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Invariant pseudodistances and pseudometrics—completeness and product property

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Abstract. A survey of properties of invariant pseudodistances and pseudometrics is given with special stress put on completeness and product property.

Introduction. Since the survey article "Intrinsic distances, measures and geometric function theory" of S. Kobayashi [31] in 1976 there has been a remarkable progress in studying properties of pseudodistances and infinitesimal pseudometrics which are "distance decreasing" under holomorphic mappings (cf., for instance, the references in [15], [19], [33], [34], [41]). Nevertheless a lot of elementary, but basic problems still remain open. Our aim in this paper is to discuss only some aspects of the whole theory of "invariant" functions, especially "completeness" and "product property".

This survey is organized as follows: the first section, besides the basic definitions, contains several explicit examples—some of them new. In the second chapter we report on completeness. The paper concludes with a discussion of the so-called product property. Each section is completed by a list of open questions. We would like to express our deep gratitude to our Universities and to DFG for valuable help during writing this paper.

I. Definitions and examples

Definition 1.1. A family $(d_G)_{G\in\mathfrak{G}}$ of pseudodistances $d_G : G\times G \to \mathbb{R}_+$ ($\mathfrak{G}$ denotes the system of all domains $G \subset \mathbb{C}^n$, n arbitrary) is called a *Schwarz–Pick system of pseudodistances* [22] if:

(i) whenever $f : G \to D$ ($D, G \in \mathfrak{G}$) is holomorphic then

$$d_D(f(z'), f(z'')) \le d_G(z', z'') \quad (z', z'' \in G),$$

(ii) $d_E = \varrho :=$ the hyperbolic distance on the unit disc $E \subset \mathbb{C}$.

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In the above definition we can restrict $\mathfrak{G}$ to be a subsystem $\mathfrak{G}'$ of $\mathfrak{G}$ with $E \in \mathfrak{G}'$. Sometimes we also will use the notion of Schwarz–Pick system in this general meaning.

It is well known that for any Schwarz–Pick system $(d_G)_{G\in\mathfrak{G}}$ of pseudodistances we have

$$c_G \le d_G \le k_G \quad (G \in \mathfrak{G})$$

where $(c_G)_G$ (resp. $(k_G)_G$) is the Schwarz–Pick system of Carathéodory (resp. Kobayashi) pseudodistances.

Since $k_G : G \times G \to \mathbb{R}_+$ is always continuous, so is $d_G : G \times G \to \mathbb{R}_+$.

Observe that for any $z, w \in G$, $G \in \mathfrak{G}$, there exists a continuous curve $\alpha : [0,1] \to G$ connecting z and w with finite k_G-length $l_{k_G}(\alpha)$.

DEFINITION 1.2. Let $(d_G)_{G\in\mathfrak{G}}$ be a Schwarz–Pick system of pseudodistances. Put

$$d^i_G(z,w) := \inf\{l_{d_G}(\alpha) : \ \alpha \text{ a continuous curve in } G \text{ connecting } z \text{ and } w\}.$$

Remark. $(d^i_G)_{G\in\mathfrak{G}}$ is again a Schwarz–Pick system of pseudodistances with $d_G \le d^i_G$. We call d^i_G the *associated inner pseudodistance*.

By [43] we know that if d_G is a distance (i.e. G is d_G*-hyperbolic*) and $d_G = d^i_G$ (i.e. d_G is *inner*) then the d_G-topology coincides with the $\| \ \|$-topology. In particular, k_G is inner; hence, if G is k_G-hyperbolic, the k_G-topology is the $\| \ \|$-topology [4].

On the other hand, c_G, in general, is not inner (cf. [5], [27], [53]; see also Examples 1.21 and 1.23, 6), and therefore the c_G-topology must be studied by different methods.

It is well known that, if G is biholomorphically equivalent to a bounded domain, the d_G-topology equals the $\| \ \|$-topology for any Schwarz–Pick system $(d_G)_{G\in\mathfrak{G}}$.

In general, the question whether the c_G-topology coincides with the initial topology seems to be open (cf. Problem 1.1) ([1]).

We mention that in the case of complex spaces the answer is negative [54]. On the other hand, for domains in $\mathbb{C}^1$ the answer is affirmative:

PROPOSITION 1.3. *For any $G \subset \mathbb{C}^1$ c_G-hyperbolic, the c_G-topology coincides with its standard topology.*

Proof (J. Wiegerinck). Fix $a \in G$ and let $G \ni z^\nu \to a$ in the c_G-topology. Let $f \in H^\infty(G)$ with $f(a) = 0$ and $f \not\equiv 0$. Write $f = (z-a)^k g$, $g(a) \ne 0$. Then $g(z^\nu) - g(a) \to 0$ and hence $z^\nu \to a$. ∎

Remark ([46]). For a domain $G \subset \mathbb{C}^1$ the following properties are equivalent:

([1]) Cf. the addendum.

(i) G is c_G-hyperbolic,
(ii) $H^\infty(G) \neq \mathbb{C}$,
(iii) the analytic capacity of $\mathbb{C} \setminus G$ is positive.

DEFINITION 1.4. A family $(F_G)_{G\in\mathfrak{G}}$ of functions $F_G : G \times G \to [0,1)$ is called a *Schwarz–Pick system of functions* if

(i) whenever $f : G \to D$ is holomorphic then

$$F_D(f(z'), f(z'')) \leq F_G(z', z'') \quad (z', z'' \in G),$$

(ii) $F_E = \tanh \varrho =:$ the *Möbius distance* in E.

Observe that if $(d_G)_{G\in\mathfrak{G}}$ is a Schwarz–Pick system in the sense of Definition 1.1 then the family $(\tanh d_G)_{G\in\mathfrak{G}}$ is a Schwarz–Pick system in the sense of Definition 1.4. In particular, the *Möbius pseudodistances* $c_G^* := \tanh c_G$, $G \in \mathfrak{G}$, form a Schwarz–Pick system of functions.

Let $k_G^*(z', z'') := \inf\{t \in [0,1) : \exists \varphi \in \mathcal{O}(E, G) : \varphi(0) = z', \varphi(t) = z''\}$ $(z', z'' \in G)$. It is clear that $(k_G^*)_{G\in\mathfrak{G}}$ gives a Schwarz–Pick system of functions, and for any Schwarz–Pick system of functions $(F_G)_{G\in\mathfrak{G}}$ one has

$$c_G^* \leq F_G \leq k_G^*, \quad G \in \mathfrak{G}.$$

Recall that k_G is the largest pseudodistance below $\tanh^{-1} k_G^*$, $G \in \mathfrak{G}$.

EXAMPLE 1.5. Let

(a) $m_G^{(p)}(a, z) := \sup\{|f(z)|^{1/p} : f \in \mathcal{O}(G, E),\ \operatorname{ord}_a f \geq p\}$ $(a, z \in G, p \in \mathbb{N})$,

(b) $g_G(a, z) := \sup\{u(z) : u \in \mathcal{K}_G(a)\}$ where

$$\mathcal{K}_G(a) := \{u : G \to [0,1) : u \text{ log-psh. and } u(z) \leq c\|z - a\| \text{ near } a\}.$$

The families $(m_G^{(p)})_{G\in\mathfrak{G}}$, $(g_G)_{G\in\mathfrak{G}}$ are Schwarz–Pick systems of functions. Since $m_G^{(1)} = c_G^*$, we call $m_G^{(p)}$ the *p-th Möbius function* on G. The function $\log g_G(a, \cdot)$ is the *pluri-complex Green function* for G with pole at a (see [29], [14], [15]). Obviously one has

$$c_G^* \leq m_G^{(p)} \leq g_G \leq k_G^*, \quad G \in \mathfrak{G}.$$

Now we collect some of the properties of $m_G^{(p)}$ and g_G:

PROPOSITION 1.6. (a) *$m_G^{(p)}(a, \cdot)$ is a continuous log-psh. function on G.*

(b) *$m_G^{(p)}$ is upper semicontinuous on $G \times G$* (cf. [28]); *$c_G^* = m_G^{(1)}$ is even continuous on $G \times G$ and, moreover, c_G is log-psh. on $G \times G$* (cf. [17], [52]). *If G is biholomorphic to a bounded domain then $m_G^{(p)}$ is continuous on $G \times G$* (cf. [28]).

PROPOSITION 1.7. (a) $g_G(a, \cdot) \in \mathcal{K}_G(a)$, $G \in \mathfrak{G}$, $a \in G$ [29].

(b) g_G *is upper semicontinuous on* $G \times G$ *whenever* G *is a domain of holomorphy* [30]; *moreover*, *if* G *is bounded hyperconvex then* G *is continuous on* $G \times G$ *and* $\lim_{z\to\zeta} g_G(a,z) = 1$, $\zeta \in \partial G$ [14].

(c) *For* $n = 1$, $-\log g_G(a,\cdot)$ *coincides with the classical Green function for* G *with pole at* a.

Example 1.8. Let $G := \{(z,w) \in \mathbb{C}^2 : |z| < 2,\ |w| < 1/2 \text{ or } 1 < |z| < 2,\ |w| < 2\}$. Then

$$g_G(0,(z,w)) = \begin{cases} |w| & \text{if } |z| \le 1,\ |z| \le 2|w|,\\ |z|/2 & \text{if } |z| \le 1,\ |z| \ge 2|w|,\\ |z|/2 & \text{if } 1 < |z| < 2,\ |w| \le |z|^2/2,\\ \sqrt{|w|/2} & \text{if } 1 < |z| < 2,\ |w| \ge |z|^2/2.\end{cases}$$

Hence $g_G(0,\cdot)$ is different from $g_{\widetilde{G}}(0,\cdot)|_G$ where $\widetilde{G} = 2E \times 2E$ is the envelope of holomorphy of G.

The above example and similar ones were obtained during our discussion with R. Zeinstra to whom we express our thanks. This example shows that the idea to obtain (b) in Proposition 1.7 for arbitrary domains in $\mathbb{C}^n$ by passing to the envelope of holomorphy fails (cf. Problem 1.2).

Example 1.9 ([28]). Let $G := \{z \in \mathbb{C}^n : |z^\alpha| < 1\}$, where $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$ and $\alpha_1,\ldots,\alpha_n$ are relatively prime, $n \ge 2$. Then:

(a) $m_G^{(p)}(a,z) = [c_E^*(a^\alpha, z^\alpha)]^{(1/p)E_+(p/r)}$ $\quad (a,z \in G, p \in \mathbb{N})$

where $r = r(a) := \operatorname{ord}_a(z \to z^\alpha)$, $E_+(t) :=$ the smallest $\nu \in \mathbb{N}$ with $\nu \ge t$. In particular, for $p \ge 2$, $m_G^{(p)}$ is neither continuous nor symmetric.

(b) $g_G(a,z) = [c_E^*(a^\alpha, z^\alpha)]^{1/r}$.

Again g_G is not continuous and not symmetric.

Definition 1.10. A family $(\delta_G)_{G\in\mathfrak{G}}$ of functions $\delta_G : G \times \mathbb{C}^n \to \mathbb{R}_+$ $(G \subset \mathbb{C}^n)$ is called a *Schwarz–Pick system of infinitesimal pseudometrics* if

$$\delta_G(z;\lambda X) = |\lambda|\delta_G(z;X) \quad (z \in G, X \in \mathbb{C}^n, \lambda \in \mathbb{C}) \quad \text{and}$$

(i) whenever $f : G \to D$ is holomorphic then

$$\delta_D(f(z); f'(z)X) \le \delta_G(z;X), \quad z \in G \subset \mathbb{C}^n \ni X\,;$$

(ii) $\delta_E(0;1) = 1$.

Example 1.11.

(a) $\gamma_G^{(p)}(z;X) := \lim_{\lambda\to 0} \frac{1}{|\lambda|} m_G^{(p)}(z, z+\lambda X)\,, \quad z \in G \subset \mathbb{C}^n \ni X$ [28];

(b) $S_G(z;X) := \sup\left\{\limsup_{\lambda\to 0}\frac{1}{|\lambda|}\sqrt{u(z,z+\lambda X)} : u\in\mathcal{S}_G(z)\right\}$,
$z\in G\subset\mathbb{C}^n\ni X$,

where $\mathcal{S}_G(a) := \{u : G\to[0,1) : u$ log-psh., $u(a)=0$ and u of class C^2 near $a\}$ [47], [50]; note that $\sqrt{\mathcal{S}_G(a)}\subset\mathcal{K}_G(a)$;

(c) $A_G(z;X) := \limsup_{\lambda\to 0}\frac{1}{|\lambda|}g_G(z,z+\lambda X)$, $z\in G\subset\mathbb{C}^n\ni X$
[1], [2], [30];

(d) $\kappa_G(z;X) := \inf\{|\alpha| : \alpha\in\mathbb{C}, \exists\varphi\in\mathcal{O}(E,G) : \varphi(0)=z, \alpha\varphi'(0)=X\}$,
$z\in G\subset\mathbb{C}^n\ni X$[45].

The families $(\gamma_G^{(p)})_{G\in\mathfrak{G}}$, $(S_G)_{G\in\mathfrak{G}}$, $(A_G)_{G\in\mathfrak{G}}$, $(\kappa_G)_{G\in\mathfrak{G}}$ are Schwarz–Pick systems of infinitesimal pseudometrics. $\gamma_G := \gamma_G^{(1)}$ is called the *Carathéodory–Reiffen pseudometric* [42]; $\gamma_G^{(p)}$ is the *p-th Reiffen pseudometric* and S_G, A_G, κ_G are known as the *Sibony, Azukawa* and *Kobayashi–Royden pseudometric*, respectively.

Observe that

$$\gamma_G\le S_G\le A_G\le\kappa_G\,,$$
$$\gamma_G\le\gamma_G^{(p)}\le A_G\le\kappa_G\,,$$
$$\gamma_G\le\delta_G\le\kappa_G$$

for any Schwarz–Pick system $(\delta_G)_G$.

In the case of convex domains all invariant objects coincide.

THEOREM 1.12 ([35], [36]). *Let $G\subset\mathbb{C}^n$ be a domain biholomorphically equivalent to a convex domain. Then the following equalities hold*:

(i) $c_G = k_G = \tanh^{-1} k_G^*$;
(ii) $c_G^* = m_G^{(p)} = g_G = k_G^*$;
(iii) $\gamma_G = \gamma_G^{(p)} = S_G = A_G = \kappa_G$.

Remark. By [37] the above results are also true if G is strictly linearly convex (cf. Problem 1.4).

Remark. For strongly pseudoconvex domains in $\mathbb{C}^n$ and for bounded smooth pseudoconvex domains of finite type in $\mathbb{C}^2$ there are a lot of comparison results for some of the above invariant objects (for example see [12], [51] and references there).

We summarize some of the properties of the pseudometrics introduced in Example 1.11:

PROPOSITION 1.13. (a) $\gamma_G^{(p)}(a;X) = \sup\{|\sum_{|\alpha|=p}(1/\alpha!)D^\alpha f(z)X^\alpha|^{1/p} : f\in\mathcal{O}(G,E), \operatorname{ord}_a f\ge p\}$ [28];

(b) $\gamma_G^{(p)}$ *is upper semicontinuous on* $G \times \mathbb{C}^n$ [28], γ_G *is even locally Lipschitz on* $G \times \mathbb{C}^n$ *and* $\gamma_G(a;\cdot)$ *is a seminorm* [42];

(c) *if* G *is biholomorphic to a bounded domain then* $\gamma_G^{(p)}$ *is continuous on* $G \times \mathbb{C}^n$ [28].

Proposition 1.14. (a) $S_G(a;X) = \sup\{[\sum_{i,j=1}^n \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(a) X_i \overline{X}_j]^{1/2} : u \in \mathcal{S}_G(a)\}$;

(b) $S_G(a;\cdot)$ *is a seminorm*;

(c) A_G *is upper semicontinuous on* $G \times \mathbb{C}^n$ *whenever* G *is a domain of holomorphy* [30] (cf. Problem 1.2);

(d) *if* $g_G^2(a,\cdot)$ *is* C^2 *near* a *then* $S_G(a;\cdot) = A_G(a;\cdot)$ [30], *in particular*, *if* $n = 1$ *then* $S_G = A_G$.

Proposition 1.15. (a) κ_G *is upper semicontinuous on* $G \times \mathbb{C}^n$ [45];

(b) κ_G *is continuous on* $G \times \mathbb{C}^n$ *whenever* G *is taut* [45].

Example 1.16 ([28]). Let G be as in Example 1.9. Then:

(a) $A_G(a;X) = [\gamma_E(a^\alpha; \Phi_r(a,X))]^{1/r}$, where $r = r(a)$ (see Example 1.9), $\Phi(z) := z^\alpha$ and $\Phi_r(a,X) := \sum_{|\beta|=r} (1/\beta!) D^\beta \Phi(a) X^\beta$; in particular, A_G is not continuous.

(b)

$$\gamma_G^{(p)}(a;X) = \begin{cases} A_G(a;X) & \text{if } r|p, \\ 0 & \text{otherwise;} \end{cases}$$

and so, for $p \ge 2$, $\gamma_G^{(p)}$ need not be continuous.

(c)

$$S_G(a;X) = \begin{cases} A_G(a;X) & \text{if } \sharp\{j : a_j = 0\} \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that here S_G is upper semicontinuous but not continuous.

Example 1.17. Let $G = G_h = \{z \in \mathbb{C}^n : h(z) < 1\}$ be a balanced domain of holomorphy (h denotes its Minkowski function). Then $A_G(0;\cdot) = \kappa_G(0;\cdot) = h$; in particular, there are G_h's for which $\kappa_{G_h}(0;\cdot)$ is not continuous and not a seminorm.

Example 1.18. Let $\varphi(\xi,\eta) = \sum_{j=1}^\infty \lambda_j \log(|\xi - a_j|^2/j + |\eta|/j)$ ($\xi,\eta \in \mathbb{C}$) where $\{a_j\}_{j=1}^\infty$ is a dense subset of E with $a_j \ne 0$ and $\lambda_j > 0$ are such that: (i) $\varphi(0) > -\infty$, (ii) φ is C^2 on $\mathbb{C} \times \mathbb{C}_*$, (iii) φ is psh. Define

$$G := \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| \exp\varphi(z_2,0) < 1\},$$

$$D := \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1| \exp\varphi(z_2,z_3) < 1\},$$

$$f : G \to D, \quad f(z_1,z_2) := (z_1,z_2,0).$$

By the construction of G we obtain $S_G = 0$ on $(G \cap (\mathbb{C} \times E)) \times \mathbb{C}^2$ and, therefore, $S_G^* = 0$ on $(G \cap (\mathbb{C} \times E)) \times \mathbb{C}^2$ where

$$S_G^*(z;X) := \limsup_{(z',X')\to(z,X)} S_G(z',X')\,.$$

On the other hand, since $z \to |z_1|^2 \exp(2\varphi(z_2,z_3))$ belongs to $\mathcal{S}_D((0,0,t))$ $(t>0)$, we get

$$\limsup_{t\searrow 0} S_D((0,0,t);(1,0,0)) \ge \lim_{t\searrow 0} \exp\varphi(0,t) = \exp\varphi(0,0) > 0\,;$$

consequently,

$$S_D^*(f(0,0); f'(0,0)(1,0)) > S_G^*((0,0);(1,0))\,.$$

This example shows that, in general, S_G is not upper semicontinuous and, even more, that the idea presented in [50] to take $(S_G^*)_{G\in\mathfrak{G}}$ in order to get a Schwarz–Pick system of upper semicontinuous pseudometrics fails (cf. Problem 1.5).

Sometimes it is useful to pass from a Schwarz–Pick system of infinitesimal pseudometrics to a Schwarz–Pick system of pseudodistances:

Let $(\delta_G)_{G\in\mathfrak{G}'}$ be a system of upper semicontinuous infinitesimal pseudometrics. Put

$$\Big(\int \delta_G\Big)(z',z'') = \varDelta_G(z',z'') := \inf\Big\{ \int_0^1 \delta_G(\alpha(t);\dot\alpha(t))\,dt : \alpha : [0,1] \to G\,,\ \alpha \text{ piecewise } C^1, \alpha(0) = z' \text{ and } \alpha(1) = z''\Big\}\,.$$

Then $(\varDelta_G)_{G\in\mathfrak{G}'}$ is a Schwarz–Pick system of pseudodistances. We mention that always $\varDelta_G = \varDelta_G^i$ $(G \in \mathfrak{G}')$.

In the case of the Carathéodory–Reiffen and the Kobayashi–Royden metrics even the following more precise results are true:

THEOREM 1.19. *Let G be a domain in $\mathbb{C}^n$. Then*:

(a) $k_G = \int \kappa_G$ [45];

(b) $c_G^i = \int \gamma_G$ *whenever any c_G-rectifiable continuous curve $\alpha : [0,1] \to G$ is $\|\ \|$-rectifiable*, *in particular*, *whenever G is biholomorphic to a bounded domain.*

In [38] the statement (b) is proved for the Bergman metric. We only mention that this proof extends to the above case. Notice that (b) without any additional assumptions is formulated in [31], Theorem 2.6(2) (cf. Problem 1.6).

The first examples of domains G with $c_G \ne c_G^i$ were given by Th. Barth

[5] and later by J.-P. Vigué [53] who obtained even a bounded complete Reinhardt domain of holomorphy with this property. From the latter paper we can extract the following useful lemma.

LEMMA 1.20. *Let G be a domain in $\mathbb{C}^n$, let $z', z'' \in G$, $z' \neq z''$, and let $f \in \mathcal{O}(G, E)$ be such that*:

(i) $f(z') = 0$ *and* $c^*_G(z', z'') = |f(z'')|$,

(ii) $\gamma_G(z'; X) > |f'(z')X|$ *for any* $X \in (\mathbb{C}^n)_*$.

Then $c_G(z', z'') < c^i_G(z', z'')$.

EXAMPLE 1.21 ([23]). Let $G := \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, 2|z_1 z_2| < 1\}$. Then there exists an open set $V \supset \partial G \cap (E \times E)$ such that for all $z \in V \cap G$ we have $c^*_G(0, z) = |2z_1 z_2|$ and, therefore, by the above lemma, $c_G(0, z) < c^i_G(0, z)$ (cf. Problem 1.7).

We like to point out that, so far, there are no sufficient criteria for $c_G = c^i_G$ (cf. Problem 1.8).

Now we would like to present the full description of all the invariant objects in the case where $G = P := \{\lambda \in \mathbb{C} : 1/R < |\lambda| < R\}$ $(R > 1)$. In order to establish the formulas we need the following lemma of R. M. Robinson [44]:

LEMMA 1.22. *Let $z^0 \in (-R, -1/R)$ and let $h : P \setminus \{z^0\} \to \mathbb{C}$ be a holomorphic function with a simple pole at z^0. If $\limsup_{z \to \partial P} |h(z)| \le 1$, then, for any $x \in (1/R, R)$, we have $|h(x)| \le 1$, and here equality holds at one point $\Leftrightarrow |h| \equiv 1$.*

For $1/R < a < R$ we define

$$f(a, \lambda) := \left(1 - \frac{\lambda}{a}\right) \Pi_R(a, \lambda) \quad \text{where}$$

$$\Pi_R(a, \lambda) := \frac{\prod_{j=1}^{\infty}\left(1 - \frac{a}{\lambda R^{4j}}\right)\left(1 - \frac{\lambda}{a R^{4j}}\right)}{\prod_{j=1}^{\infty}\left(1 - \frac{\lambda a}{R^{4j-2}}\right)\left(1 - \frac{1}{\lambda a R^{4j-2}}\right)} \quad \text{(cf. [13], 335–336).}$$

We only mention that $f(a, \cdot)$ is meromorphic on $\mathbb{C}_*$, holomorphic on $\overline{P}$ and the only zero of $f(a, \cdot)$ in $\overline{P}$ is $\lambda = a$; moreover,

$$|f(a, \lambda)| = \begin{cases} 1 & \text{if } |\lambda| = 1/R, \\ R/a & \text{if } |\lambda| = R. \end{cases}$$

EXAMPLE 1.23.

(a) $c_P^*(a,\lambda) = \dfrac{1}{R|\lambda|}|f(a,\lambda)|f\left(\dfrac{1}{a}, -|\lambda|\right)$;

$$\gamma_P(a;1) = \frac{1}{Ra^2}\Pi_R(a,a)f\left(\frac{1}{a}, -a\right) \quad [8], [49].$$

(b) $g_P(a,\lambda) = \left(\dfrac{1}{R|\lambda|}\right)^{s(a)} |f(a,\lambda)| \quad \text{where} \quad s(a) = \dfrac{1}{2}\left(1 - \dfrac{\log a}{\log R}\right)$;

$$A_P(a;1) = S_P(a,1) = \frac{1}{a}\left(\frac{1}{Ra}\right)^{s(a)} \Pi_R(a,a) \quad [2], [15].$$

(c) $m_P^{(k)}(a,\lambda) = |f(a,\lambda)|\left[\left(\dfrac{1}{R|\lambda|}\right)^{l_k(a)} f(b_k(a), -|\lambda|)\right]^{1/k}$

where $l_k(a) := E_+(ks(a))$, $b_k(a) := R^{1-2(l_k(a)-ks(a))}$ and $f(R,\cdot) :\equiv 1$;

$$\gamma_P^{(k)}(a;1) = \frac{1}{a}\Pi_R(a,a)\left[\left(\frac{1}{Ra}\right)^{l_k(a)} f(b_k(a), -a)\right]^{1/k}.$$

(d) $\tanh k_P(a,\lambda) = k_P^*(a,\lambda) = \left[\dfrac{x^2+1-2x\cos(\pi(s-t))}{x^2+1-2x\cos(\pi(s+t))}\right]^{1/2}$

where $a = R^{1-2s}$ (i.e. $s = s(a)$ in (b)), $\lambda = e^{i\varphi}R^{1-2t}$ with $-\pi < \varphi \le \pi$ and $x := \exp(\pi\varphi/(2\log R))$;

$$\kappa_P(a;1) = \frac{\pi}{4a\log R\sin(\pi s)} \quad [2].$$

In order to prove (c) observe that, by Lemma 1.22, the function

$$h(\zeta) := [f(a,\zeta)]^k \left(\frac{1}{R\zeta}\right)^{l_k(a)} f(b_k(a), -e^{-i\varphi}\zeta)$$

is an extremal function for $m_P^{(k)}(a, |\lambda|e^{i\varphi})$.

The proof of (d) only uses the explicit form for the universal covering of P.

The above formulas imply the following remarks:

1) $m_P^{(k)} \to g_P$ and $\gamma_P^{(k)} \to A_P$ as $k \to \infty$.

2) For fixed $k \in \mathbb{N}$ and a the following conditions are equivalent:

(i) $\exists \lambda_0 \in P \setminus \{a\} : m_P^{(k)}(a,\lambda_0) = g_P(a,\lambda_0)$,
(ii) $m_P^{(k)}(a,\cdot) = g_P(a,\cdot)$,
(iii) $\gamma_P^{(k)}(a;1) = A_P(a;1)$,
(iv) $k \ge 2$ and $ks(a) \in \mathbb{N}$.

3) $c_P^*(a,\cdot) < m_P^{(k)}(a,\cdot)$ in $P \setminus \{a\}$ and $\gamma_P(a;1) < \gamma_P^{(k)}(a;1)$ if $k \ge 2$ (use Lemma 1.22).

4) For $k, k' \ge 2$, $k \ne k'$, the following statements are equivalent:

(i) $m_P^{(k)}(a,\cdot) = m_P^{(k')}(a,\cdot)$,
(ii) $m_P^{(k)}(a,\cdot) = m_P^{(k')}(a,\cdot) = g_P(a,\cdot)$,
(iii) $ks(a)$, $k's(a) \in \mathbb{N}$.

5) For $k, k' \ge 2$, $k \ne k'$:

(i) for $\lambda_0 \in P$ there exists $\lambda \in P \setminus \{\lambda_0\}$ with
$$m_P^{(k)}(\lambda,\lambda_0) = m_P^{(k')}(\lambda,\lambda_0)\,;$$
(ii) there exists a with $\gamma_P^{(k)}(a;1) = \gamma_P^{(k')}(a;1)$.

6) For a and any $\lambda = |\lambda|e^{i\varphi}$, $0 < \varphi < 2\pi$,
$$c_P(a,\lambda) < c_P^i(a,\lambda)$$
(use Lemmas 1.20 and 1.22; cf. Problem 1.9 (²)).

Besides Example 1.9 the higher order Möbius functions are known in the following case.

Example 1.24 ([23]). Let G be a complete Reinhardt domain in $\mathbb{C}^n$ with $(|z_1|^t, \ldots, |z_n|^t) \in G$ whenever $(z_1, \ldots, z_n) \in G$ and $t > 0$. Put $T(G) := \{\alpha \in (\mathbb{Z}_+^n)_* : z^\alpha \in H^\infty(G)\}$. Then we have
$$m_G^{(k)}(0,z) = \max\{|z^\alpha| : \alpha \in T(G), |\alpha| \ge k\}\,.$$

For more concrete examples of this type compare [23], [3].

Problems. 1.1. Decide whether for any domain $G \in \mathfrak{G}$ which is c_G-hyperbolic, the c_G-topology coincides with the $\|\ \|$-topology of G (²).

1.2. Is g_G upper semicontinuous for arbitrary $G \in \mathfrak{G}$?

1.3. In Example 1.8 calculate $g_G(a,\cdot)$ for all $a \in G$. Describe g_G for $G := \{z \in \mathbb{C}^n : 1 < \|z\| < 2\}$ $(n \ge 2)$.

1.4. Let $G_0 := \{z \in \mathbb{C}^n : \|z\|^2 + (\operatorname{Re} z_1^2)^2 < 1\}$; observe G_0 is strictly linearly convex but not convex. According to an information by M. Passare this example is due to V. A. Stepanenko. Is G_0 biholomorphic to a convex domain? If yes, give an example of a domain G, not biholomorphic to a convex domain, with $c_G = k_G$.

1.5. Under what conditions is S_G upper semicontinuous?

1.6. Is $c_G^i = \int \gamma_G$ for any $G \in \mathfrak{G}$?

1.7. Let G be as in 1.21. Calculate $c_G^*(0,\cdot)$ on G.

1.8. Find criteria under which $c_G = c_G^i$ holds.

(²) Cf. the addendum.

1.9. Calculate c^i_P.

II. Completeness. First we consider the general situation of a pair (G, d) where $G \subset \mathbb{C}^n$ is an arbitrary domain and where $d : G \times G \to \mathbb{R}_+$ is a continuous distance on G (i.e. G is d-hyperbolic)—for example $d = c_G$ or $d = k_G$.

Definition 2.1. (a) G is called *d-Cauchy complete* if (G, d) is a complete metric space (in the sense of functional analysis);

(b) G is said to be *d-complete* if, for any d-Cauchy sequence $\{z^\nu\} \subset G$, there exists a point $z^0 \in G$ with $z^\nu \to z^0$ in the $\| \|$-topology;

(c) G is called *d-finitely compact* if, whenever $z^0 \in G$ and $R > 0$, the d-ball $B_d(z^0, R) := \{z \in G : d(z, z^0) < R\}$ is relatively compact in G w.r.t. the $\| \|$-topology.

Observe that the condition in (c) implies that the d-topology of G coincides with the $\| \|$-topology of G and that G is d-complete and d-Cauchy complete.

Moreover, there is the following general result due to Hopf–Rinow [43] (see also [10]).

Theorem 2.2. *Let $d : G \times G \to \mathbb{R}_+$ be a continuous inner distance on the domain $G \subset \mathbb{C}^n$. Then the following properties are equivalent*:

(i) *G is d-Cauchy complete*;
(ii) *G is d-complete*;
(iii) *G is d-finitely compact*;
(iv) *any half-segment $\alpha : [0, b) \to G$ (i.e. α is a continuous curve with $d(\alpha(t'), \alpha(t'')) = t'' - t'$ whenever $0 \le t' < t'' < b$) has a continuous extension $\overline{\alpha} : [0, b] \to G$.*

Since k_G is inner we obtain

Corollary 2.3. *Let $G \subset \mathbb{C}^n$ be k_G-hyperbolic. Then all notions of Definition* 2.1 *w.r.t.* (G, k_G) *coincide.*

Therefore, in the sequel, we will only use the term k_G-complete or Kobayashi complete. On the other hand, c_G is not always inner. Nevertheless there is the following equivalence statement due to N. Sibony [46].

Theorem 2.4. *Let G be a c_G-hyperbolic domain in the complex plane. Then the following properties are equivalent*:

(i) *G is c_G-Cauchy complete*;
(ii) *G is c_G-finitely compact.*

Observe it is still an open problem whether this result extends to higher dimensions (cf. Problem 2.1).

To understand the notion of c_G-finite compactness better from the point of view of complex analysis we quote the following reformulation [40].

PROPOSITION 2.5. *For a c_G-hyperbolic domain $G \subset \mathbb{C}^n$ the following properties are equivalent*:

(i) *G is c_G-finitely compact*;

(ii) *for any $z^0 \in G$ and for any sequence $\{z^\nu\} \subset G$ without accumulation points in G, there exists $f \in \mathcal{O}(G, E)$ with $f(z^0) = 0$ and* $\sup |f(z^\nu)| = 1$.

Remarks. (a) Hence any c_G-finitely compact domain $G \subset \mathbb{C}^n$ is $H^\infty(G)$-convex and an $H^\infty(G)$-domain of holomorphy.

(b) If G is c_G-complete then G is an $H^\infty(G)$-domain of holomorphy (cf. Problem 2.2).

(c) Observe [11], [21] that any bounded smooth pseudoconvex domain $G \subset \mathbb{C}^n$ is even $A^\infty(G)$-convex and an $A^\infty(G)$-domain of holomorphy (cf. Problem 2.3).

(d) There is a pseudoconvex taut domain smooth except at one point which is not k_G-complete and therefore not c_G-finitely compact. This example is due to N. Sibony (personal communication) (cf. (c) and Problem 2.3).

Using the existence of peak functions [7], [18] and Proposition 2.5 we obtain the following examples of c_G-finitely compact domains:

(i) bounded convex or bounded strongly pseudoconvex domains in $\mathbb{C}^n$,

(ii) bounded smooth pseudoconvex domains in $\mathbb{C}^2$ of finite type.

Moreover, we have

THEOREM 2.6 ([40]). *Any bounded Reinhardt domain $G \subset \mathbb{C}^n$ of holomorphy, with $0 \in G$, is c_G-finitely compact.*

Observe that the assumption $0 \in G$ is important; for example the Hartogs triangle $G = \{(z, w) \in \mathbb{C}^2 : |z| < |w| < 1\}$ is not c_G-complete.

In [46] there is an example of a domain $G \subsetneq E \times E$ for which any bounded holomorphic function f extends holomorphically to the bidisc. Hence G is not c_G-Cauchy complete. But its construction implies that G is locally c_G-finitely compact.

On the other hand, there is the following result for the Kobayashi completeness.

THEOREM 2.7 ([16]). *Let G be a bounded domain in $\mathbb{C}^n$ and assume that for any $z^0 \in \partial G$ there exists a neighborhood $U = U(z^0)$ such that $U \cap G$ is Kobayashi complete. Then G is k_G-complete.*

Now we are going to discuss the class of balanced domains.

Let $G = G_h$ be a balanced domain in $\mathbb{C}^n$ (cf. Example 1.17). We recall the following properties of h which reflect the properties of $G = G_h$ (cf. [6], [32], [48]):

(i) $G = G_h$ is pseudoconvex $\Leftrightarrow$ h is log-psh.;
(ii) $G = G_h$ is taut $\Leftrightarrow$ h is log-psh. and continuous;
(iii) $G = G_h$ is a $H^\infty(G)$-domain of holomorphy $\Leftrightarrow$ h is log-psh. and $\{z \in \mathbb{C}^n : h \text{ is not continuous at } z\}$ is pluripolar;
(iv) if $G = G_h$ is k_G-complete then G is bounded and taut.

Observe that any bounded Reinhardt domain G of holomorphy, with $0 \in G$, is a taut balanced domain. But, in contrast to Theorem 2.6, the following result is true.

THEOREM 2.8 ([26]). *For $n \ge 3$, there exists a bounded balanced pseudoconvex domain $G = G_h$ with continuous Minkowski function h which is not k_G-complete.*

In dimension $n = 2$ it is still unclear whether such an example can exist (cf. Problems 2.4 and 2.5).

We conclude Section 2 with some results on completeness w.r.t. the Bergman distance. During this discussion we always assume that G is a bounded domain in $\mathbb{C}^n$.

The *Bergman kernel* function of G will be denoted by $K_G : G \times G \to \mathbb{C}$, the *Bergman metric* by

$$\beta_G(z;X) := \left[\sum_{\nu,\mu=1}^{n} \frac{\partial^2 \log K_G(z,z)}{\partial z_\nu \partial \overline{z}_\mu} X_\nu \overline{X}_\mu \right]^{1/2}$$

and its integrated distance—the *Bergman distance*—by $b_G : G \times G \to \mathbb{R}_+$.

Observe that (β_G) and (b_G) are, in general, not distance decreasing under holomorphic mappings but they are invariant under biholomorphic mappings. In addition, $b_G = b_G^i$, hence all completeness notions of Definition 2.1 coincide.

Remark. $c_G \le b_G$ [10], [20] and therefore any bounded c_G-complete domain is b_G-complete.

The class of b_G-complete domains is fairly large as the following two results show.

THEOREM 2.9 ([39]). *Any bounded pseudoconvex domain $G \subset \mathbb{C}^n$ with C^1-boundary is b_G-complete.*

THEOREM 2.10 ([25]). *Any bounded balanced domain of holomorphy with continuous Minkowski function is b_G-complete.*

Remark. To prove b_G-completeness the following two properties have to be verified: (i) $H^\infty(G)$ is (locally) dense in $L^2_h(G)$, and (ii) $K_G(z,z) \to \infty$ whenever $z \to \zeta \in \partial G$ (cf. Problem 2.6).

Comparing Theorems 2.8 and 2.10 we obtain

Corollary 2.11. *For any $n \ge 3$ there exists a bounded balanced domain of holomorphy $G \subset \mathbb{C}^n$ for which there is no estimate $b_G \le Ck_G$ $(C > 0)$.*

Note it is not known whether such an estimate is true in the two-dimensional case (cf. Problem 2.7).

Problems. 2.1. Does Theorem 2.4 remain true if G is an arbitrary c_G-hyperbolic domain in $\mathbb{C}^n$ $(n > 1)$?

2.2. Prove that any c_G-complete domain $G \subset \mathbb{C}^n$ is $H^\infty(G)$-convex.

2.3. Does there exist a bounded smooth pseudoconvex domain G which is not c_G-finitely compact or, even more, which is not k_G-complete?

2.4. Does Theorem 2.8 still hold in dimension $n = 2$?

2.5. Describe "completeness" of $G = G_h$ using the properties of the Minkowski function h.

2.6. Is there a bounded pseudoconvex domain $G \subset \mathbb{C}^n$, with $\operatorname{int}(\overline{G}) = G$, for which $\lim_{z\to\partial G} K_G(z,z) \ne \infty$?

2.7. Describe sufficient criteria in data of h which imply $b_G \le Ck_G$, $G = G_h$.

III. Product property

Definition 3.1 ([28]). Let $F = (F_G)_{G\in\mathfrak{G}}$ be a Schwarz–Pick system of functions or pseudodistances (cf. Definitions 1.1, 1.4). Let $G_1, G_2 \in \mathfrak{G}$. We say that F has the *product property* on $G_1 \times G_2$ if for any $z'_j, z''_j \in G_j$

$$F_{G_1\times G_2}((z'_1, z'_2), (z''_1, z''_2)) = \max\{F_{G_1}(z'_1, z''_1), F_{G_2}(z'_2, z''_2)\}\,. \tag{3.1}$$

We shortly say that F has the *product property* if (3.1) holds for *any* $G_1, G_2 \in \mathfrak{G}$ and $z'_j, z''_j \in G_j$.

If $\delta = (\delta_G)_{G\in\mathfrak{G}}$ is a Schwarz–Pick system of pseudometrics (cf. Definition 1.10) then we say that δ has the *product property on* $G_1 \times G_2$ if for any $z_j \in G_j \subset \mathbb{C}^{n_j} \ni X_j$

$$\delta_{G_1\times G_2}((z_1, z_2); (X_1, X_2)) = \max\{\delta_{G_1}(z_1; X_1), \delta_{G_2}(z_2; X_2)\}. \tag{3.2}$$

δ has the *product property* if (3.2) is fulfilled for *any* $G_1, G_2 \in \mathfrak{G}$ and $z_j \in G_j \subset \mathbb{C}^{n_j} \ni X_j$.

Note that in (3.1) (resp. (3.2)) the inequality "$\ge$" is always fulfilled. Moreover, if $z'_1 = z''_1$ or $z'_2 = z''_2$ (resp. $X_1 = 0$ or $X_2 = 0$) then the equality is trivially satisfied.

The following elementary example shows that in the class of all Schwarz–Pick systems the product property is very exceptional.

Example 3.2. Let $F^{(0)}$, $F^{(1)}$ be Schwarz–Pick systems of functions. Put $F_G^{(t)} := (1-t)F_G^{(0)} + tF_G^{(1)}$, $F^{(t)} := (F_G^{(t)})_{G\in\mathfrak{G}}$, $0 < t < 1$. Note that $F^{(t)}$ is also a Schwarz–Pick system of functions. Suppose that for some $G_0 \in \mathfrak{G}$, $F_{G_0}^{(0)} \not\equiv F_{G_0}^{(1)}$ (e.g. $F^{(0)} = c^*$, $F^{(1)} = k^*$, $G_0 = P =$ the annulus—cf. Example 1.23). Then for every $0 < t < 1$, $F^{(t)}$ does not have the product property on $G_0 \times E$.

Notice that similar examples may easily be produced for Schwarz–Pick systems of pseudodistances and pseudometrics.

The product property is inherited by inner pseudodistances (Proposition 3.3) and by integrated forms (Proposition 3.5)—cf. Definition 1.2 and the definition before Theorem 1.19.

Proposition 3.3. *Let $d = (d_G)_{G\in\mathfrak{G}}$ be a Schwarz–Pick systems of pseudodistances. If d has the product property on $G_1 \times G_2$ then so does $d^i = (d_G^i)_{G\in\mathfrak{G}}$.*

For the proof we need the following elementary lemma:

Lemma 3.4. *Let $G \in \mathfrak{G}$ and let $\alpha : [0,1] \to G$ be a continuous curve with $l := l_{d_G}(\alpha) < \infty$. Then for every $\varepsilon > 0$ there exists an increasing bijection $p : [0,1] \to [0,1]$ such that*

$$l_{d_G}((\alpha \circ p)|_{[t_1,t_2]}) \le (l+\varepsilon)(t_2 - t_1)\,, \qquad 0 \le t_1 < t_2 \le 1\,.$$

Proof. Take $p(t) := q^{-1}(t(l+\varepsilon))$, $0 \le t \le 1$, where $q(u) := \varepsilon u + l_{d_G}(\alpha|_{[0,u]})$, $0 \le u \le 1$.

Proof of Proposition 3.3. Fix $z_j', z_j'' \in G_j$, $\varepsilon > 0$ and let $\alpha_j : [0,1] \to G_j$ be a continuous curve such that $\alpha_j(0) = z_j', \alpha_j(1) = z_j''$ and $l_j - d_{G_j}^i(z_j', z_j'') \le \varepsilon$, where $l_j := l_{d_{G_j}}(\alpha_j)$. In view of Lemma 3.4, we may assume that $l_{d_{G_j}}(\alpha_j|_{[t_1,t_2]}) \le (l_j+\varepsilon)(t_2-t_1), 0 \le t_1 < t_2 \le 1$.

Suppose that $l_1 \ge l_2$. We only need to show that $l_{d_{G_1\times G_2}}(\alpha_1 \times \alpha_2) \le l_1 + \varepsilon$. Take $N \in \mathbb{N}$ and $0 = t_0 < \ldots < t_N = 1$. Then

$$\begin{aligned}\sum_{j=1}^N & d_{G_1\times G_2}((\alpha_1(t_{j-1}), \alpha_2(t_{j-1})), (\alpha_1(t_j), \alpha_2(t_j)))\\ &= \sum_{j=1}^N \max\{d_{G_1}(\alpha_1(t_{j-1}), \alpha_1(t_j)), d_{G_2}(\alpha_2(t_{j-1}), \alpha_2(t_j))\}\\ &\le \sum_{j=1}^N \max\{(l_1+\varepsilon)(t_j - t_{j-1}), (l_2+\varepsilon)(t_j - t_{j-1})\} = l_1 + \varepsilon\,. \ \blacksquare\end{aligned}$$

Proposition 3.5. *Let $\delta = (\delta_G)_{G\in\mathfrak{G}}$ be a Schwarz–Pick system of pseudometrics. Suppose that for some $G_1, G_2 \in \mathfrak{G}$, δ has the product property* *on* $G_1 \times G_2$ *and that* δ_{G_j} *is upper semicontinuous* $(j = 1, 2)$ (*in particular*, $\delta_{G_1\times G_2}$ *is also upper semicontinuous*). *Then for any* $z_j', z_j'' \in G_j$

$$\Big(\int \delta_{G_1\times G_2}\Big)((z_1', z_2'), (z_1'', z_2'')) = \max\Big\{\Big(\int \delta_{G_1}\Big)(z_1', z_1''), \Big(\int \delta_{G_2}\Big)(z_2', z_2'')\Big\}.$$

Proof. Fix $z_j', z_j'' \in G_j$, $\varepsilon > 0$ and let $\alpha_j : [0, 1] \to G_j$ be a C^1 curve with $\alpha_j(0) = z_j'$, $\alpha_j(1) = z_j''$ and $\int_0^1 \delta_{G_j}(\alpha_j(t); \dot\alpha_j(t))\,dt - l_j < \varepsilon$, where $l_j := (\int \delta_{G_j})(z_j', z_j'')$. Suppose that $l_1 \ge l_2$. Let $b_j := [0, 1] \to \mathbb{R}_{>0}$ be a continuous function such that $b_j \ge \delta_{G_j}(\alpha_j; \dot\alpha_j)$ $(j = 1, 2)$ and $\int_0^1 b_1(t)\,dt = \int_0^1 b_2(t)\,dt \le l_1 + \varepsilon$.

Set $B_j(s) := \int_0^s b_j(t)\,dt$, $0 \le s \le 1$ $(j = 1, 2)$ and $B := B_2^{-1} \circ B_1 : [0, 1] \to [0, 1]$, $\widetilde\alpha_2 := \alpha_2 \circ B$. It is enough to prove that

$$\int_0^1 \delta_{G_1\times G_2}((\alpha_1(t), \widetilde\alpha_2(t)); (\dot\alpha_1(t), \dot{\widetilde\alpha}_2(t)))\,dt \le l_1 + \varepsilon\,.$$

We have

$$\begin{aligned}\int_0^1 &\delta_{G_1\times G_2}((\alpha_1(t), \widetilde\alpha_2(t)); (\dot\alpha_1(t), \dot{\widetilde\alpha}_2(t)))\,dt\\ &= \int_0^1 \max\{\delta_{G_1}(\alpha_1(t); \dot\alpha_1(t)), B'(t)\delta_{G_2}(\alpha_2(B(t)); \dot\alpha_2(B(t)))\}\,dt\\ &\le \int_0^1 \max\{b_1(t), B'(t)b_2(B(t))\}\,dt = \int_0^1 b_1(t)\,dt \le l_1 + \varepsilon\,,\end{aligned}$$

which concludes the proof. ∎

Now we are going to discuss the product properties for $c, c^*, c^i, k, k^*, \gamma$ and κ.

Theorem 3.6 ([45]). κ *and* k^* *have the product property. In consequence*, *in view of Theorem* 1.19(a) *and Proposition* 3.5, k *has the product property.*

In view of Theorem 1.12, we get the following important

Corollary 3.7. *If* G_1, G_2 *are biholomorphically equivalent to convex domains then any Schwarz–Pick system has the product property on* $G_1\times G_2$.

Theorem 3.8 ([24]). c *has the product property. In particular*,

c^* *has the product property*,
c^i *has the product property* (Proposition 3.3),
γ *has the product property* (Example 1.11(a)).

The question whether the pluri-complex Green function and the Azukawa pseudometric have the product properties is open (cf. Problem 3.1). We only have the following partial result.

THEOREM 3.9 ([28]). *For any domains of holomorphy* G_1, G_2, g *has the product property on* $G_1 \times G_2$. *In consequence*, *in view of Example* 1.11(c), *if* G_1, G_2 *are domains of holomorphy then* A *has the product property on* $G_1 \times G_2$.

The product property for the Sibony pseudometric is unknown (cf. Problem 3.2).

We pass to the product properties for $m^{(p)}$ and $\gamma^{(p)}$, $p \geq 2$. By Example 1.24, we get

EXAMPLE 3.10. Let G_j be a complete Reinhardt domain in $\mathbb{C}^{n_j}$ with $(|z_1|^t, \ldots, |z_{n_j}|^t) \in G_j$ for $(z_1, \ldots, z_{n_j}) \in G_j$, $t > 0$, $j = 1, 2$. Then

$$m^{(p)}_{G_1\times G_2}((0,0),(z_1,z_2)) = \max\{\{[m^{(k)}_{G_1}(0,z_1)]^k[m^{(p-k)}_{G_2}(0,z_2)]^{p-k}\}^{1/p} : k = 0,\ldots,p\}, \quad (z_1,z_2) \in G_1 \times G_2$$

(where $m^{(0)} :\equiv 1$). In particular, $m^{(p)}$ and $\gamma^{(p)}$ with $p \geq 3$ do not have the product property (take $G_1 := \{(z_1,z_2) \in \mathbb{C}^2 : |z_1 z_2^{p-2}| < 1\}$, $G_2 := E$—cf. [28]).

In view of the above example, we conjectured in [28] that the "correct" forms of the product properties for $m^{(p)}$ and $\gamma^{(p)}$ are the following :

$$m^{(p)}_{G_1\times G_2}((z_1',z_2'),(z_1'',z_2'')) = \max\{\{[m^{(k)}_{G_1}(z_1',z_1'')]^k[m^{(p-k)}_{G_2}(z_2',z_2'')]^{p-k}\}^{1/p} : k = 0,\ldots,p\},$$

$$\gamma^{(p)}_{G_1\times G_2}((z_1,z_2);(X_1,X_2)) = \max\{\{[\gamma^{(k)}_{G_1}(z_1;X_1)]^k[\gamma^{(p-k)}_{G_2}(z_2;X_2)]^{p-k}\}^{1/p} : k = 0,\ldots,p\}.$$

Note that the inequalities "$\geq$" are always satisfied and that for $p = 1, 2$ the above "product properties" coincide with the standard ones.

Unfortunately, for $p \geq 2$, even these general product properties are not true (cf. Problem 3.3), namely:

EXAMPLE 3.11. Let $P = P(R) := \{\lambda \in \mathbb{C} : 1/R < |\lambda| < R\}$ $(R > 1)$. Then for every $p \geq 2$ and for any $R \gg 1$,

$$\gamma^{(p)}_{P\times E}((a,0);(1,Y)) > \max\{\{[\gamma^{(k)}_P(a;1)]^k Y^{p-k}\}^{1/p} : k = 0,\ldots,p\} = \gamma^{(p)}_P(a;1), \tag{3.3}$$

where $a = a(R,p) := R^{(p-1)/(p+1)}$, $Y = Y(R,p) := \gamma^{(p)}_P(a;1)$.

Proof. Fix $p \geq 2$. In view of Example 1.23(a),

$$\gamma_P^{(k)}(a;1) = \frac{1}{a}\Pi_R(a,a)\left[\frac{1}{Ra}f(b_k,-a)\right]^{1/k}, \quad 1 \leq k \leq p, \tag{3.4}$$

where $b_k = b_k(R,p) := R^{2k/(p+1)-1}$ (in particular, $b_p = a$). In view of (3.4), we get

$$\begin{aligned}
&[\max\{[\gamma_P^{(k)}(a;1)]^k Y^{p-k} : k = 1,\dots,p-1\}]^p \\
&\quad = \left[\frac{1}{a}\Pi_R(a,a)\right]^{p^2} \\
&\qquad \times \max\left\{\left[\frac{1}{Ra}f(b_k,-a)\right]^p\left[\frac{1}{Ra}f(a,-a)\right]^{p-k} : k = 1,\dots,p-1\right\}.
\end{aligned}$$

Observe that

$$\left[\frac{1}{Ra}f(b_k,-a)\right]^p\left[\frac{1}{Ra}f(a,-a)\right]^{-k} \to 2^{-k} \quad \text{as } R \to \infty \quad (1 \leq k \leq p-1).$$

Hence

$$\max\{\{[\gamma_P^{(k)}(a;1)]^k Y^{p-k}\}^{1/p} : k = 0,\dots,p\} = \gamma_P^{(p)}(a;1) \quad \text{if } R \gg 1.$$

For the proof of the strict inequality in (3.3), let

$$h(\lambda,\xi) := \alpha_1 h_1(\lambda)\xi^{p-1} + \alpha_p h_p(\lambda), \quad \lambda \in \overline{P},\ \xi \in E,$$

where

$$h_1(\lambda) := \frac{1}{R\lambda}f(a,\lambda)f\left(\frac{2}{R},-\lambda\right), \quad h_p(\lambda) := \frac{1}{R\lambda}[f(a,\lambda)]^p f\left(\frac{R}{2},-\lambda\right),$$

$$\alpha_1 := \frac{2}{2+R^{2/(p+1)}}, \quad \alpha_p := \frac{R^{2/(p+1)}}{2+R^{2/(p+1)}} \quad (R > 2^{(p+1)/2}).$$

Then $\operatorname{ord}_{(a,0)} h = p$ and $\alpha_1|h_1| + \alpha_p|h_p| = 1$ on ∂P. Consequently,

$$\begin{aligned}
&[\gamma_{P\times E}^{(p)}((a,0);(1,Y))]^p \\
&\geq \left[\frac{1}{a}\Pi_R(a,a)\right]^p \frac{1}{Ra}\left\{\alpha_1 f\left(\frac{2}{R},-a\right)\left[\frac{1}{Ra}f(a,-a)\right]^{(p-1)/p} + \alpha_p f\left(\frac{R}{2},-a\right)\right\}.
\end{aligned}$$

To conclude the proof, it remains to observe that

$$\begin{aligned}
&\left\{\alpha_1 f\left(\frac{2}{R},-a\right)\left[\frac{1}{Ra}f(a,-a)\right]^{(p-1)/p} + \alpha_p f\left(\frac{R}{2},-a\right)\right\}[f(a,-a)]^{-1} \\
&\qquad\qquad \to \frac{1+2^{(p-1)/p}}{2} \quad \text{as } R \to \infty.
\end{aligned}$$

Problems. 3.1. Decide whether $(g_G)_{G\in\mathfrak{G}}$ and $(A_G)_{G\in\mathfrak{G}}$ have the product properties.

3.2. Does $(S_G)_{G\in\mathfrak{G}}$ have the product property?

3.3. What are "product properties" for $m^{(p)}$ and $\gamma^{(p)}$ with $p \geq 2$?

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Addendum. In order to update this survey article we mention the following two recent results:

THEOREM ([55]). *For $n \geq 3$, there exists a domain of holomorphy $G \subset \mathbb{C}^n$, c_G-hyperbolic, whose c_G-topology is different from its euclidean topology.*

THEOREM ([56]). *For two points $\lambda', \lambda'' \in P$ the following equivalence is true: $c_P(\lambda', \lambda'') = c^i_P(\lambda', \lambda'')$ if and only if λ', λ'' lie on the same radius.*

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