ANNALES
POLONICI MATHEMATICI
55 (1991)

On the spectral properties of translation operators in one-dimensional tubes

by Wojciech Hyb (Warszawa)

Abstract. We study the spectral properties of some group of unitary operators in the Hilbert space of square Lebesgue integrable holomorphic functions on a one-dimensional tube (see formula (1)). Applying the Genchev transform ([2], [5]) we prove that this group has continuous simple spectrum (Theorem 4) and that the projection-valued measure for this group has a very explicit form (Theorem 5).

Let $D = \{z \in \mathbb{C} : \operatorname{Im} z \in J\}$ be a tube over an open interval $J \subset \mathbb{R}$, $J \neq$ R. Let $L^2H(D)$ be the Hilbert space of all functions which are holomorphic and square integrable in D with the standard inner product.

We define the group of unitary operators $\{U^t\}$, $t \in \mathbb{R}$, in $L^2H(D)$ as follows:

$$U^t(f)(z) = f(z+t) \tag{1}$$

for $f \in L^2H(D)$ and $z \in D$. This is called the *group of translation operators*. The aim of this paper is to present some of its spectral properties.

We shall use the following notations.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}$ and let m denote the Lebesgue measure on $\mathcal{B}$. If m_j, $j = 1, 2$, are two measures on $\mathcal{B}$, we write $m_1 \ll m_2$ to mean that m_1 is absolutely continuous with respect to m_2. Let $\widehat{f}$ denote the Fourier transform of $f \in L^2(\mathbb{R})$ ([6], p. 17).

For $t \in \mathbb{R}$ we define

$$w(t) = \int_J e^{-4\pi t y}\, dy$$

and let $B_J = \{t \in \mathbb{R} : w(t) < \infty\}$. An easy computation shows that we have three possibilities: $B_J = \mathbb{R}$, $B_J = \mathbb{R}^+$ or $B_J = \mathbb{R}^-$.

Let $L^2(\mathbb{R}, w)$ be the Hilbert space of all measurable functions g on $\mathbb{R}$

1991 *Mathematics Subject Classification*: 47D10

such that

$$\int_{\mathbb{R}} |g(t)^2| w(t)\, dt < \infty.$$

It is easy to see that for $g \in L^2(\mathbb{R}, w)$, $g(t) = 0$ for m-a.e. $t \notin B_J$.

We recall briefly the definition of the Genchev transform (see [2], [5] for more details).

For $f \in L^2H(D)$ and for $y \in J$ we set $f_y(x) = f(x + iy)$, $x \in \mathbb{R}$. Then $f_y \in L^2(\mathbb{R})$ and we define the *Genchev transform* G as follows:

$$G(f)(t) = e^{2\pi ty} \widehat{f_y}(t). \tag{2}$$

Note that the right side of (2) does not depend on $y \in J$.

The following theorem follows from the results of [2] and [5].

THEOREM 1. *The Genchev transform is a unitary mapping of $L^2H(D)$ onto $L^2(\mathbb{R}, w)$. Moreover, for $g \in L^2(\mathbb{R}, w)$ and for $z \in D$ the function $t \mapsto e^{2\pi itz} g(t)$ belongs to $L^1(\mathbb{R})$ and the inverse of the Genchev transform has the form*

$$G^{-1}(g)(z) = \int_{\mathbb{R}} e^{2\pi itz} g(t)\, dt. \tag{3}$$

We define the group of unitary operators $\{V^t\}$, $t \in \mathbb{R}$, in $L^2(\mathbb{R}, w)$ by

$$V^t(g)(s) = e^{2\pi ist} g(s) \tag{4}$$

for $g \in L^2(\mathbb{R}, w)$. The fundamental property of the Fourier transform gives

THEOREM 2. $G \circ U^t = V^t \circ G$ *for all* $t \in \mathbb{R}$.

This shows that the groups $\{U^t\}$ and $\{V^t\}$ have the same spectral properties. Therefore we shall investigate $\{V^t\}$.

THEOREM 3. *The projection-valued measure P for the group $\{V^t\}$ is*

$$P(E)g = \chi_{(2\pi)^{-1}E}\, g \tag{5}$$

for $E \in \mathcal{B}$ and $g \in L^2(\mathbb{R}, w)$, where χ_A denotes the characteristic function of the set A.

Proof. It is easy to see that (5) defines a projection-valued measure on $\mathcal{B}$ ([3], p. 12).

For $g_k \in L^2(\mathbb{R}, w)$, $k = 1, 2$, we define the finite complex measure $m_{g_1 g_2}$ by

$$m_{g_1 g_2}(e) = \langle P(E) g_1, g_2 \rangle \tag{6}$$

for $E \in \mathcal{B}$. The definition of the inner product in $L^2(\mathbb{R}, w)$ and a change of variables gives

$$m_{g_1 g_2}(E) = \int_E \frac{1}{2\pi} g_1\left(\frac{s}{2\pi}\right) \overline{g_2\left(\frac{s}{2\pi}\right)} w\left(\frac{s}{2\pi}\right) ds$$

for $E \in \mathcal{B}$ and from (4) we have

$$\langle V^t g_1, g_2 \rangle = \int_{\mathbb{R}} e^{2\pi i s t} g_1(s) \overline{g_2(s)} w(s)\, ds = \int_{\mathbb{R}} e^{iut}\, dm_{g_1 g_2}(u).$$

Thus our statement follows from the spectral theorem ([3], p 23).

For $g \in L^2(\mathbb{R}, w)$ we denote by $C(g)$ the smallest closed subspace of $L^2(\mathbb{R}, w)$ containing $V^t g$ for all $t \in \mathbb{R}$. Moreover, let $m_g = m_{gg}$ (see (6)) and set $M(g) = \{h \in L^2(\mathbb{R}, w) : m_h \ll m_g\}$.

Remark 1. It is easy to show that $M(g)$ is a closed subspace of $L^2(\mathbb{R}, w)$ and $C(g) \subset M(g)$.

Definition. The measure m_g is called a *maximal spectral type* for P if $m_h \ll m_g$ for every $h \in L^2(\mathbb{R}, w)$ (see also [1], p. 913).

Lemma 1. *For $g \in L^2(\mathbb{R}, w)$ the following statements are equivalent*:

(a) $g(t) \neq 0$ *for m-a.e. $t \in B_j$*,
(b) $C(g) = L^2(\mathbb{R}, w)$,
(c) m_g *is a maximal spectral type for P*.

Proof. (a)$\Rightarrow$(b). Suppose that there exists $h \in L^2(\mathbb{R}, w)$ such that h is orthogonal to $C(g)$. Then for all $t \in \mathbb{R}$

$$0 = \langle V^t g, h \rangle = \int_{\mathbb{R}} e^{2\pi i t s} g(s) \overline{h(s)} w(s)\, ds. \tag{7}$$

The function $g_1(s) = g(s)\overline{h(s)}w(s)$ belongs to $L^1(\mathbb{R})$ and (7) shows that its Fourier transform is 0. Thus $g_1(s) = 0$ for m-a.e. $s \in \mathbb{R}$ and we conclude that $h(s) = 0$ for m-a.e. $s \in \mathbb{R}$.

(b)$\Rightarrow$(c) follows from Remark 1.

(c)$\Rightarrow$(a). Let $A = \{t \in B_j : g(t) = 0\}$. Let E be any Borel subset of A such that $m(E)$ is finite. Then the function $h(t) = \chi_E(t)(w(t))^{-1/2}$, $t \in \mathbb{R}$, belongs to $L^2(\mathbb{R}, w)$. An immediate calculation shows that $m_h((2\pi)E) = m(E)$ and $m_g((2\pi)E) = 0$. Then (c) yields $m(E) = 0$ and we conclude that $m(A) = 0$.

Remark 2. The function $g(t) = e^{-t^2}(w(t))^{-1/2}$, $t \in \mathbb{R}$, belongs to $L^2(\mathbb{R}, w)$ and satisfies the condition (a) in Lemma 1.

We notice that $m_g \ll m$ for all $g \in L^2(\mathbb{R}, w)$. Thus from Lemma 1 and Remark 2 we obtain

COROLLARY 1. *The group* $\{V^t\}$ *has continuous simple spectrum* (for definition see [3], p. 21).

Remark 3. The simple spectrum of $\{V^t\}$ implies that the set of spectral types for $\{V^t\}$ has only one element ([1], p. 916).

Now we are going to prove some spectral properties of the group $\{U^t\}$ in $L^2H(D)$. Let $K_D(z,w)$ $(z,w \in D)$ denote the Bergman function of the domain D ([4], p. 332).

THEOREM 4. 1° *The projection-valued measure* Q *for the group* $\{U^t\}$ *is*

$$Q(E) = G^{-1}P(E)G \tag{8}$$

for $E \in \mathcal{B}$.

2° *The group* $\{U^t\}$ *has continuous simple spectrum.*

3° *For fixed* $y \in J$ *let* $f(z) = K_D(z, iy)$ *for* $z \in D$. *Then* $C(f) = L^2H(D)$ *and* m_f *is a maximal spectral type for* Q.

Proof. 1° and 2° follow immediately from Theorem 2 and Corollary 1.

3° Suppose that there exists $h \in L^2H(D)$ such that h is orthogonal to $C(f)$. Then for all $t \in \mathbb{R}$ we have

$$\langle h, U^t f\rangle = 0. \tag{9}$$

Since $K_D(\cdot, w)$ has the reproducing property and U^t is a unitary operator, from (9) and (1) we obtain

$$0 = \langle U^{-t}h, f\rangle = U^{-t}h(iy) = h(-t+iy).$$

This implies that $G(h) = 0$ and by Theorem 1 we conclude that $h = 0$. Hence $C(f) = L^2H(D)$ and Remark 1 shows that m_f is a maximal spectral type for Q.

The projection-valued measure for the group $\{U^t\}$ has the following property:

THEOREM 5. *For* $E \in \mathcal{B}$ *such that* $m(E)$ *is finite*, *for every* $f \in L^2H(D)$ *and every* $z \in D$,

$$Q(E)(f)(z) = \int\limits_{\mathbb{R}} f(z+t)\widehat{\chi}_{(2\pi)^{-1}E}(t)\,dt \tag{10}$$

where the integrand belongs to $L^1(\mathbb{R})$.

Proof. For fixed $E \in \mathcal{B}$ let

$$g(t) = \chi_{(2\pi)^{-1}E}(t)$$

Assume that $m(E)$ is finite. Then g and $\widehat{g}$ belong to $L^2(\mathbb{R})$.

Let $z = x + iy \in D$. Then the function $f(z+t) \cdot \widehat{g}(t) = f_y(x+t)\widehat{g}(t)$, $t \in \mathbb{R}$, belongs to $L^1(\mathbb{R})$.

From (8), (5), (3) and (2) we obtain

$$Q(E)(f)(z) = G^{-1}P(E)G(f)(z) = \int_{\mathbb{R}} e^{2\pi i t z} g(t) e^{2\pi t y} \widehat{f}_y(t)\, dt$$
$$= \int_{\mathbb{R}} e^{2\pi i t x} \widehat{f}_y(t) g(t) dt.$$

It is easy to see that the function $e^{2\pi i t x}\widehat{f}_y(t)$ is the Fourier transform of $h(t) = f_y(t+x)$. Applying the multiplication formula for h and g ([6], p. 17) we obtain

$$Q(E)(f)(z) = \int_{\mathbb{R}} \widehat{h}(t) g(t)\, dt = \int_{\mathbb{R}} h(t)\widehat{g}(t)\, dt = \int_{\mathbb{R}} f_y(t+x)\widehat{g}(t)\, dt\,,$$

which completes the proof.

Remark 4. For every bounded interval $E = (a,b) \subset \mathbb{R}$, (10) has the form

$$Q(E)(f)(z) = \int_{\mathbb{R}} f(z+t)\frac{e^{-iat} - e^{-ibt}}{2\pi i t}\, dt$$

for $f \in L^2H(D)$ and $z \in D$.

Acknowledgment. The author is greatly indebted to Professor M. Skwarczyński for suggesting the problem.

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DEPARTMENT OF APPLIED MATHEMATICS
AGRICULTURAL ACADEMY
NOWOURSYNOWSKA 166
02-766 WARSZAWA, POLAND

Reçu par la Rédaction le 14.9.1990