On approximation of analytic functions
and generalized orders

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Abstract. A characterization of a generalized order of analytic functions of several complex variables by means of polynomial approximation and interpolation is established.

We say that a differentiable function $\alpha$ defined on $[0, \infty)$ is slowly growing if it is positive, strictly increases to infinity and for every positive constant $c$

$$\lim_{x \to \infty} \frac{\alpha(cx)}{\alpha(x)} = 1.$$  

In the sequel $\alpha$ and $\beta$ are two fixed slowly growing functions.

Let $K$ be a compact set in $\mathbb{C}^N$, $N \geq 1$, such that the Siciak extremal function of $K$ ([6])

$$\Phi_K(z) := \sup \{|p(z)|^{1/\deg p} : p \text{ a polynomial, } \deg p \geq 1, \|p\| \leq 1, z \in \mathbb{C}^N, \|\| \text{ being the supremum norm on } K.$$  

given a function $g$ analytic in $K_R := \{z \in \mathbb{C}^N : \Phi_K(z) < R\}$ for some $R > 1$, we put

$$M(r) := \sup \{|g(z)| : \Phi_K(z) = r\}, \quad 1 < r < R.$$  

The quantity

$$\varrho := \limsup_{r \to R} \frac{\alpha \left( \log^+ M(r) \right)}{\beta \left( R/(R - r) \right)}$$

is called the $(\alpha, \beta)$-order of $g$ in the sense of Sheremeta ([4], [3]). If $\alpha = \beta = \log^+$ (suitably modified near 0) and $K$ is a ball, we obtain the classical definition of the order of an analytic function.

The aim of this paper is to characterize the $(\alpha, \beta)$-order of a function $g$ analytic in $K_R$ by means of polynomial approximation and interpolation to

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A characterization of a similar generalized order of entire functions was established in [2].

Given a function $f$ defined and bounded on $K$, we put for $n \in \mathbb{N}$

$E_n^{(1)} = E_n^{(1)}(f, K) := \|f - t_n\|,$

$E_n^{(2)} = E_n^{(2)}(f, K) := \|f - l_n\|,$

$E_{n+1}^{(3)} = E_{n+1}^{(3)}(f, K) := \|l_{n+1} - t_n\|,$

where $t_n$ denotes the $n$th Chebyshev polynomial of the best approximation to $f$ on $K$ and $l_n$ denotes the $n$th Lagrange interpolation polynomial for $f$ with nodes at extremal points of $K$ ([5]).

**Theorem.** Let $K$ be a balanced compact set in $\mathbb{C}^N$ such that $\Phi_K$ is continuous. For positive $x$ and $c$ write

$$F(x, c) := \beta^{-1}(\alpha(x)).$$

Assume that for every positive $c$

$$\limsup_{x \to \infty} \frac{d \log F(x, c)}{d \log x} < 1,$$

$$\alpha(x/F(x, c)) = (1 + o(x))\alpha(x) \quad \text{as } x \to \infty.$$

Then a function $f$ defined and bounded on $K$ is the restriction to $K$ of a function $g$ analytic in $K_R$ for some $R$ and of finite $(\alpha, \beta)$-order $\varrho$ if and only if

$$g = \limsup_{x \to \infty} \frac{\alpha(n)}{\beta(n/\log^+(E_n^{(j)}(R^n)))}, \quad j = 1, 2, 3$$

(with the obvious conventions $1/0 = \infty$ and $1/\infty = 0$).

We begin by proving the following

**Lemma.** Let the assumptions of the Theorem hold and let $(p_n)_{n \in \mathbb{N}}$ be a sequence of polynomials in $\mathbb{C}^N$. Assume that

(i) $\deg p_n \leq n$, $n \in \mathbb{N}$,

(ii) there exist $n_0 \in \mathbb{N}$, $\mu > 0$ and $R > 1$ such that

$$\log^+(\|p_n\|R^n) \leq n/F(n, 1/\mu) \quad \text{provided } n \geq n_0.$$

Then $\sum_{n=0}^{\infty} p_n$ is an analytic function in $K_R$ and its $(\alpha, \beta)$-order $\varrho$ does not exceed $\mu$.

**Proof.** From (ii)

$$\log^+(\|p_n\|R^n) \leq n \log(r/R) + n/F(n, 1/\mu)$$

provided $n \geq n_0$ and $1 < r < R$. By the methods of calculus we find that the maximum of the function

$$\mathbb{R}_+ \ni x \to x \log(r/R) + x/F(x, 1/\mu)$$
is reached for \( x = x_r \), where \( x_r \) is the solution of the equation

\[
x = \alpha^{-1} \left( \frac{1 - d \log F(x, 1/\mu)}{d \log R/r} \right).
\]

From the assumptions of the Theorem and the properties of \( \alpha \) and \( \beta \) we obtain

\[
x_r = (1 + o(1)) \alpha^{-1}(\mu \beta/(R - r)) \quad \text{as } r \to R.
\]

Thus for \( r \) sufficiently close to \( R \)

\[\log^+ (\|p_n\| r^n) \leq \text{const.} \alpha^{-1}(\mu \beta/(R - r)), \quad n \in \mathbb{N}.\]

For every polynomial \( p \) we have ([6])

\[|p(z)| \leq \|p\|_{\Phi} \|p_n\|, \quad z \in \mathbb{C}^N.\]

So for every \( r \in (1, R) \) the series \( \sum_{n=0}^{\infty} p_n \) is convergent in \( K_r \), whence \( \sum_{n=0}^{\infty} p_n \) is analytic in \( K_R \).

Write

\[M^*(r) := \sup\{\|p_n\| r^n : n \in \mathbb{N}\}, \quad r \geq 0,
\]

\[\varrho^* := \limsup_{r \to R} \frac{\alpha(\log^+ M^*(r))}{\beta(R/(R - r))}.
\]

According to inequality (1) we have

\[\log^+ M^*(r) \leq \text{const.} \alpha^{-1}(\mu \beta/(R - r))\]

for \( r \) sufficiently close to \( R \). This immediately yields \( \varrho^* \leq \mu \). Moreover (see [1], 2.3(1)),

\[\log^+ M(r) \leq \log^+ M^*(\sqrt{rR}) - \log(1 - \sqrt{r/R}).\]

Thus

\[
\frac{\alpha(\log^+ M(r))}{\beta \left( \frac{R}{R - r} \right)} \leq \frac{\alpha(\log^+ M^*(\sqrt{rR}) - \log(1 - \sqrt{r/R}))}{\beta \left( \frac{R}{R - \sqrt{rR}} \right)} \cdot \frac{\beta \left( \frac{R}{R - r} \right)}{\beta \left( \frac{R}{R - \sqrt{rR}} \right)},
\]

which gives (after passing to the upper limit) \( \varrho \leq \varrho^* \) and consequently \( \varrho \leq \mu \).

**Proof of Theorem.** Let \( g \) be a function analytic in \( K_R \), of \( (\alpha, \beta) \)-order \( \varrho \). Write

\[\gamma_j := \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(n/\log^+ (E_n^{(j)} R^n))}, \quad j = 1, 2, 3;
\]
here $E^{(j)}_n$ stands for $E^{(j)}_n(g, K)$. We claim that $\gamma_j = \varrho$, $j = 1, 2, 3$. It is known (see e.g. [7]) that

\[(2)\]
\[
E^{(1)}_n \leq E^{(2)}_n \leq (n_* + 2)E^{(1)}_n, \quad n \geq 0,
\]

\[(3)\]
\[
E^{(3)}_n \leq 2(n_* + 2)E^{(1)}_{n-1}, \quad n \geq 1,
\]

where $n_* := \left( \frac{n+N}{n} \right)$. Thus $\gamma_3 \leq \gamma_2 = \gamma_1$ and it suffices to prove that $\gamma_1 \leq \varrho \leq \gamma_3$.

We first prove $\gamma_1 \leq \varrho$. By definition of the $(\alpha, \beta)$-order we have for every $\mu > \varrho$

\[
\log^+ M(r) \leq \alpha^{-1}(\mu \beta(R)/(R - r))
\]

provided $r$ is sufficiently close to $R$. By Lemma 3.4 of [1]

\[
E^{(1)}_n \leq \frac{M(r)}{(r - 1)r^n}, \quad 1 < r < R,
\]

so

\[
\log^+(E^{(1)}_n R^n) \leq -\log(r - 1) - n \log(r/R) + \alpha^{-1}(\mu \beta(R/(R - r)))
\]

for every $n \in \mathbb{N}$ and for $r$ sufficiently close to $R$. Substituting $r = r_n$, where $r_n := R[1 - 1/F(n/F(n, 1/\mu), 1/\mu)]$, yields

\[
\log^+(E^{(1)}_n R^n) \leq -\log(r_n - 1) - n \log[1 - 1/F(n/F(n, 1/\mu), 1/\mu)] + n/F(n, 1/\mu).
\]

On account of the assumptions and the properties of the logarithm we obtain

\[
\log^+(E^{(1)}_n R^n) \leq 4n/F(n, 1/\mu)
\]

for sufficiently large $n$. Hence, by the properties of slowly growing functions, for every $\varepsilon > 0$ and for sufficiently large $n$

\[
\frac{\alpha(n)}{\beta(n/ \log^+(E^{(1)}_n R^n))} \leq \mu + \varepsilon.
\]

Owing to the arbitrariness of $\varepsilon > 0$ and $\mu > \varrho$ we get after passing to the upper limit $\gamma_1 \leq \varrho$.

Next we claim $\varrho \leq \gamma_3$. Suppose $\gamma_3 < \varrho$. Then for every $\mu \in (\gamma_3, \varrho)$

\[
\frac{\alpha(n)}{\beta(n/ \log^+(E^{(3)}_n R^n))} \leq \mu
\]

provided $n$ is sufficiently large. Thus

\[
\log^+(E^{(3)}_n R^n) \leq n/F(n, 1/\mu)
\]

and by the Lemma $\varrho \leq \mu$, which contradicts the assumption $\mu < \varrho$. 
Now let $f$ be a function defined and bounded on $K$. Put
\[ \gamma_j := \limsup_{n \to \infty} \frac{\alpha(n)}{\beta(n/\log^+(E_{n}^{(j)}R^n))}, \quad j = 1, 2, 3. \]
We claim that if $\gamma_k$ is finite for $k = 1, 2$ or 3, then
\[ g := l_0 + \sum_{n=0}^{\infty} (l_{n+1} - l_n) \]
is the required analytic continuation of $f$ to $K_R$ and its $(\alpha, \beta)$-order $\rho$ is $\gamma_j, j = 1, 2, 3$. Indeed, for every $\mu > \gamma_k$
\[ \frac{\alpha(n)}{\beta(n/\log^+(E_{n}^{(k)}R^n))} \leq \mu \]
provided $n$ is sufficiently large. Hence
\[ E_{n}^{(k)}R^n \leq \exp(n/F(n, 1/\mu)). \]
By (2), (3) and the Lemma, $g$ is analytic in $K_R$ and its $(\alpha, \beta)$-order $\rho$ is finite. So by the first part of the proof $\rho = \gamma_j, j = 1, 2, 3$, as claimed.

References


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