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A counterexample to subanalyticity of an arc-analytic function

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Abstract. We construct an arc-analytic function (i.e. a function analytic on each analytic arc) whose graph is not subanalytic.

Let $f : U \to \mathbb{R}$ be a function, where U is open in $\mathbb{R}^n$. We say that f is *arc-analytic* iff for each analytic arc $\gamma : (-1, 1) \to U$, the composition $f \circ \gamma$ is analytic (see [K2], [BM] for examples). If we suppose moreover that f has subanalytic graph it turns out that such an f has some interesting properties. For example if we compose f with a suitable finite composition of local blowing-ups we get an analytic function (see [BM]). In Spring 1985, during the Warsaw Semester on Singularities, after discussions with E. Bierstone, P. Milman and B. Teissier the following conjecture was stated.

CONJECTURE. Every arc-analytic function is locally subanalytic. More precisely, given an arc-analytic function $f : U \to \mathbb{R}$, where U is open in $\mathbb{R}^n$, for each $x \in U$ there is a neighborhood V_x of x such that the restriction $f|V_x$ has subanalytic graph.

In this paper we give a counterexample to this conjecture. The idea of our construction was suggested by an example, due to G. Dloussky, of a mapping which is meromorphic in the sense of Stoll but not in the sense of Remmert (see 5.5 in [D]). I wish to express my gratitude to G. Dloussky for enlightening discussions.

We are going to construct by induction an infinite composition of blowing-ups. Put $X_0 = \mathbb{R}^2$, $P_0 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, $c_0 = (0, 0)$. We denote by $\pi_{1,0} : X_1 \to X_0$ a blowing-up of X_0 centered at c_0. Suppose we have already constructed a blowing-up $\pi_{n,n-1} : X_n \to X_{n-1}$ centered at c_{n-1}. We denote by P_n a strict transform of P_{n-1}, and we put $D_n = \pi^{-1}_{n,n-1}(c_{n-1})$, $\{c_n\} = P_n \cap D_n$. We take for $\pi_{n+1,n} : X_{n+1} \to X_n$ a blowing-up of X_n centered at c_n. If $n > m$ we put $\pi_{n,m} = \pi_{n,n-1} \circ \ldots \circ \pi_{m+1,m} : X_n \to X_m$,

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and $\pi_{n,n} = \mathrm{id}_{X_n}$, for $n \in \mathbb{N}$. Clearly $\pi_{n,k} = \pi_{n,m} \circ \pi_{m,k}$ for $n \geq m \geq k$. Hence $\pi_{n,m} : X_n \to X_m$, $n, m \in \mathbb{N}$, $n \geq m$, is an inverse system. Consider its limit:

$$\varprojlim X_n = \Big\{(x_n) \in \prod_{n\in\mathbb{N}} X_n ; \pi_{n+1,n}(x_{n+1}) = x_n\Big\}.$$

We put $L = \varprojlim X_n \setminus \{c\}$, where $c = (c_n)$, $n \in \mathbb{N}$. Set $p_k : L \ni (x_n)_{n\in\mathbb{N}} \mapsto x_k \in X_k$. We have an induced topology on $\varprojlim X_n$, hence also on L, such that all p_n, $n \in \mathbb{N}$, are continuous. Clearly the topology of L has a countable basis.

Let $x = (x_n)_{n\in\mathbb{N}} \in L$. Then there is an integer $k \in \mathbb{N}$ such that $x_n \neq c_n$ for all $n \geq k$. Hence there is a neighborhood U of x (in L) such that $y_n \neq c_n$ for all $n \geq k$ and all $y = (y_n)_{n\in\mathbb{N}} \in U$.

Notice that $\pi_{n,m} \mid U_n : U_n \to U_m$, $n \geq m \geq k$, is an analytic diffeomorphism, where $U_i = p_i(U)$, $i \in \mathbb{N}$. Thus $p_n|U : U \to U_n$, $n \geq k$, is a homeomorphism on U_n which is a neighborhood of $x_n \in X_n$. The family of all such projections defines on L the structure of a real analytic manifold such that all projections $p_n : L \to X_n$, $n \in \mathbb{N}$, are analytic, and moreover each p_n has an analytic inverse on $X_n \setminus \{c_n\}$.

Consider $p = p_0 : L \to \mathbb{R}^2$; clearly p has an analytic inverse on $\mathbb{R}^2 \setminus (0,0)$.

Now take an analytic arc $\gamma : (-1,1) \to \mathbb{R}^2$, $\gamma(0) = (0,0)$, $\gamma = (\gamma_1, \gamma_2)$, and suppose $\gamma_2(t) \neq 0$ for $t \neq 0$. Assume that γ_2 has multiplicity k at 0. The mapping $\widetilde{\gamma}_n(t) = \pi_{n,0}^{-1} \circ \gamma(t)$, for $t \neq 0$, can be extended analytically to 0. Notice that if $n \geq k$, then $\lim_{t\to 0} \widetilde{\gamma}_n(t) \neq c_n$. Hence the arc $\widetilde{\gamma} = p^{-1} \circ \gamma$, for $t \neq 0$, can be extended analytically to 0, since $p = p_n \circ \pi_{n,0}$.

Let now $\gamma = (\gamma_1, \gamma_2)$ be an arc such that $\gamma_2 \equiv 0$. Then for each compact K in L we can find $\varepsilon > 0$ such that $p^{-1} \circ \gamma(t) \notin K$ for all t with $0 < |t| < \varepsilon$.

By the Grauert embedding theorem (see [G]) the analytic manifold L admits a proper analytic embedding $\alpha : L \to \mathbb{R}^N$, for some $N \in \mathbb{N}$ (by construction the topology of L has a countable basis). Put $G = \alpha \circ p^{-1} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^N$, $G = (G_1, \ldots, G_N)$ and $g = \sum_{i=1}^N G_i^2$. By the previous remarks it is obvious that g satisfies the following conditions:

(i) if $\gamma : (-1,1) \to \mathbb{R}^2$, $\gamma(0) = (0,0)$, is an analytic arc such that $\gamma_2(t) \neq 0$ for $t \neq 0$, then the function $g \circ \gamma(t)$, for $t \neq 0$, has an analytic extension to 0.

(ii) $\lim_{t\to 0} g(t,0) = +\infty$.

Finally, we define $f : \mathbb{R}^2 \to \mathbb{R}$, putting $f(0,0) = 0$, $f(x,y) = yg(x,y)$ for $(x,y) \neq (0,0)$. By property (i) of g it is clear that f is an arc-analytic function.

Assume now that the restriction $f|V$ has subanalytic graph for some neighborhood V of $(0,0)$. Then by the "curve selecting lemma" f is continuous on V (see [K2], [BM]), thus we can assume that f is bounded on V.

Denote by $\tau : \mathbb{R} \to \mathbb{P}^1$ the natural embedding of $\mathbb{R}$ in $\mathbb{P}^1$, $\tau(t) = (t, 1) \in \mathbb{P}^1$. Let $\varphi : A \to \mathbb{R}$, $A \subset \mathbb{R}^n$, be a function. We say that φ is *subanalytic at infinity* ($\varphi \in \mathrm{SUBB}(\mathbb{R}^N)$ in the notation of [K1]) iff the graph of $\tau \circ \varphi$ is subanalytic in $\mathbb{R}^n \times \mathbb{P}^1$.

Clearly our f, being bounded, is subanalytic at infinity; also $h(x, y) = 1/y$, defined for $(x, y) \in \mathbb{R}^2 \setminus \{y = 0\}$, is subanalytic at infinity. Hence the product $g' = f \cdot h$ (defined on $V \setminus \{y = 0\}$) is subanalytic at infinity (cf. [K2]).

Clearly $g' = g$ on $V \setminus \{y = 0\}$. Since g is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$ we have

$$\lim_{z \to (0,0)} \sup g(z) = \lim_{z \to (0,0)} \sup g'(z) = +\infty\,.$$

From the curve selecting lemma applied to the graph of $\tau \circ g'$ at $((0, 0), \infty)$ we obtain an analytic arc $\gamma = (\gamma_1, \gamma_2)$, $\gamma : (-1, 1) \to \mathbb{R}^2$, such that $\gamma(0) = (0, 0)$, $\gamma_2(t) \neq 0$ for $t \neq 0$ and $\lim_{t \to 0} g' \circ \gamma(t) = \lim_{t \to 0} g \circ \gamma(t) = \infty$. This gives a contradiction with property (i) of g.

In [BM] Bierstone and Milman asked whether every arc-analytic function is continuous. It is not clear whether or not our function f is continuous.

Addendum. After writing this paper I have learned that an example of an arc-analytic function which is not continuous was constructed by E. Bierstone, P. D. Milman and A. Parusiński in their preprint *A function which is arc-analytic but not continuous*, Univ. of Toronto, 1990. They also constructed a continuous arc-analytic function whose graph is not subanalytic. Their constructions differ from ours.

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