ANNALES
POLONICI MATHEMATICI
55 (1991)

The kaehlerian structures and reproducing kernels

by Anna Krok and Tomasz Mazur (Radom)

Abstract. It is shown that one can define a Hilbert space structure over a kaehlerian manifold with global potential in a natural way.

Introduction. S. Bergman introduced and developed some methods of functional analysis and differential geometry in the theory of several complex variables [2, 3]. In this approach important role is played by the Hilbert space $L^2H(D)$ of all functions which are holomorphic and Lebesgue square integrable on a domain $D \subset \mathbb{C}^N$. The evaluation functional

$$\chi_z^* : L^2H(D) \to \mathbb{C}\,, \qquad \chi_z^*(f) = f(z)\,,$$

is continuous and can be represented by $\chi_z \in L^2H(D)$ as follows:

$$f(z) = (f, \chi_z)$$

(for details see [2, 10, 12]). The well-known Bergman function [2, 12]

$$K_D(z,w) = (\chi_w, \chi_z) \tag{0.1}$$

generates a geometric structure on D, given by a tensor g of the form

$$g(z) = \sum_{i,j=1}^{N} (g_{i\bar{j}}(z) dz_i \otimes d\bar{z}_j + g_{\bar{i}j}(z) d\bar{z}_i \otimes dz_j) \tag{0.2}$$

where

$$g_{i\bar{j}}(z) := \partial^2 \log K_D(z,z)/\partial z_i \partial \bar{z}_j\,.$$

The tensor g defines a kaehlerian structure on every bounded domain $D \subset \mathbb{C}^N$ [2, 12].

The situation described above was the starting point of fruitful investigations exhibiting fine connections between different branches of mathematics: spectral theory of operators in Hilbert space [8, 10], ergodic theory

1991 *Mathematics Subject Classification*: Primary 58A46, 58A32.
Key words and phrases: kaehlerian manifold, kaehlerian potential, positive definite function, Bergman function, reproducing kernel.

[5], group representations [7] and mathematical physics [6, 8, 9]. The link between pseudo-riemannian geometry and Hilbert space methods is very interesting from mathematical-physics point of view. Both these subjects are tools of large parts of physics: general relativity and quantum mechanics. In this context, we try to explain that the notions of kaehlerian manifold and of a Hilbert space with a reproducing kernel are very strongly related. In [9], the problem of when a reproducing kernel in a Hilbert space of functions $f : X \to \mathbb{C}$ generates a kaehlerian structure on X is solved. In the present paper we consider a case when a kaehlerian potential produces a reproducing kernel in some Hilbert space.

1. Kaehlerian manifolds, reproducing kernels and positive definite functions. We recall the basic notions which will be used in this paper.

A complex manifold $\mathbf{M}$ with a tensor g is *kaehlerian* if:

1) g is a riemannian metric tensor on $\mathbf{M}$ as a real manifold,

2) the $\mathbb{C}$-linear extension of g to the complex tangent bundle $\mathbf{TM}$ is invariant w.r.t. to the operator J of complex structure,

3) the exterior form

$$\omega(Z,W) := g(JZ,W) \tag{1.1}$$

is closed i.e. $d\omega = 0$ (Z, W are sections of $\mathbf{TM}$).

A complex manifold $\mathbf{M}$ is kaehlerian if and only if there exists a locally defined complex-valued C^∞-function F on $\mathbf{M}$ such that

$$\omega = \partial\overline{\partial}(F - \overline{F}) \tag{1.2}$$

(for details see [4], pp. 59–60). The function

$$p(z) := -i(F(z) - \overline{F(z)})\,, \qquad z \in \mathbf{M}\,, \tag{1.3}$$

is called the *kaehlerian potential* on $\mathbf{M}$.

Let $\mathbf{X}$ be an arbitrary set. A non-zero function $k : \mathbf{X}\times\mathbf{X} \to \mathbb{C}$ is *positive definite* if for any $t_1, \ldots, t_n \in \mathbf{X}$ and any $c_1, \ldots, c_n \in \mathbb{C}$

(a) $\sum_{i,j=1}^{n} c_i\overline{c}_j k(t_i, t_j) \ge 0$,

(b) $k(t_i, t_j) = \overline{k(t_j, t_i)}$.

Let $(H, (\ ,\ ))$ be a Hilbert space of complex functions defined on $\mathbf{X}$. A function $K : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$ is called a *reproducing kernel* for $(H\,, (\cdot\,,\cdot))$ if:

(a) $K(\cdot\,, y) \in H$ for each $y \in \mathbf{X}$,

(b) $h(y) = (h, \mathbf{K}(\cdot\,, y))$ for each $h \in H$ and $y \in \mathbf{X}$.

1.1. Remark. The Bergman function (0.1) is a reproducing kernel in $L^2H(D)$.

1.2. Remark. Every reproducing kernel is positive definite.

1.3. Theorem (Aronszajn). *A Hilbert space* $(H,(\cdot\,,\cdot))$ *has a reproducing kernel if and only if for any* $y\in\mathbf{X}$ *there exists a constant* $a(y)$ *such that for any* $h\in H$

$$|h(y)|\le a(y)\|h\|\,.$$

2. How does a kaehlerian potential produce a reproducing kernel? The main purpose of this paper is to prove the following

2.1. Theorem. *Let* $\mathbf{M}$ *be a kaehlerian manifold with potential* p *of the form* (1.3). *If the function* F (*and so* p) *is defined globally on* $\mathbf{M}$, *then*

$$K(z,w):=e^{-i(F(z)-\overline{F(w)})},\qquad z,w\in\mathbf{M}\,, \tag{2.1}$$

is a reproducing kernel in some Hilbert space.

Proof. First we will show that K is positive definite; then we use a method due to N. Aronszajn [1] (see also [11]) to construct the desired space. Indeed,

$$\overline{K(z_j,z_i)}=\overline{e^{-i(F(z_j)-\overline{F(z_i)})}}=e^{-i(F(z_i)-\overline{F(z_j)})}=K(z_i,z_j)\,,$$

$$\sum_{i,j=1}^{n}c_i\overline{c}_jK(z_i,z_j)=\sum_{i,j=1}^{n}c_i\overline{c}_je^{-i(F(z_i)-\overline{F(z_j)})}$$

$$=\Big(\sum_{i=1}^{n}c_ie^{-iF(z_i)}\Big)\Big(\sum_{j=1}^{n}\overline{c}_j\overline{e^{-iF(z_j)}}\Big)=\Big|\sum_{i=1}^{n}c_ie^{-iF(z_i)}\Big|^2\ge 0\,.$$

Set

$$H_0:=\Big\{f:\mathbf{M}\to\mathbb{C};\ f(z)=\sum_{i=1}^{n}a_iK(z,t_i)\,,\ a_i\in\mathbb{C},\ z,t_i\in\mathbf{M},\ i=1\ldots n\,,\ n=1,2,\ldots\Big\}\,.$$

If $f(\cdot)=\sum_{i=1}^{n_1}a_iK(\cdot,t_i)$ and $g(\cdot)=\sum_{j=1}^{n_2}b_jK(\cdot,\tau_j)$ set

$$(f,g)_0=\sum_{i,j=1}^{n}a_i\overline{b}_j\overline{K(\tau_j,t_i)}\,,\qquad n=\min(n_1,n_2)\,. \tag{2.2}$$

Clearly, $(\cdot\,,\cdot)_0$ is a scalar product in H_0. Let $(H(K)\,,(\cdot\,,\cdot))$ be the completion of $(H_0\,,(\cdot\,,\cdot)_0)$. Then $(H(K)\,,(\cdot\,,\cdot))$ is a Hilbert space for which K is a reproducing kernel.

3. Examples. 1. Let $\mathbf{M}=D$ be a bounded domain in $\mathbb{C}^N$. In this case the kaehlerian potential (1.2) has the form $p(z)=\log K_D(z,z)$. Then (2.1) is exactly the Bergman function, and moreover $H(K_D) = L^2H(D)$, up to isomorphism.

2. Let $\mathbf{M} = \mathbb{C}^1$. Consider the geometry given by the tensor

$$g(z) = 1dz \otimes d\overline{z} + 1d\overline{z} \otimes dz\,.$$

g describes a kaehlerian geometry on the plane, which is in fact euclidean in the real sense. In this case (see [9]) $K_{\mathbb{C}}(z,w) = e^{z\overline{w}}$ and $H(K_{\mathbb{C}})$ is the well-known Fock space.

References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1956), 337–404.
[2] S. Bergman, *The Kernel Function and Conformal Mapping*, 2nd ed., Math. Surveys 5, Amer. Math. Soc., 1970.
[3] —, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math. 169 (1933), 1–42.
[4] S. Chern, *Complex Manifolds without Potential Theory*, 2nd ed., Springer, 1978.
[5] W. Chojnacki, *On some holomorphic dynamical systems*, Quart. J. Math. Oxford Ser. (2) 39 (1988), 159–172.
[6] S. Janson, J. Peetre and R. Rochberg, *Hankel forms and the Fock space*, Rev. Mat. Iberoamericana 3 (1987), 61–138.
[7] S. Kobayashi, *On the automorphism group of a homogeneous complex manifold*, Proc. Amer. Math. Soc. 12 (3) (1961), 359–361.
[8] T. Mazur, *Canonical isometry on weighted Bergman spaces*, Pacific J. Math. 136 (2) (1989), 303–310.
[9] —, *On the complex manifolds of Bergman type*, preprint.
[10] T. Mazur and M. Skwarczyński, *Spectral properties of holomorphic automorphism with fixed point*, Glasgow Math. J. 28 (1986), 25–30.
[11] W. Mlak, *Introduction to Hilbert Space Theory*, PWN, Warszawa 1991.
[12] M. Skwarczyński, *Biholomorphic invariants related to the Bergman functions*, Dissertationes Math. 173 (1980).

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF RADOM
MALCZEWSKIEGO 29
26-600 RADOM, POLAND

Reçu par la Rédaction le 14.9.1990