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On approximation of analytic functions and generalized orders

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Abstract. A characterization of a generalized order of analytic functions of several complex variables by means of polynomial approximation and interpolation is established.

We say that a differentiable function α defined on $[0,\infty)$ is *slowly growing* if it is positive, strictly increases to infinity and for every positive constant c

$$\lim_{x\to\infty} \alpha(cx)/\alpha(x) = 1\,.$$

In the sequel α and β are two fixed slowly growing functions.

Let K be a compact set in $\mathbb{C}^N$, $N \ge 1$, such that the Siciak extremal function of K ([6])

$$\Phi_K(z) := \sup\{|p(z)|^{1/\deg p} : p \text{ a polynomial, } \deg p \ge 1\,, \|p\| \le 1\},\ z \in \mathbb{C}^N,$$

is continuous, $\|\ \|$ being the supremum norm on K. Given a function g analytic in

$$K_R := \{z \in \mathbb{C}^N : \Phi_K(z) < R\}$$

for some $R > 1$, we put

$$M(r) := \sup\{|g(z)| : \Phi_K(z) = r\}\,, \qquad 1 < r < R\,.$$

The quantity

$$\varrho := \limsup_{r\to R} \frac{\alpha\left(\log^+ M(r)\right)}{\beta(R/(R-r))}$$

is called the (α,β)-*order* of g in the sense of Sheremeta ([4], [3]). If $\alpha = \beta = \log^+$ (suitably modified near 0) and K is a ball, we obtain the classical definition of the order of an analytic function.

The aim of this paper is to characterize the (α,β)-*order* of a function g analytic in K_R by means of polynomial approximation and interpolation to

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g on K. A characterization of a similar generalized order of entire functions was established in [2].

Given a function f defined and bounded on K, we put for $n \in \mathbb{N}$

$$E_n^{(1)} = E_n^{(1)}(f, K) := \|f - t_n\|,$$
$$E_n^{(2)} = E_n^{(2)}(f, K) := \|f - l_n\|,$$
$$E_{n+1}^{(3)} = E_{n+1}^{(3)}(f, K) := \|l_{n+1} - t_n\|,$$

where t_n denotes the nth Chebyshev polynomial of the best approximation to f on K and l_n denotes the nth Lagrange interpolation polynomial for f with nodes at extremal points of K ([5]).

THEOREM. *Let K be a balanced compact set in $\mathbb{C}^N$ such that Φ_K is continuous. For positive x and c write*

$$F(x, c) := \beta^{-1}(c\alpha(x)).$$

Assume that for every positive c

$$\limsup_{x\to\infty} \frac{d \log F(x, c)}{d \log x} < 1,$$

$$\alpha(x/F(x, c)) = (1 + o(x))\alpha(x) \quad \text{as } x \to \infty.$$

Then a function f defined and bounded on K is the restriction to K of a function g analytic in K_R for some R and of finite (α, β)-order ϱ if and only if

$$\varrho = \limsup_{x\to\infty} \frac{\alpha(n)}{\beta(n/\log^+(E_n^{(j)} R^n))}, \quad j = 1, 2, 3$$

(*with the obvious conventions* $1/0 = \infty$ *and* $1/\infty = 0$).

We begin by proving the following

LEMMA. *Let the assumptions of the Theorem hold and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of polynomials in $\mathbb{C}^N$. Assume that*

(i) $\deg p_n \le n,\ n \in \mathbb{N}$,
(ii) *there exist $n_0 \in \mathbb{N}, \mu > 0$ and $R > 1$ such that*

$$\log^+(\|p_n\| R^n) \le n/F(n, 1/\mu) \quad \text{provided } n \ge n_0.$$

Then $\sum_{n=0}^{\infty} p_n$ is an analytic function in K_R and its (α, β)-order ϱ does not exceed μ.

Proof. From (ii)

$$\log^+(\|p_n\| r^n) \le n \log(r/R) + n/F(n, 1/\mu)$$

provided $n \ge n_0$ and $1 < r < R$. By the methods of calculus we find that the maximum of the function

$$\mathbb{R}_+ \ni x \to x \log(r/R) + x/F(x, 1/\mu)$$

is reached for $x = x_r$, where x_r is the solution of the equation

$$x = \alpha^{-1}\left(\mu\beta\left(\frac{1 - d\log F(x, 1/\mu)/d\log x}{\log(R/r)}\right)\right).$$

From the assumptions of the Theorem and the properties of α and β we obtain

$$x_r = (1 + o(1))\alpha^{-1}(\mu\beta(R/(R - r))) \quad \text{as } r \to R.$$

Thus for r sufficiently close to R

(1) $$\log^+(\|p_n\|r^n) \le \text{const.}\,\alpha^{-1}(\mu\beta(R/(R - r))), \quad n \in \mathbb{N}.$$

For every polynomial p we have ([6])

$$|p(z)| \le \|p\|\Phi_K^{\deg p}(z), \quad z \in \mathbb{C}^N.$$

So for every $r \in (1, R)$ the series $\sum_{n=0}^{\infty} p_n$ is convergent in K_r, whence $\sum_{n=0}^{\infty} p_n$ is analytic in K_R.

Write

$$M^*(r) := \sup\{\|p_n\|r^n : n \in \mathbb{N}\}, \quad r \ge 0,$$

$$\varrho^* := \limsup_{r\to R} \frac{\alpha(\log^+ M^*(r))}{\beta(R/(R - r))}.$$

According to inequality (1) we have

$$\log^+ M^*(r) \le \text{ const.}\,\alpha^{-1}(\mu\beta(R/(R - r)))$$

for r sufficiently close to R. This immediately yields $\varrho^* \le \mu$. Moreover (see [1], 2.3(1)),

$$\log^+ M(r) \le \log^+ M^*(\sqrt{rR}) - \log(1 - \sqrt{r/R}).$$

Thus

$$\frac{\alpha(\log^+ M(r))}{\beta\left(\frac{R}{R - r}\right)} \le \frac{\alpha(\log^+ M^*(\sqrt{rR}) - \log(1 - \sqrt{r/R}))}{\beta\left(\frac{R}{R - \sqrt{rR}}\right)} \cdot \frac{\beta\left(\frac{R}{R - \sqrt{rR}}\right)}{\beta\left(\frac{R}{R - r}\right)},$$

which gives (after passing to the upper limit) $\varrho \le \varrho^*$ and consequently $\varrho \le \mu$.

P r o o f o f T h e o r e m. Let g be a function analytic in K_R, of (α, β)-order ϱ. Write

$$\gamma_j := \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(n/\log^+(E_n^{(j)} R^n))}, \quad j = 1, 2, 3;$$

here $E_n^{(j)}$ stands for $E_n^{(j)}(g,K)$. We claim that $\gamma_j = \varrho$, $j = 1,2,3$. It is known (see e.g. [7]) that

$$E_n^{(1)} \le E_n^{(2)} \le (n_* + 2)E_n^{(1)}, \quad n \ge 0, \tag{2}$$

$$E_n^{(3)} \le 2(n_* + 2)E_{n-1}^{(1)}, \quad n \ge 1, \tag{3}$$

where $n_* := \binom{n+N}{n}$. Thus $\gamma_3 \le \gamma_2 = \gamma_1$ and it suffices to prove that $\gamma_1 \le \varrho \le \gamma_3$.

We first prove $\gamma_1 \le \varrho$. By definition of the (α,β)-order we have for every $\mu > \varrho$

$$\log^+ M(r) \le \alpha^{-1}(\mu\beta(R)/(R-r))$$

provided r is sufficiently close to R. By Lemma 3.4 of [1]

$$E_n^{(1)} \le \frac{M(r)}{(r-1)r^n}, \quad 1 < r < R,$$

so

$$\log^+(E_n^{(1)}R^n) \le -\log(r-1) - n\log(r/R) + \alpha^{-1}(\mu\beta(R/(R-r)))$$

for every $n \in \mathbb{N}$ and for r sufficiently close to R. Substituting $r = r_n$, where

$$r_n := R[1 - 1/F(n/F(n,1/\mu),1/\mu)],$$

yields

$$\begin{aligned}\log^+(E_n^{(1)}R^n) \le\ & -\log(r_n - 1) - n\log[1 - 1/F(n/F(n,1/\mu),1/\mu)] \\ & + n/F(n,1/\mu).\end{aligned}$$

On account of the assumptions and the properties of the logarithm we obtain

$$\log^+(E_n^{(1)}R^n) \le 4n/F(n,1/\mu)$$

for sufficiently large n. Hence, by the properties of slowly growing functions, for every $\varepsilon > 0$ and for sufficiently large n

$$\frac{\alpha(n)}{\beta(n/\log^+(E_n^{(1)}R^n))} \le \mu + \varepsilon.$$

Owing to the arbitrariness of $\varepsilon > 0$ and $\mu > \varrho$ we get after passing to the upper limit $\gamma_1 \le \varrho$.

Next we claim $\varrho \le \gamma_3$. Suppose $\gamma_3 < \varrho$. Then for every $\mu \in (\gamma_3, \varrho)$

$$\frac{\alpha(n)}{\beta(n/\log^+(E_n^{(3)}R^n))} \le \mu$$

provided n is sufficiently large. Thus

$$\log^+(E_n^{(3)}R^n) \le n/F(n,1/\mu)$$

and by the Lemma $\varrho \le \mu$, which contradicts the assumption $\mu < \varrho$.

Now let f be a function defined and bounded on K. Put

$$\gamma_j := \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(n/\log^+(E_n^{(j)} R^n))}, \qquad j = 1, 2, 3.$$

We claim that if γ_k is finite for $k = 1, 2$ or 3, then

$$g := l_0 + \sum_{n=0}^{\infty} (l_{n+1} - l_n)$$

is the required analytic continuation of f to K_R and its (α, β)-order ϱ is $\gamma_j, j = 1, 2, 3$. Indeed, for every $\mu > \gamma_k$

$$\frac{\alpha(n)}{\beta(n/\log^+(E_n^{(k)} R^n))} \le \mu$$

provided n is sufficiently large. Hence

$$E_n^{(k)} R^n \le \exp(n/F(n, 1/\mu)).$$

By (2), (3) and the Lemma, g is analytic in K_R and its (α, β)-order ϱ is finite. So by the first part of the proof $\varrho = \gamma_j$, $j = 1, 2, 3$, as claimed.

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