partial/\partial z_n, \partial/\partial \overline{z}_1, \ldots, \partial/\partial \overline{z}_n$ it can be written in the form

$$X = \sum_{j=1}^{n} f_j \partial/\partial z_j + \sum_{j=1}^{n} \overline{f}_j \partial/\partial \overline{z}_j \,. \tag{1}$$

X is said to be *holomorphic* if for any function h holomorphic on D the function Xh is holomorphic on D. The components f_j, $j = 1, \ldots, n$, of a real holomorphic vector field (1) are holomorphic on D. We say that a real vector field X on D *generates* a one-parameter group $\{g_t : t \in \mathbb{R}\} \subset G$ if for all $z \in D$, $(g_t)_{*t=0}(d/dt) = X(z)$.

Denote by $\underline{G}$, $\underline{K}$ and $\underline{T}$ respectively the Lie algebras of all real vector fields on D generating one-parameter subgroups of G, K and T. $\underline{G}$, $\underline{K}$ and $\underline{T}$ are isomorphic to the Lie algebras of the Lie groups G, K and T respectively (see [3]). Since in the case of circular domains containing the origin we have some information about elements of $\underline{G}$, $\underline{K}$ and $\underline{T}$ (see Theorem (5)) we operate in $\underline{G}$ rather than in the Lie algebra of the Lie group G.

Assume that D is a bounded circular domain in $\mathbb{C}^n$ containing the origin. The proof of the following two theorems can be found in [3].

(2) Theorem. *Assume that H is an s-dimensional connected compact abelian Lie group* (*not necessarily maximal*) *whose elements are restrictions to D of linear transformations of $\mathbb{C}^n$. Then $1 \le s \le n$ and there exists a $\mathbb{C}$-linear change of coordinates in $\mathbb{C}^n$ and real numbers a_k^j, $j = 1, \ldots, s$, $k = s+1, \ldots, n$, satisfying $\sum_{j=1}^{s} a_k^j = 1$ for $k = s+1, \ldots, n$ such that in the new coordinates $w_1, \ldots, w_n$*

(i) *for any $h \in H$ there exists an s-tuple $(\theta_1, \ldots, \theta_s) \in \mathbb{R}^s$ such that the matrix of h is* $\mathrm{diag}[\exp(i\widehat{\theta}_1), \ldots, \exp(i\widehat{\theta}_n)]$ *with*

$$\widehat{\theta}_p = \begin{cases} \theta_p \,, & 1 \le p \le s, \\ \sum_{j=1}^{n} a_p^j e_j \,, & p > s, \end{cases} \tag{2a}$$

(ii) *the vector fields*

$$X_j = i\Big(z_j \partial/\partial z_j + \sum_{k=s+1}^{n} a_k^j z_k \partial/\partial z_k - \overline{z}_j \partial/\partial \overline{z}_j - \sum_{k=s+1}^{n} a_k^j \overline{z}_k \partial/\partial \overline{z}_k\Big) \tag{2b}$$

for $j = 1, \ldots, s$, form a frame for the Lie algebra $\underline{H}$ of all real vector fields on D generating one-parameter subgroups of H. ■

(3) Theorem. *Let T_1 and T_2 be any pair of maximal tori in $K = \{f \in \mathrm{Aut}(D) : f(0) = 0\}$. Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are linear frames in $\mathbb{C}^n$ such that the matrices of elements of T_1 in $\mathcal{B}_1$ are of the form* $\mathrm{diag}[\exp(i\widehat{\theta}_1), \ldots, \exp(i\widehat{\theta}_n)]$ *with*

$$\widehat{\theta}_k = \begin{cases} \theta_k\,, & k \le r, \\ \sum_{j=1}^{r} a_k^j \theta_j\,, & k > r, \end{cases}$$

and the matrices of elements of T_2 in $\mathcal{B}_2$ are of the form $\mathrm{diag}[\exp(i\widehat{\phi}_1), \ldots, \exp(i\widehat{\phi}_n)]$ *with*

$$\widehat{\phi}_k = \begin{cases} \phi_k\,, & k \le r, \\ \sum_{j=1}^{r} b_k^j \phi_j\,, & k > r. \end{cases}$$

Set

$$A = \begin{bmatrix} a_{r+1}^1 & \ldots & a_n^1 \\ \ldots & \ldots & \ldots \\ a_{r+1}^r & \ldots & a_n^r \end{bmatrix}, \qquad B = \begin{bmatrix} b_{r+1}^1 & \ldots & b_n^1 \\ \ldots & \ldots & \ldots \\ b_{r+1}^r & \ldots & b_n^r \end{bmatrix}.$$

Then

(1) *$\mathcal{B}_1$ and $\mathcal{B}_2$ are the same up to the order of elements.*

(2) *If C is the transition matrix from $\mathcal{B}_2$ to $\mathcal{B}_1$ then there exists a real $r \times r$ matrix E satisfying the following conditions*:

(a) *$E[\mathbf{1}_r, A]C = [\mathbf{1}_r, B]$ with $\mathbf{1}_r$ the $r \times r$ identity matrix, and $[\mathbf{1}_r, A]$ the $r \times n$ real matrix whose first r columns are those of $\mathbf{1}_r$ and the other are those of A.*

(b) *If $v = [1, \ldots, 1] \in \mathbb{R}^r$ then $vE = v$.*

(3) *If $\mathcal{B}_1 = \mathcal{B}_2$, then $A = B$.* ■

(4) Theorem. *In the notation of Theorem* (2) *the map*

$$[\mathbb{R}/2\pi\mathbb{Z}]^s \ni (\theta_1, \ldots, \theta_s) \to \phi = \mathrm{diag}[\exp(i\widehat{\theta}_1), \ldots, \exp(i\widehat{\theta}_n)]$$

is a homomorphism of groups if and only if a_k^j is an integer for all $j = 1, \ldots, s$, $k = s+1, \ldots, n$.

Proof. For any $k \in \{1, \ldots, n\}$ the map $\mathbb{R}^s \ni (\theta_1, \ldots, \theta_s) \to \widehat{\theta}_k(\theta_1, \ldots, \theta_s) \in \mathbb{R}$ is linear. One can easily check that $\{(\theta_1, \ldots, \theta_s) \in \mathbb{R}^s : \widehat{\theta}_k(\theta_1, \ldots, \theta_s) = 0 \bmod 2\pi$ for $k = 1, \ldots, n\} = \{(\theta_1, \ldots, \theta_s) \in \mathbb{R}^s : \theta_j = 0 \bmod 2\pi$ for $j = 1, \ldots, s\}$ if and only if for any $(m_1, \ldots, m_s) \in \mathbb{Z}^s$ and for any $k = s+1, \ldots, n$, $\sum_{j=1}^{s} a_k^j m_j \in \mathbb{Z}$. This is equivalent to the condition $a_k^j \in \mathbb{Z}$ for $j = 1, \ldots, s$, $k = s+1, \ldots, n$. ■

3. Properties of the set of roots of the algebra $\underline{G}$. It can be checked that the Lie algebra $\underline{G}$ of all real vector fields on D generating one-parameter subgroups in $\mathrm{Aut}(D)$ is real, i.e. for any nonzero X in $\underline{G}$, iX is not in $\underline{G}$ (see for instance [3]). It is easy to see that for any $X \in \underline{G}$ the map $\underline{G} \ni Y \to \mathrm{ad}(X)Y = [X,Y] \in \underline{G}$ is linear.

Denote by $\underline{G}^c$, $\underline{K}^c$ and $\underline{T}^c$ the complexifications of the algebras $\underline{G}$, $\underline{K}$ and $\underline{T}$ respectively. In a natural way the map $\underline{G}\times\underline{G} \ni (X,Y) \to \mathrm{ad}(X)Y$ extends to a $\mathbb{C}$-bilinear map $\underline{G}\times\underline{G} \ni (X,Y) \to \mathrm{ad}(X)Y \in \underline{G}^c$. Denote by J the real vector field on D generating the one-parameter group $\{\exp(it)\,\mathrm{id}_D : t \in \mathbb{R}\}$. One easily checks that in the standard frame on $\mathbb{C}^n$

$$J = i\Big(\sum_{k=1}^{n} z_k\partial/\partial z_k - \sum_{k=1}^{n} \overline{z}_k\partial/\partial \overline{z}_k\Big),$$

(5) Theorem. *In the above notation*

(i) $\underline{K} = \ker[\mathrm{ad}(J)']$, *where* $'$ *denotes the restriction of a map to* $\underline{G}$. *If* $\underline{P} = \ker\{\mathrm{id}' + [\mathrm{ad}(J)']^2\}$, *then* $\underline{G} = \underline{K} + \underline{P}$ (*direct sum*) *and* $[\underline{K},\underline{P}] \subset \underline{P}$, $[\underline{P},\underline{P}] \subset \underline{K}$.

(ii) *If* $\underline{P}^+ = \{X \in \underline{P}^c : \mathrm{ad}(J)X = iX\}$, $\underline{P}^- = \{X \in \underline{P}^c : \mathrm{ad}(J)X = -iX\}$, *then* $\underline{G}^c = \underline{K}^c + \underline{P}^+ + \underline{P}^-$ (*direct sum*) *and* $[\underline{P}^+,\underline{P}^+] = [\underline{P}^-,\underline{P}^-] = \{0\}$.

(iii) *For any* $X \in \underline{G}^c$ *there exists a unique decomposition* $X = X^{(0)} + X^{(1)} + X^{(2)}$ *such that in the standard frame on* $\mathbb{C}^n$

$$X^{(m)} = \sum_{j=1}^{n} p_j^m(z)\partial/\partial z_j + \sum_{j=1}^{n} \overline{p_j^m(z)}\partial/\partial \overline{z}_j\,, \qquad m = 0,1,2\,,$$

and p_j^m *are homogeneous polynomials in* $z_1,\ldots,z_n$ *of degree* m *for* $j = 1,\ldots,n$. *Moreover*, $X^{(0)} \in \underline{P}^-$, $X^{(1)} \in \underline{K}^c$, $X^{(2)} \in \underline{P}^+$.

See [2] for a proof. ∎

Since T is abelian, $\underline{T}^c$ is commutative. By the Jacobi identity, for all $X,Y \in \underline{T}^c$, $\mathrm{ad}(X)\,\mathrm{ad}(Y) = \mathrm{ad}(Y)\,\mathrm{ad}(X)$. A nonzero linear functional $\alpha : \underline{T}^c \to \mathbb{C}$ is said to be a *root* of the algebra $\underline{G}$ if there exists a nonzero Y in $\underline{G}^c$ such that for all $X \in \underline{T}^c$, $\mathrm{ad}(X)Y = \alpha(X)Y$. Assume that α is a root of $\underline{G}$. Then the set $\underline{G}^\alpha = \{Y \in \underline{G}^c : \mathrm{ad}(X)Y = \alpha(X)Y$, for all $X \in \underline{G}^c\}$ is a complex linear space and is called the *root subspace* in $\underline{G}^c$ corresponding to the root α. For any complex vector space $V \subset \underline{G}^c$ we put $\Delta(V) = \{\alpha : \alpha$ is a root of $\underline{G}$, $\underline{G}^\alpha \subset V\}$; $\Delta := \Delta(\underline{G}^c)$.

Assume that in the coordinates $z_1,\ldots,z_n$ in $\mathbb{C}^n$ all elements of T have diagonal matrices satisfying the conditions of Theorem (2). It can be shown that the vector fields

$$(6)\qquad Z_m = z_m\partial/\partial z_m + \sum_{j=1}^{r} a_j^m z_j\partial/\partial z_j + \overline{z}_m\partial/\partial \overline{z}_m + \sum_{j=1}^{r} a_j^m \overline{z}_j\partial/\partial \overline{z}_j$$

for $m = 1, \dots, r$ form a frame of the complex space $\underline{T}^c$. Denote by α_m, $m = 1, \dots, r$, the elements of the dual frame, i.e. the $\mathbb{C}$-linear functionals on $\underline{T}^c$ such that $\alpha_k(Z_m) = \delta_{km}$ (Kronecker's delta) for $k, m = 1, \dots, r$.

Since $\underline{G}$ is a real vector space, for any $Z \in \underline{G}^c$ there exist unique $X, Y \in \underline{G}$ such that $Z = X + iY$. Denote by σ the map $\underline{G}^c \ni X + iY \to \sigma(X + iY) = X - iY$. σ is called the *conjugation* in $\underline{G}^c$ with respect to the real algebra $\underline{G}$. Below we list some properties of σ.

(7) Lemma. *In the above notation*:

1° $\sigma^2 = \mathrm{id}$.
2° $\forall X, Y \in \underline{G}^c\, \forall a, b \in \mathbb{C}$, $\sigma(aX + bY) = \overline{a}\sigma(X) + \overline{b}\sigma(Y)$.
3° $\forall X, Y \in \underline{G}^c$, $\sigma([X, Y]) = [\sigma(X), \sigma(Y)]$.
4° $\forall X \in \underline{G}^c$, $X \in \underline{G} \Leftrightarrow \sigma(X) = X$.
5° $\sigma(\underline{P}^-) = \underline{P}^+$, $\sigma(\underline{P}^+) = \underline{P}^-$, $\sigma(\underline{K}^c) = \underline{K}^c$.
6° $\sigma(\underline{G}^\alpha) = \underline{G}^\beta$, *with* $\beta = \overline{\alpha \circ \sigma}$.

Proof. 1° is obvious. For 2°, let $a, b \in \mathbb{R}$, $X, Y \in \underline{G}$. Then $\sigma((a + bi)(X + iY)) = \sigma((aX - bY) + i(bX + aY)) = aX - bY - i(bX + aY) = (a - bi)\sigma(X + iY)$. 3° can be checked by a direct computation similar to that of 2°. 4° is obvious. For 5°, we first show that $\sigma(\underline{P}^-) \subset \underline{P}^+$. Assume that $X \in \underline{P}^-$. Then $\mathrm{ad}(J)\sigma(X) = [J, \sigma(X)] = \sigma([\sigma(J), X]) = \sigma([J, X]) = \sigma(-iX) = i\sigma(X)$ (we have used the fact that $J \in \underline{T} \subset \underline{G}$). Hence $\sigma(X) \in \underline{P}^+$. The converse inclusion can be checked in the same way. For 6° assume that Y is a nonzero element of $\underline{G}^\alpha$. Let X be any element of $\underline{T}^c$. Then $\mathrm{ad}(X)\sigma(Y) = [X, \sigma(Y)] = \sigma([\sigma(X), Y]) = \sigma(\alpha(\sigma(X))Y) = \overline{\alpha(\sigma(X))}\sigma(Y)$. Hence $\sigma(Y) \in \underline{G}^\beta$ with $\beta = \overline{\alpha \circ \sigma}$. In the same way one checks that $\sigma(\underline{G}^\beta) \subset \underline{G}^\alpha$. Since σ is bijective, this completes the proof. ∎

(8) Lemma. *In the above notation*, *for any* $\alpha \in \Delta$ *exactly one of the following conditions holds*:

$$\text{(a) } \underline{G}^\alpha \subset \underline{P}^- , \quad \text{(b) } \underline{G}^\alpha \subset \underline{K}^c , \quad \text{(c) } \underline{G}^\alpha \subset \underline{P}^+ .$$

Proof. Let $\alpha \in \Delta$ and let Y be any nonzero element of $\underline{G}^\alpha$. By Theorem (5) there are unique $Y^{(0)} \in \underline{P}^-$, $Y^{(1)} \in \underline{K}^c$, $Y^{(2)} \in \underline{P}^+$ such that $Y = Y^{(0)} + Y^{(1)} + Y^{(2)}$ and $\mathrm{ad}(J)Y = \alpha(J)Y$. On the other hand, $\mathrm{ad}(J)Y = -iY^{(0)} + 0Y^{(1)} + iY^{(2)}$. It follows that at most one of the components $Y^{(0)}, Y^{(1)}, Y^{(2)}$ is nonzero. ∎

Assume that in the coordinates $z_1, \dots, z_n$ in $\mathbb{C}^n$ the matrices of all elements of T have the form $\mathrm{diag}[\exp(i\widehat{\theta}_1), \dots, \exp(\widehat{\theta}_n)]$ with $\widehat{\theta}_k$ of the form (2a) with $s = r$. Let α_k for $k = 1, \dots, r$ be the elements of the dual frame to (6) in $\underline{T}^c$ and define

$$\widehat{\alpha}_k = \begin{cases} \alpha_k , & 1 \le k \le r, \\ \sum_{j=1}^n a_k^j \alpha_j , & k > r, \end{cases} \qquad \text{for } k = 1, \dots, n . \tag{9}$$

Below we investigate some properties of $\underline{G}$ associated to root subspaces contained in $\underline{P}^-$, $\underline{P}^+$ and $\underline{K}^c$ respectively. Z_m are the vector fields defined in (6) for $m = 1, \ldots, r$.

(10) LEMMA. *In the above notation, if* $\alpha \in \Delta(\underline{P}^-)$ *then*

(a) *there exists* $k \in \{1, \ldots, n\}$ *such that* $\alpha = -\widehat{\alpha}_k$.

(b) $\underline{G}^\alpha \subset \sum_{j,\hat\theta_j=\hat\theta_k} \mathbb{C}\partial/\partial z_j + \sum_{j,\hat\theta_j=\hat\theta_k} \mathbb{C}\partial/\partial \overline{z}_j$ (*direct sum of one-dimensional subspaces*), *where both sums are over all* $j \in \{1, \ldots, n\}$ *such that* $\widehat{\theta}_j = \widehat{\theta}_k$.

P r o o f. Let Y be a nonzero element of $\underline{G}^\alpha$. By Theorem (5) there exist $y_j, y'_j \in \mathbb{C}$ for $j = 1, \ldots, n$ such that $Y = \sum_{j=1}^n y_j \partial/\partial z_j + \sum_{j=1}^n y'_j \partial/\partial \overline{z}_j$. Let $X = \sum_{j=1}^r x_j Z_j$ be an arbitrary element of $\underline{T}^c$ with $x_j \in \mathbb{C}$ for $j = 1, \ldots, r$. Assume that $\alpha = \sum_{j=1}^r b_j \alpha_j$ with $b_j \in \mathbb{C}$ for $j = 1, \ldots, r$. By a direct computation one finds that $\mathrm{ad}(X)Y = \alpha(X)Y$ if and only if

$$y_m \Big[\sum_{j=1}^r (b_j + \delta_{jm}) x_j\Big] = y'_m \Big[\sum_{j=1}^r (b_j + \delta_{jm}) x_j\Big] = 0 \quad \text{for } m = 1, \ldots, r\,,$$

$$y_m \Big[\sum_{j=1}^r (b_j + a_j^m) x_j\Big] = y'_m \Big[\sum_{j=1}^r (b_k + a_j^m) x_j\Big] = 0 \quad \text{for } m = r+1, \ldots, n\,.$$

The above equations are satisfied for any X in $\underline{T}^c$ if and only if $y_m(\alpha + \widehat{\alpha}_m) = y'_m(\alpha + \widehat{\alpha}_m) = 0$ for $m = 1, \ldots, n$. Since $Y \neq 0$, there exists $k \in \{1, \ldots, n\}$ such that $y_k \neq 0$ or $y'_k \neq 0$. Hence $\alpha + \widehat{\alpha}_k = 0$ and $y_m = y'_m = 0$ for $m \in \{1, \ldots, n\}$ such that $\widehat{\alpha}_m \neq \widehat{\alpha}_k$ or equivalently such that $\widehat{\theta}_m \neq \widehat{\theta}_k$. ∎

(11) LEMMA. *In the above notation, if* $\alpha \in \Delta(\underline{K}^c)$ *then*

(a) *there exist* $p, q \in \{1, \ldots, n\}$ *such that* $\alpha = \widehat{\alpha}_p - \widehat{\alpha}_q$,

(b) $\underline{G}^\alpha \subset \sum_{j,k,\hat\theta_j-\hat\theta_k=\hat\theta_p-\hat\theta_q} \mathbb{C} z_j \partial/\partial z_k + \sum_{j,k,\hat\theta_j-\hat\theta_k=\hat\theta_p-\hat\theta_q} \mathbb{C} \overline{z}_j \partial/\partial \overline{z}_k$ (*direct sum of one-dimensional subspaces*).

P r o o f. Let $Y = \sum_{j,k=1}^n y_{jk} z_j \partial/\partial z_k + \sum_{j,k=1}^n y'_{jk} \overline{z}_j \partial/\partial \overline{z}_k$ be any nonzero element of $\underline{G}^\alpha$, let $X = \sum_{j=1}^n x_j Z_j$ be an arbitrary element of $\underline{T}^c$, and assume that $\alpha = \sum_{j=1}^r b_j \alpha_j$. By a direct computation one finds that $\mathrm{ad}(X)Y = \alpha(X)Y$ for all $X \in \underline{T}^c$ if and only if $y_{km}[\widehat{\alpha}_m - \widehat{\alpha}_k + \alpha] = y'_{km}[\widehat{\alpha}_m - \widehat{\alpha}_k + \alpha] = 0$ for all $k, m \in \{1, \ldots, n\}$. A reasoning similar to that in the proof of Lemma (10) completes the proof. ∎

(12) LEMMA. *In the above notation, if* $\alpha \in \Delta(\underline{P}^+)$ *then*

(a) *there exists* $p \in \{1, \ldots, n\}$ *such that* $\alpha = \widehat{\alpha}_p$,

(b) $\underline{G}^\alpha \subset \sum_{j,k,l,\hat\theta_j+\hat\theta_k-\hat\theta_l=\hat\theta_p} \mathbb{C} z_j z_k \partial/\partial z_l + \sum_{j,k,l,\hat\theta_j+\hat\theta_k-\hat\theta_l=\hat\theta_p} \mathbb{C} \overline{z}_j \overline{z}_k \partial/\partial \overline{z}_l$ (*direct sum of one-dimensional subspaces*).

P r o o f. Let $Y = \sum_{j,k,l=1}^{n} y_{jkl} z_j z_k \partial/\partial z_l + \sum_{j,k,l=1}^{n} y'_{jkl} \overline{z}_j \overline{z}_k \partial/\partial \overline{z}_l$ be a nonzero element of $\underline{G}^{\alpha}$ and let $X = \sum_{j=1}^{n} x_j Z_j$ be an arbitrary element of $\underline{T}^c$. Assume that $\alpha = \sum_{j=1}^{r} b_j \alpha_j$. By a direct computation one finds that $\operatorname{ad}(X)Y = \alpha(X)Y$ for all $X \in \underline{T}^c$ if and only if $y_{jkl}[\widehat{\alpha}_l - \widehat{\alpha}_j - \widehat{\alpha}_k + \alpha] = y'_{jkl}[\widehat{\alpha}_l - \widehat{\alpha}_j - \widehat{\alpha}_k + \alpha] = 0$ for $j,k,l \in \{1,\ldots,n\}$. Since $Y \neq 0$, there exists a triple $(p,q,s) \in \{1,\ldots,n\}^3$ such that $y_{pqs} \neq 0$ or $y'_{pqs} \neq 0$. Hence $\alpha = \widehat{\alpha}_p + \widehat{\alpha}_q - \widehat{\alpha}_s$ and $y_{jkl} = y'_{jkl} = 0$ for all $j,k,l \in \{1,\ldots,n\}$ such that $\widehat{\theta}_j + \widehat{\theta}_k - \widehat{\theta}_l \neq \widehat{\theta}_p + \widehat{\theta}_q - \widehat{\theta}_s$.

On the other hand, by Lemma 7(6°), $\sigma(\underline{G}^{\alpha}) = \underline{G}^{\beta}$ with $\beta = \overline{\alpha \circ \sigma}$. Let $X = X' + iX''$ with $X', X'' \in \underline{T}$. Then $\beta(X) = \overline{\alpha(X' - iX'')} = \overline{\alpha(X')} + \overline{i\alpha(X'')}$. Since for any $\alpha \in \Delta(\underline{P}^-)$, $\alpha = \sum_j b_j \alpha_j$ with b_j real for $j = 1,\ldots,r$ and $\alpha(\underline{T}) \subset i\mathbb{R}$, we have $\beta = -\alpha$. Hence there exists $p \in \{1,\ldots,n\}$ such that $\alpha = \widehat{\alpha}_p$ and $y_{jkl} = y'_{jkl} = 0$ for $j,k,l \in \{1,\ldots,n\}$ such that $\widehat{\theta}_j + \widehat{\theta}_k - \widehat{\theta}_l \neq \widehat{\theta}_p$. ■

(13) Corollary. *The assertion* 6° *in Lemma* (7) *can be formulated as follows*: $\sigma(\underline{G}^{\alpha}) = \underline{G}^{-\alpha}$.

(14) Theorem. *Let D be a bounded circular domain in $\mathbb{C}^n$ containing the origin, and let T be any maximal torus in* $\operatorname{Aut}(D)$. *Let $\underline{G}$, $\underline{T}$ be the Lie algebras of real vector fields generating all one-parameter subgroups in* $\operatorname{Aut}(D)$ *and T respectively. Let Δ be the set of all roots of $\underline{G}$. Then*

(i) *For any $\alpha \in \Delta$ we have $-\alpha \in \Delta$.*

(ii) *There exists a set Π in the dual space to $\underline{T}^c$ with the following properties*:

(a) *Π has at most n elements.*

(b) *For any $\alpha \in \Delta$ one of the following holds*:

$$\alpha = \beta\,, \quad \alpha = -\beta\,, \quad \alpha = \beta - \gamma \quad \textit{for some } \beta, \gamma \in \Pi\,.$$

(iii) *If in the coordinates $z_1, \ldots, z_n$ in $\mathbb{C}^n$ all elements of T have diagonal matrices satisfying the conditions of Theorem* (2), *then $\Pi = \{\widehat{\alpha}_k : k = 1,\ldots,n\}$ with $\widehat{\alpha}_k$ of the form* (9).

P r o o f. This is an easy consequence of Lemmas 8, 10, 11, 12. ■

(15) R e m a r k. For $r = n$ we obtain the n-circular case studied by T. Sunada [5].

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Reçu par la Rédaction le 14.9.1990
Révisé le 10.1.1991