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A geometric approach to the Jacobian Conjecture in $\mathbb{C}^2$

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Abstract. We consider polynomial mappings (f,g) of $\mathbb{C}^2$ with constant nontrivial jacobian. Using the Riemann–Hurwitz relation we prove among other things the following: If $g-c$ (resp. $f-c$) has at most two branches at infinity for infinitely many numbers c or if f (resp. g) is proper on the level set $g^{-1}(0)$ (resp. $f^{-1}(0)$), then (f,g) is bijective.

Introduction. In 1939 O.-H. Keller [11] raised the following question: If $f,g\in\mathbb{Z}[x,y]$ and

$$\mathrm{Jac}(f,g) := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 1\,.$$

then is it possible to represent x and y as polynomials of f and g with integral coefficients?

It is known ([4], [5], [16]) that the solution of the Keller problem follows from the solution of the two-dimensional case of the *Jacobian Conjecture* (for short JC):

If $f,g\in\mathbb{C}^2[x,y]$ *and* $\mathrm{Jac}(f,g)=\mathrm{const}\neq 0$*, then the mapping* $(f,g):\mathbb{C}^2\to\mathbb{C}^2$ *is injective.*

The above case of the general Jacobian Conjecture is sometimes called *Keller's Jacobian Conjecture*. If $(f,g):\mathbb{C}^2\to\mathbb{C}^2$ is injective, then it is bijective [5] and its inverse is a polynomial map of the same degree ([4], [19]).

For some history and a brief exposition of the basic facts on the Jacobian Conjecture we refer the reader to [4]. A short review of the results on JC in the twodimensional case and a presentation of the method of weighted gradings is given in [3] and [17].

Another approach to JC was presented in Abhyankar's and Moh's papers. S. S. Abhyankar proved [1] that if f (or g) has always one point at infinity and $\mathrm{Jac}(f,g)=\mathrm{const}\neq 0$, then (f,g) is injective (see also [17]). In 1975

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S. Abhyankar and T. T. Moh using complicated techniques of characteristic pairs proved [2] that if $\mathrm{Jac}(f,g) = 1$ and f or (or g) has exactly one branch at infinity, then (f,g) is injective (see also [8]). In 1983 T. T. Moh checked [14] that JC is true when $\max\{\deg f, \deg g\} < 100$.

1. A geometric approach to JC

(i) First we quote a nice theorem on polynomials ([10], Prop. A.1). We say that a polynomial $f = f(x,y)$ is *primitive* iff there exists a finite set $E \subset \mathbb{C}$ such that the polynomial $h(x,y) := f(x,y) - c$ is irreducible for every $c \in \mathbb{C} \setminus E$. If $c \in \mathbb{C} \setminus E$, then we call c a *typical value* (for the polynomial f).

THEOREM 1 ([10]). *Let $f = f(x,y)$ be a polynomial. Then there exists a primitive polynomial $p \in \mathbb{C}[x,y]$ and a polynomial $T \in \mathbb{C}[t]$ such that $f = T \circ p$.*

COROLLARY 1.1. *If $f \in \mathbb{C}[x,y]$ and $\mathrm{grad}\, f(x,y) \neq (0,0)$ for $(x,y) \in \mathbb{C}^2$, then f is primitive.*

Proof. By Theorem 1 we get $f = T \circ p$, where $T \in \mathbb{C}[t]$, $p \in \mathbb{C}[x,y]$ and p is primitive. Since

$$\mathrm{grad}\, f(x,y) = \left(T'[p(x,y)]\frac{\partial p}{\partial x}, T'[p(x,y)]\frac{\partial p}{\partial y}\right) \neq (0,0)\,,$$

we have $T'[p(x,y)] \neq 0$ for $(x,y) \in \mathbb{C}^2$ and $T(t) = at + b$, $a, b \in \mathbb{C}$, $a \neq 0$. But p is primitive iff $ap + b$ is primitive, so the corollary follows.

From Corollary 1.1 we immediately have

COROLLARY 1.2. *If $\mathrm{Jac}(f,g) = 1$, then f and g are primitive and for each $c \in \mathbb{C}$ the polynomials $f + c$ and $g + c$ are reduced (i.e. without multiple factors).*

(ii) Let $f = f_0 + \ldots + f_m$, $g = g_0 + \ldots + g_n$, where $m = \deg f$, $n = \deg g$ and f_j, g_k are homogeneous polynomials of degree j, k respectively. It can be easily checked that without loss of generality we can assume in JC that

$$(*) \qquad f(x,y) = x^m + f_{m-1} + \ldots + f_0\,, \ g(x,y) = x^n + g_{n-1} + \ldots + g_0\,.$$

(If $(*)$ does not hold we take a polyautomorphism $T(x,y) = (ax, by + cx^2)$, $abc \neq 0$, and then $f \circ T$ and $g \circ T$ have the form $(*)$).

From now on we assume that the considered polynomials have the form $(*)$. Notice that it is sufficient to consider only the case $\deg f = \deg g$ because (f,g) is injective iff $(f+g, g)$ is injective iff $(f, f+g)$ is injective.

(iii) Let us recall a well-known fact about polynomial dominating mappings.

THEOREM 2 ([15]). *If* $h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n$ *and* $\operatorname{Jac} h(x) \neq 0$ *for some* $x \in \mathbb{C}^n$, *then there exists a nontrivial polynomial* D_h *such that if* $D_h(y) \neq 0$, *then* $\#h^{-1}(y) = \sup\{\#h^{-1}(z) : z \in \mathbb{C}^n, \#h^{-1}(z) < \infty\} < \infty$.

The last number, i.e. the number of points in the general fibre of h, is called the *geometrical degree* of h (for short $\operatorname{g.deg} h$).

If $\operatorname{Jac} h = \text{const} \neq 0$, then $\{y \in \mathbb{C}^n : \#h^{-1}(y) = \infty\} = \emptyset$ because for fixed $y \in \mathbb{C}$ the equation $h(x) = y$ has only isolated solutions and by the Bézout inequalities [12] the set $h^{-1}(y)$ is finite. Thus $\operatorname{g.deg} h = \#h^{-1}(y)$ when $D_h(y) \neq 0$.

Let $h = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ and $\operatorname{Jac} h = 1$. The algebraic set $Z = \{(u, v) \in \mathbb{C}^2 : h^{-1}(u, v) = \emptyset\}$ is finite because if there were a nontrivial polynomial $q \in \mathbb{C}[u, v]$ such that $Z = q^{-1}(0)$, then there would exist a nontrivial polynomial $p(x, y) = q[f(x, y), g(x, y)]$ having $p^{-1}(0) = \emptyset$.

(iv) By definition, the homogenizations of f and g are given by

$$F(x, y, z) := z^m f_0 + \ldots + z^1 f_{m-1} + f_m, \quad \text{i.e.}$$

$$F(x, y, z) = z^m f\left(\frac{x}{z}, \frac{y}{z}\right) \text{for } z \neq 0\,,$$

$$G(x, y, z) := z^n g_0 + \ldots + z^1 g_{n-1} + g_n, \quad \text{i.e.}$$

$$G(x, y, z) = z^n g\left(\frac{x}{z}, \frac{y}{z}\right) \text{for } z \neq 0\,.$$

By the above formulas, F (resp. G) is irreducible iff f (resp. g) is irreducible. So Corollary 1.2 yields that if $\operatorname{Jac}(f, g) = 1$, then F and G are primitive and reduced.

(v) Choose $c \in \mathbb{C}$ such that $g - c$ is irreducible and $\mathbb{C} \times \{c\} \not\subset D^{-1}_{(f,g)}(0)$. Then by Th. 2 there exists $u \in \mathbb{C}$ such that $\#(f, g)^{-1}(u, c) = \operatorname{g.deg}(f, g) =: d$. Put

$$M := \{(x, y, z) \in \mathbb{P}^2 : G(x, y, z) - cz^n = 0\}\,.$$

Since $\operatorname{grad} G(x, y, 1) = \operatorname{grad} g(x, y) \neq 0$ for $(x, y) \in \mathbb{C}^2$, the affine part of M is smooth. The curve M has exactly one point at infinity, namely $M \cap \{(x, y, z) : z = 0\} = (0, 1, 0) =: S$.

We have two possibilities: either

1) M is smooth at S, i.e. $\operatorname{grad} G(S) \neq (0, 0, 0)$, or
2) M has a singularity at S.

If 2) holds, then we take a normalization (desingularization) of M and we have a smooth algebraic curve $\widetilde{M}$ in $\mathbb{P}^3$ and a holomorphic mapping $\pi : \widetilde{M} \to M$ such that π has finite fibres (i.e. $\#\pi^{-1}(P) < \infty$ for $P \in M$) and $\pi : \widetilde{M} \setminus \pi^{-1}(S) \to M \setminus S$ is biholomorphic. (In case 1) we put $\widetilde{M} := M$, $\pi := \text{id}$.)

(vi) Let M, N be compact Riemann surfaces of genus a and b, respectively, and let h be a nonconstant holomorphic map from M to N. Then, of course, h has to be surjective. Let $\mathrm{mult}_x h$ denote the multiplicity of h at $x \in M$. Take $y \in N$ and put

$$d(y) := \sum_{x \in h^{-1}(y)} \mathrm{mult}_x h\,.$$

Evidently $d(y) = \mathrm{const}$ for $y \in N$ ([9]). Call $d := d(y) =$ the *geometrical degree* of h. We recall the following fundamental theorem.

Riemann–Hurwitz Relation ([9]).

$$2a = 2d(b-1) + 2 + B\,, \tag{1}$$

where $B := \sum_{x \in M} (\mathrm{mult}_x h - 1) =$ *the total branching number of* h.

(vii) Define $h : \widetilde{M} \setminus \pi^{-1}(S) \to \widehat{\mathbb{C}}$ by

$$h(P) := f \circ \pi(P), \qquad \text{where } \ \pi(P) = (x, y, z) \in M \setminus S\,.$$

By (v) there exists $u \in \mathbb{C}$ such that $\#(f,g)^{-1}(u,c) = \mathrm{g.deg}(f,g) = d$. Let $h^{-1}(u) = \{P_1, \ldots, P_d\}$. For sufficiently small neighbourhoods $D(P_1, \varepsilon), \ldots, D(P_d, \varepsilon)$ of $P_1, \ldots, P_d$, respectively, there exists $\delta > 0$ such that

$$h\Big[\widetilde{M} \setminus \Big(\bigcup_{j=1}^{d} D(P_j, \varepsilon) \cup \pi^{-1}(S)\Big)\Big] \subset \mathbb{C} \setminus B(u, \delta)\,.$$

Then by the Riemann theorem on removable singularities it is possible to holomorphically extend h to each point $P \in \pi^{-1}(S) =: \{S_1, \ldots, S_r\}$.

Notice that by the normalization process we have $r := \#\pi^{-1}(S) =$ number of irreducible holomorphic germs of $G(x,y,z)$ at S, i.e. the number of holomorphically irreducible factors of $G(x,1,z)$ at $(0,0)$. Remember that $h|_{\widetilde{M} \setminus \pi^{-1}(S)} = f|_{M \setminus S} \circ \pi$ is a locally biholomorphic map because of the jacobian assumption (if $g(x,y) - c = 0$ and $y = y(x)$, then $y' = -g_x/g_y$ when $g_y \neq 0$ and $(d/dx) f(x, y(x)) = f_x + f_y y'(x) = \mathrm{Jac}(f,g)/g_y(x, y(x)) \neq 0$). Therefore

$$B = \sum_{j=1}^{r} (\mathrm{mult}_{S_j} h - 1), \ \{S_1, \ldots, S_r\} = \pi^{-1}(S)\,. \tag{2}$$

Since $b =$ genus of $\widehat{\mathbb{C}} = 0$, by the Riemann–Hurwitz Relation we get $0 \le 2a = -2d + 2 + B$, hence

$$2d \le 2 + B\,. \tag{3}$$

2. Main theorem. We are now ready to prove the following:

Theorem. *Assume that* $f, g \in \mathbb{C}[x,y]$ *and* $\mathrm{Jac}(f,g) = 1$. *Then*:

1° *If g (or f) has one branch at infinity (i.e., e.g., g has one point at infinity and g is holomorphically irreducible at this point), then (f,g) is injective.*

2° *If $g-v$ (or $f-v$) has at most two branches at infinity for an infinite number of $v \in \mathbb{C}$, then (f,g) is injective.*

3° *If f is proper on $g^{-1}(c)$ for some c (i.e. $\lim f(x,y)=\infty$ as $|(x,y)| \to \infty$, $g(x,y)=c$), then (f,g) is injective.*

4° *If* $\mathrm{g.deg}(f,g) \le 2$, *then (f,g) is injective.*

Notice that 1° is Abhyankar and Moh's result [2], but our proof is extremely easy and elementary. 4° is well known in JC and holds for every dimension [4]. 2° and 3° seem to be new. 2° generalizes 1°, because of the result of [13] stating that holomorphic irreducibility at infinity of $g-c$ for some $c \in \mathbb{C}$ is equivalent to irreducibility for every $c \in \mathbb{C}$.

Proof of the theorem. Let M, $\widetilde{M}$, π, h and S be defined as above.

1° Since $r=1$, we derive from (2) that $B = \mathrm{mult}_{S_1} h - 1 = d-1$. By (3) we get $2d \le 2+d-1$, that is, $d \le 1$, so (f,g) is injective.

2° Assume $d \ge 2$. By the assumption we can choose $c \in \mathbb{C}$ such that $g-c$ is irreducible and $\{(u,c) \in \mathbb{C}^2 : (f,g)^{-1}(u,c) = \emptyset\} = \emptyset$. If $\pi^{-1}(S) = \{S_1, S_2\}$, then we have two possibilities: either

(a) $h^{-1}(\infty) = \{S_1, S_2\}$, or

(b) $h^{-1}(\infty) = \{S_1\}$ and $h(S_2) = h(A) \in \mathbb{C}$ for some $A \in \widetilde{M} \setminus \pi^{-1}(S)$.

From (2) we derive that $B \le d-2$ or $B \le 2d-3$. Hence by (3) we obtain a contradiction.

3° 1) Let c be a typical value for g. Then $g-c$ is an irreducible polynomial and since f is proper we have $h^{-1}(\infty) = \pi^{-1}(S) = \{S_1, \ldots, S_r\}$. From (2) we get $B = \sum_{j=1}^{r} \mathrm{mult}_{S_j} h - r = d(\infty) - r = d - r$ and by (3) we obtain $2d \le 2 + d - r$, so $d = r = 1$.

2) Assume that c is not a typical value for g and let $g - c = PQ$, where P and Q are nontrivial polynomials, P is irreducible and P does not divide Q. Put $M' := \{(x,y,z) \in \mathbb{P}^2 : z^p P(x/z, y/z) = 0\}$, $p = \deg P$, and consider $h = f \circ \pi : \widetilde{M} \to \widehat{\mathbb{C}}$. By 1), the map h is biholomorphic, so $f : M' = P^{-1}(0) \to \mathbb{C}$ is also biholomorphic. Since M' is biregularly equivalent to $\mathbb{C}$ (cf. [18]), there exists a bijective polynomial map $T = (R,S) : \mathbb{C} \to M'$ such that $T'(t) \ne (0,0)$ for each $t \in \mathbb{C}$. Put $x = R(t)$, $y = S(t)$ into the equation $\mathrm{Jac}(f,g)(x,y) = 1$. Because $P \circ T(t) = 0$ for each $t \in \mathbb{C}$, we get

$$Q \circ T(t) \cdot \mathrm{Jac}(f,P)(T(t)) = 1, \quad t \in \mathbb{C}.$$

Thus $Q \circ T(t) = \mathrm{const} \ne 0$ for $t \in \mathbb{C}$. From the classical facts [15] and the irreducibility of P we get $Q = W(P)$ for some nontrivial polynomial

$W \in \mathbb{C}[t]$. Hence $g = PW(P)$, which contradicts Corollary 1.2 asserting that g is primitive.

4° Assume $d = 2$. By (iii) we can choose a typical value c for g such that $\{(u,c) \in \mathbb{C}^2 : (f,g)^{-1}(u,c) = \emptyset\} = \emptyset$, i.e. $\#(f,g)^{-1}(u,c) \ge 1$ for each $u \in \mathbb{C}$. If $h^{-1}(\infty) = \{S_1, S_2\}$, then $B = 0$. If $h^{-1}(\infty) = S_1$, then $B = 1$. In both cases we have $B \le 1$. By (3) we get $2d \le 3$, thus $d = 1$, a contradiction.

3. Remarks on a geometric approach to the Keller problem. A geometric way of proving the Jacobian Conjecture in $\mathbb{C}^2$ could be the following. Let $\mathrm{Jac}(f,g) = 1$ and take a typical value $c \in \mathbb{C}$ for the polynomial g. Consider the Riemann surface $M := g^{-1}(c) \subset \mathbb{C}^2$. If one could prove that $H_1(M) = 0$ (resp. $\pi_1(M) = 0$), then M would be biholomorphically equivalent to $\mathbb{P}^1$, $\mathbb{C}$ or $U = \{z \in \mathbb{C} : |z| < 1\}$ ([9]). Since M is a Liouville space, M is biholomorphic to $\mathbb{C}$. In this case M is biregularly equivalent to $\mathbb{C}$ ([18]), so, in particular, there exists a polynomial map $T = (P,Q) \in (\mathbb{C}[t])^2$ such that $T : \mathbb{C} \to M$ is bijective and $T'(t) \ne (0,0)$ for $t \in \mathbb{C}$. If we assume that (f,g) is not injective, then the polynomial map $h := f \circ T : \mathbb{C} \to \mathbb{C}$ is not injective. Hence

$$0 = h(t') = \frac{\partial f}{\partial x}[T(t')]P'(t') + \frac{\partial f}{\partial y}[T(t')]Q'(t') \quad \text{for some } t' \in \mathbb{C}\,. \tag{4}$$

Since $g \circ T(t) = c = \text{const}$ for $t \in \mathbb{C}$, we have

$$0 = \frac{\partial g}{\partial x}[T(t')]P'(t') + \frac{\partial g}{\partial y}[T(t')]Q'(t')\,. \tag{5}$$

By (4), (5) and $\mathrm{Jac}(f,g) = 1$ we get $T'(t') = (P'(t'), Q'(t')) = (0,0)$, a contradiction.

A nice formula for $H_1(F^{-1}(c))$, where $F = F(x_1, \ldots, x_n)$ is a polynomial, is given in [6], but it is very hard to check that $H_1(g^{-1}(c)) = 0$ having "only" the assumption $\mathrm{Jac}(f,g) = 1$.

Note. The result 3° of our Theorem has been obtained independently and by quite different methods in [7].

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