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Jung's type theorem for polynomial transformations of $\mathbb{C}^2$

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Abstract. We prove that among counterexamples to the Jacobian Conjecture, if there are any, we can find one of lowest degree, the coordinates of which have the form $x^m y^n$ + terms of degree $< m+n$.

Introduction. In this note we prove the following

Theorem. *Let $\Phi = (f, g) : \mathbb{C}^2(x, y) \to \mathbb{C}^2(z, w)$ be a polynomial mapping of degree $m > 1$ with constant (non-zero) Jacobian and let*

$$f = \sum_{j=0}^{m} f_j, \qquad \deg f_j = j\,,$$

be the expansion of f into homogeneous polynomials. If the set $\{f_m = 0\}$ is a complex line then there exists a polynomial automorphism Ψ such that $\deg \Phi \circ \Psi < m = \deg \Phi$.

To give a context for this result we recall the famous Jacobian Conjecture [5] (see also [2], [7], [8]) saying that any polynomial transformation of $\mathbb{C}^n$ which has constant non-zero Jacobian is an automorphism. The theorem implies that if there exist counterexamples to the conjecture in $\mathbb{C}^2$ then those of the lowest degree among them fail to satisfy our assumption on the set $\{f_m = 0\}$ (it is known [1], [6] that this set contains at most two complex lines).

As long as the Jacobian Conjecture is not proved the present theorem generalizes Jung's theorem [3]:

Any polynomial automorphism of $\mathbb{C}^2$ can be represented by means of a finite superposition of linear and triangular transformations defined by $z = x + cy^m$, $w = y$, where c is a constant and m is a positive integer.

Indeed, polynomial automorphisms satisfy the assumptions of our theorem and Ψ from the statement is in fact a superposition of linear and triangular mappings.

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The proof of the theorem. After making a linear transformation in $\mathbb{C}^2(z,w)$ (resp. $\mathbb{C}^2(x,y)$) we may assume that $\deg g = \deg f = m$ (resp. $f(x,y) = y^m + \sum_{j+k<m} c_{jk}x^j y^k$). Set

$$\frac{\alpha}{\beta} = \max\left\{\frac{j}{m-k} : c_{jk} \neq 0\right\},$$

α, β coprime positive integers. It is clear that $0 < \alpha/\beta < 1$. Write f in yet another form:

$$f = \widehat{f} + f_1, \quad \text{where } \widehat{f}(x,y) = y^m W(x^\alpha/y^\beta),$$

W a polynomial of one variable x^α/y^β, $W(0) \neq 0$, and

$$f_1 = \sum_{j/(m-k)<\alpha/\beta} c_{jk}x^j y^k.$$

The Jacobian condition implies that if $\widehat{g}$ is defined in the same way as $\widehat{f}$ then

$$\widehat{g} = \text{const.}\widehat{f}. \tag{1}$$

Indeed, the Jacobian of $(\widehat{f}, \widehat{g})$ must be zero since $\widehat{f}$ (resp. $\widehat{g}$) is the sum of those monomials in the Taylor expansion of $\widehat{f}$ (resp. $\widehat{g}$) where $j/(m-k)$ is maximal, with j being the power of x, and k the power of y. If $\widehat{g} = y^m V(x^\alpha/y^\beta)$ then

$$\text{Jac}(\widehat{f}, \widehat{g}) = \alpha m x^{\alpha-1} y^{2m-\beta-1}(VW' - WV')(x^\alpha/y^\beta).$$

Thus $V = \text{const.}W$.

For any non-zero polynomial $P(z,w) \in \mathcal{P}(\mathbb{C}^2(z,w))$ (the set of all polynomials in (z,w)) we adopt the following notation:

$$\widetilde{P}(x,y) := P \circ (f,g)(x,y), \quad \widetilde{P} = \widehat{P} + P_1, \quad \text{where}$$

$$\widehat{P}(x,y) = x^N y^M \varPhi_P(x^\alpha/y^\beta), \quad \varPhi_P \text{ a polynomial, } \varPhi_P(0) \neq 0,$$

$$P_1(x,y) = \sum_{(j-N)/(M-k)<\alpha/\beta} d_{jk}x^j y^k.$$

(To get $\widehat{P}$ we sum up those monomials $d_{jk}x^j y^k$ in the Taylor expansion of $\widetilde{P}$ for which $j + (\alpha/\beta)k$ is maximal. The monomial $x^N y^M$ is their greatest common divisor, which is guaranteed by the condition $\varPhi_P(0) \neq 0$.)

We now define a subfamily $\mathcal{A}$ of $\mathcal{P}(\mathbb{C}^2(z,w))$ by

$$\mathcal{A} = \{P \in \mathcal{P}(\mathbb{C}^2(z,w)) : \widehat{P} = \text{const.}\widehat{f}^\varrho,\ \varrho \text{ a rational number}\}$$

(the coefficient of $y^{m\varrho}$ in $\widehat{f}^\varrho$ is assumed to be 1). First note that the constants do belong to $\mathcal{A}$. Next we exhibit a polynomial P_0 not in $\mathcal{A}$. The image of the line $\{x = 0\}$ under $\varPhi$ is algebraic and hence it is the zero set of some polynomial $P_0(z,w)$. Since x divides $\widehat{P}_0$ but not $\widehat{f}$, P_0 does not belong to $\mathcal{A}$.

These remarks ensure the existence of a non-constant polynomial $Q(z,w)$ of the lowest degree among those from $\mathcal{P}(\mathbb{C}^2(z,w)) - \mathcal{A}$.

A close look at this polynomial and its partial derivative $(\partial/\partial w)Q = Q_w$ (which by definition of Q is a member of $\mathcal{A}$) will help us to prove the theorem. By the chain rule we get

$$\widetilde{Q_w} = (\mathrm{Jac}(f,g))^{-1}\,\mathrm{Jac}(f,\widetilde{Q}) = c_0\,\mathrm{Jac}(f,\widetilde{Q})\,, \tag{2}$$

where $\mathrm{Jac}(\phi,\psi)$ stands for the Jacobian of the mapping (ϕ,ψ). Fix $c_1 \in \mathbb{C}$ and $\varrho \in \mathbb{Q}$ satisfying $\widehat{Q_w} = c_1\widehat{f}^{\varrho}$ and put $V := \Phi_Q$. So

$$\widehat{Q}(x,y) = x^N y^M V(x^\alpha/y^\beta)\,.$$

Considering three possible cases:

$$\text{(a) } N > 1, \quad \text{(b) } N = 1, \quad \text{(c) } N = 0\,,$$

we first check that neither (a) nor (b) can really occur, and then we show how to reduce the degree of Φ for $N = 0$.

(a) Suppose $N > 1$. Let us take into account only those monomials in the expansions of $\widehat{f}, \widehat{Q}$ and $\widehat{Q_w}$ whose y-degree is maximal. These are $y^m W(0)$, $x^N y^M V(0)$ and $c_1 y^{\varrho m} W^{\varrho}(0)$ respectively. Since $\mathrm{Jac}(W(0)y^m, V(0)x^N y^M) \neq 0$ the following equality must hold:

$$\begin{aligned} c_1 y^{\varrho m} W^{\varrho}(0) &= c_0\,\mathrm{Jac}(W(0)y^m, V(0)x^N y^M) \\ &= -W(0)V(0)Nmx^{N-1}y^{M+m-1}\,. \end{aligned}$$

This is not true for $N > 1$.

(b) Suppose $N = 1$. Note that $\widehat{\mathrm{Jac}(f,\widetilde{Q})} = \mathrm{Jac}(\widehat{f},\widehat{Q})$ unless the right hand side is 0. Hence with u standing for x^α/y^β we may write

$$\begin{aligned} &(1/c_0)\widehat{Q_w} = \widehat{\mathrm{Jac}(f,\widetilde{Q})} \\ &= \alpha(y^m/x)uW'(u)(-\beta x y^{M-1}uV'(u) + Mxy^{M-1}V(u)) \\ &\quad - (-\beta y^{m-1}uW'(u) + my^{m-1}W(u))(y^M V(u) + \alpha y^M uV'(u)) \\ &= y^{M+m-1}(-mW(u)V(u) + (\alpha M + \beta)uW'(u)V(u) - \alpha m u W(u)V'(u))\,. \end{aligned}$$

Hence

$$-mW(u)V(u) + (\alpha M+\beta)uW'(u)V(u) - \alpha m u W(u)V'(u) = c_2 W^{\varrho}(u)\,, \tag{3}$$

with $c_2 = c_1/c_0$ and $\varrho = (M-1)/m + 1$.

Set $A := \deg W$ and $B := \deg V$. Since $\widehat{f} = y^m W(x^\alpha/y^\beta)$ is a polynomial, obviously

$$m \geq \beta A\,. \tag{4}$$

The degree of the polynomial on the left of (3) does not exceed $A+B$. It is less than $A+B$ iff $m - A(\alpha M+\beta) + \alpha m B = 0$. So either

(i) $A+B=A\varrho=A((M-1)/m+1)$, or
(ii) $m-A(\alpha M+\beta)+\alpha mB=0$.

It is easy to check that in both cases

$$B/A\le M/m\,. \tag{5}$$

Indeed, if (i) is true then $B/A=M-1/m<M/m$, and (ii) may be rewritten in the form

$$B/A=(\alpha M+\beta)/m\alpha-1/A\alpha=M/m+1/\alpha(\beta/m-1/A)\,,$$

where the term in brackets does not exceed zero (see (4)). From (5) we deduce that there exists $a\in\mathbb{C}$ such that $W(a)=0$ and if W (resp. V) has zero of multiplicity μ (resp. ν) at this point then

$$\nu/\mu\le B/A\le M/m\,.$$

We shall prove that these inequalities lead to a contradiction. Write

$$W(u)=(u-a)^{\mu}W_1(u),\qquad V(u)=(u-a)^{\nu}V_1(u),\qquad u-a=\lambda\,.$$

With this notation (3) takes the form

$$\begin{aligned}m\lambda^{\mu+\nu}W_1(u)V_1(u)-(\alpha M+\beta)uV_1(u)(\mu\lambda^{\mu+\nu-1}W_1(u)+\lambda^{\mu+\nu}W_1'(u))\\+\alpha muW_1(u)(\nu\lambda^{\mu+\nu-1}V_1(u)+\lambda^{\mu+\nu}W_1'(u))\\=c_2\lambda^{\mu((M-1)/m+1)}(W_1(u))^{(M-1)/m+1}\,.\end{aligned}$$

Hence either $\nu\alpha m-(\alpha M+\beta)\mu=0$ or the polynomial on the left hand side has zero of multiplicity $\mu+\nu-1$ at a and thus $\mu+\nu-1=\mu((M-1)/m+1)$. In the former case we have

$$\nu/\mu=(\alpha M+\beta)/\alpha m=M/m+\beta/\alpha m>M/m\,,$$

in the latter

$$\nu/\mu=(M-1)/m+1/\mu=M/m+(1/\mu-1/m)>M/m$$

(since $\mu\le A\le m/\beta<m$).

The above inequalities contradict the choice of a. Thus we have proved that

$$\widehat{Q}(x,y)=y^mV(x^{\alpha}/y^{\beta})\,.$$

Then

$$\widehat{Q_w}(x,y)=c_0\alpha x^{\alpha-1}y^{M+m-\beta-1}(mW(u)V'(u)-MW'(u)V(u))\,.$$

Since on the other hand $\widehat{Q_w}(x,y)=c_1\widehat{f}^{\varrho}(x,y)=c_1y^{\varrho m}W^{\varrho}(u)$ it follows that $\alpha=1$.

Now we are ready to define a polynomial automorphism Ψ satisfying $\deg\Phi\circ\Psi<\deg\Phi$. Take $a\in\mathbb{C}$ such that $W(a)=0$ and set

$$\phi(x,y)=x-ay^{\beta},\qquad \psi(x,y)=y\,.$$

Then $(\phi,\psi):\mathbb{C}^2(x,y)\to\mathbb{C}^2(s,t)$ is clearly an automorphism. To prove that the degree of $F:=f\circ(\phi,\psi)^{-1}$ is less than that of f we may apply the Jensen formula, or to be more explicit let us define for any $c\in\mathbb{C}$

$$\Psi_c(x):=\prod_{\phi_c(x,y)=0} f(x,y)\,,\qquad \text{where } \phi_c=\phi-c\psi\,.$$

This function is well defined and holomorphic outside the finite set $\{x:(\partial/\partial y)\phi_c(x,y)=0\}$. Since it is locally bounded as well we conclude that Ψ_c has a unique extension to an entire function. Let us estimate the growth of Ψ_c. Take $(x,y)\in\{\phi_c=0\}$; then

$$|x/y^\beta-a|=|cy^{1-\beta}|\le C_1|x|^{(1/\beta)-1}+C_2$$

for some positive constants C_1,C_2. Setting $W(u)=(u-a)W_1(u)$ we have for (x,y) from the zero set of ϕ_c

$$\begin{aligned}|f(x,y)|&\le|y^m|\,|x/y^\beta-a|\,|W_1(x/y^\beta)|+|f_1(x,y)|\\&\le C_3|x/y^\beta-a||x|^{m/\beta}+C_4|x|^{(m-1)/\beta}+C_5\,.\end{aligned}$$

Combine the above two estimates to get

$$|f(x,y)|\le C_6|x|^{(m-1)/\beta}+C_7\qquad\text{whenever } \phi_c(x,y)=0\,.$$

Therefore

$$|\Psi_c(x)|\le C_8|x|^{m-1}+C_9\,.$$

It follows that $\deg\Psi_c\le m-1$. By definition, the degree of Ψ_c is equal to the number of common zeros (counting multiplicities) of f and ϕ_c, which is the same as the degree of $F=f\circ(\phi,\psi)^{-1}$ restricted to the line $\{s=ct\}$. So for every $c\in\mathbb{C},\deg F_{|s=ct}\le m-1$ and consequently $\deg F\le m-1$. Since $\widehat{g}=c\widehat{f}$ the same argument works for $G:=g\circ(\phi,\psi)^{-1}$ and thus we obtain

$$\deg\Phi\circ(\phi,\psi)^{-1}<m=\deg\Phi\,,$$

which completes the proof.

Addendum. After submitting this paper the author has learned that the present result has recently been proved by R. C. Heitmann [3] in a more general setting and by a different method.

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