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On the disc theorem

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Abstract. Ahlfors' disc theorem for Riemann covering surfaces is extended to normally exhaustible Klein coverings.

The Rolf Nevanlinna second main theorem gives information not only on the exceptional values but also on the ramification, in particular on the totally ramified values of a meromorphic function [10, Chap. X, §3].

In 1935, L. V. Ahlfors considered in his metrical-topological value distribution theory, instead of totally ramified values, totally ramified Jordan regions called discs [1], [10, Chap. XIII, §6].

Let X and Y be Riemann surfaces and $T : X \to Y$ an analytic map. The triple (X, T, Y) is called a Riemann covering.

A Jordan region $\mathit{\Delta}$ in Y is a *totally ramified disc* if there are no relatively compact components of $T^{-1}(\mathit{\Delta})$ covering $\mathit{\Delta}$ with a single sheet by means of T, i.e. if there are no one-sheeted islands over $\mathit{\Delta}$. Sometimes $\overline{\mathit{\Delta}}$ has been called a totally ramified disc [12], [4]–[8].

Ahlfors' theory applies to regularly exhaustible Riemann covering surfaces and his celebrated disc theorem asserts in particular that for entire (resp. meromorphic) functions T, there are $h \le 2$ (resp. $h \le 4$) mutually disjoint totally ramified discs on $\mathbb{C}$ (resp. $\widehat{\mathbb{C}}$).

In 1938, S. Stoilow proved a topological disc theorem, this time for normally exhaustible Riemann covering surfaces, a topological equivalent of regularly exhaustible ones. For entire functions T generating a normally exhaustible covering, $h \le 1$ instead of 2 [11], [12].

S. Stoilow only considered simply connected normally exhaustible Riemann coverings, but in 1952 we established the disc theorem for arbitrary such coverings [4], [5] and afterwards we proved this theorem for more and more general classes of Riemann coverings: the L. I. Volkovyskiĭ class A_∞ [5], the E-quasinormally exhaustible coverings including the T. Kuroda class

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[6], the R. Osserman coverings, the coverings with a partially regular (in Stoilow's sense) exhaustion [7], the quasitotally exhaustible coverings [8], and even the general case of the polyhedrally exhaustible coverings [8, II, Chap. IV, §1]. As this last theorem refers to the most general case of coverings, it is expressed by a rather complicated inequality corresponding to the great complexity of the situation considered, but it includes as special cases all previous disc theorems obtained by developing Stoilow's method.

The aim of our present paper is to extend the disc theorem to normally exhaustible Klein coverings. More precisely, we shall determine an upper bound for the number h of totally ramified discs of a normally exhaustible, in particular a total, Klein covering. At the same time, a discussion of the Hurwitz formula will put in evidence new necessary conditions for the normal exhaustibility of a Riemann or a Klein covering.

§ 1. Definitions and notations. A *Klein covering* (X, T, Y) is a triple where X and Y are Klein surfaces [3, Chap. 1, §2] and $T : X \to Y$ is a non-constant morphism [3, Chap. 1, §4]. Due to the topological character of the ramification problem and of the method used, which is based on the Hurwitz formula and its generalizations [9], in what follows we suppose X and Y endowed only with topological structure. Thus X and Y will be orientable or non-orientable surfaces (connected two-manifolds with countable basis) with or without border and $T : X \to Y$ an interior transformation in Stoilow's sense (a continuous, open and light mapping) [13], [9]. The borders of X and Y will be denoted by BX and BY respectively. Evidently, $T(BX) \subset BY$.

Let $\varphi(z) = x + i|y|$ be the folding map, $\varphi : \mathbb{C} \to \mathbb{C}^+ = \{z : y \geq 0\}$.

In a neighborhood of a point $P \in X$, the mapping T is topologically equivalent to a mapping $w = \psi(z)$ in a neighborhood of $z = 0$ with $k \in \mathbb{N} - \{0\}$, as follows:

— if $P \in \operatorname{int} X$ $(= X \setminus BX)$ and $p = T(P) \in \operatorname{int} Y$ then $\psi(z) = z^k$,
— if $P \in \operatorname{int} X$ but $p \in BY$ then $\psi(z) = \varphi(z^{k/2})$, k even.
— if $P \in BX$ and $p \in BY$ then $\psi(z) = \varphi(z^k)$ or $\varphi(-z^k)$.

By definition T takes at P the value p with *multiplicity* k and has at P the *ramification order* $k - 1$, $k/2 - 1$, $(k - 1)/2$ respectively.

We also recall some definitions of Stoilow's theory [13, Chaps. V and VI], which we directly extend from Riemann to Klein coverings.

The Klein covering (X, T, Y) is *total* if for each infinite sequence of points $P_\nu \in X$ which tends to the ideal boundary ∂X of X (i.e. has no accumulation point in X) its projection $p_\nu = T(P_\nu)$ tends to the ideal boundary ∂Y of Y. A Klein covering (X, T, Y) is total iff T is proper. For any total Klein covering there exists a natural number n, called the number of sheets, such that T takes every value $p \in Y$, counting multiplicities, n times.

If Δ is a region of Y and P a point in X with $p = T(P) \in \Delta$, then the component of $T^{-1}(\Delta)$ which contains P is called the *maximal region* of Δ with respect to P. A *normal region* is a relatively compact maximal region and its closure is a *normal domain.* For a region Δ in X (resp. Y) the boundary of Δ with respect to X (resp. Y) will be denoted by $\partial\Delta$ and its closure by $\overline{\Delta}$.

If X is a compact surface every Klein covering (X, T, Y) is total, hence $T(X) = Y$.

A Klein covering (X, T, Y) is called *normally exhaustible* iff there exists an exhaustion sequence of X by normal polyhedral regions $\{D_i\}_{i\in\mathbb{N}}$, i.e. D_i is a polyhedral region, $\overline{D}_i \subset D_{i+1}$, $\bigcup D_i = X$ [2, Chap. I, 29] and D_i is a normal region for T [12].

Evidently, it is possible to choose the regions D_i so that ∂D_i do not contain any ramification point of the covering and we shall suppose this condition fulfilled.

Total coverings are always normally exhaustible; in particular, for a compact surface X we take $D_i = X$, $i \in \mathbb{N}$.

§ 2. Remarks on normal exhaustibility. In all the rest of the paper (X, T, Y) will be a normally exhaustible Klein covering, $\{D_i\}$ an exhaustion sequence as before and n_i the number of sheets of D_i over $T(D_i)$. The covering (X, T, Y) has $n = \lim n_i \le \infty$ sheets over $T(X)$. A normally exhaustible covering is total iff n is finite.

Denote by r_i the ramification order of the covering $(D_i, T|, T(D_i))$, by c_i the connectivity of $D_i \setminus BD_i$ ($c_i - 2$ is the Euler characteristic of $D_i \setminus BD_i$), by μ_i the number of its boundary components and by g_i and $\mathfrak{g}_i$ its genus according as it is orientable or not. In order to uniformize the results we write $\mathfrak{g}_i = 2g_i$ if D_i is orientable. The notations c_i', μ_i', g_i', and $\mathfrak{g}_i'$ for $T(D_i)$ will have similar meanings.

2.1. The Klein covering $(D_i, T|, T(D_i))$ being total, the Hurwitz formula implies

$$r_i \le (c_i - 2) - n_i(c_i' - 2), \tag{1}$$

because of the possible presence of borders or folds [9].

As $r_i \ge 0$, we deduce that

$$c_i' \le 2 + \frac{1}{n_i}(c_i - 2) \tag{2}$$

and from this inequality we derive a first series of results concerning n, the connectivity c of X and the connectivity c' of $T(X)$.

PROPOSITION 1. (i) *$c' \le c$ except for the case $c = 0$, $c' = 1$.*

(ii) *If c and n are finite, then* $c' \le 2 + (1/n)(c-2)$; $c = 0 \Rightarrow c' \le 1$; $c = 1 \Rightarrow c' = 1$; $c \ge 2$ *and* $n \ge 2 \Rightarrow c' \le 1 + c/2$; *etc.*

(iii) *If c is finite but $n = \infty$, then* $c = 1 \Rightarrow c' = 1$ *and* $c \ge 2 \Rightarrow c' \le 2$.

(iv) *If* $c = \infty$ *and* $n = \infty$ *but* $\overline{\lim}\, c_i/n_i = L < \infty$, *then* $c' \le 2 + L$.

Proof of (i). If $c \ge 2$, then $c_i \ge 2$ for sufficiently large i and (2) implies $c'_i \le c_i$, whence $c' \le c$. If $c = 1$, then $c'_i \le 2 - 1/n_i$, hence $c'_i \le 1$ and $c' \le c$. If $c = 0$, then $c' \le 2 - 2/n$, hence $c' \le 1$. The case $c = 0$, $c' = 1$ corresponds to the unramified double covering of the projective plane or of the closed disc by the sphere.

2.2. Let us now remark that $\mu_i \le n_i \mu'_i$. It follows from (2) that

$$\mathfrak{g}'_i \le 2 + \frac{1}{n_i}(\mathfrak{g}_i - 2), \tag{3}$$

with the special cases:

$$g'_i \le 1 + \frac{1}{n_i}(g_i - 1) \tag{3'}$$

for X and $T(X)$ orientable,

$$\mathfrak{g}'_i \le 2 + \frac{2}{n_i}(g_i - 1) \tag{3''}$$

for X orientable and $T(X)$ non-orientable, and (3) for X and $T(X)$ non-orientable. The notations g and $\mathfrak{g}$ (g' and $\mathfrak{g}'$) will be used for the genus of X (resp. $T(X)$).

Proposition 2. (i) $\mathfrak{g}' \le \mathfrak{g}$ *except for the case* $g = 0$, $\mathfrak{g}' = 1$; *hence* $g' \le g$ *for X and $T(X)$ orientable*, $\mathfrak{g}' \le 2g$ *for X orientable and $T(X)$ non-orientable with exception of* $g = 0$, $\mathfrak{g}' = 1$, *and* $\mathfrak{g}' \le \mathfrak{g}$ *for X and $T(X)$ non-orientable.*

(ii) *If $\mathfrak{g}$ and n are finite, then* $\mathfrak{g}' \le 2 + (1/n)(\mathfrak{g} - 2)$; $g = 0 \Rightarrow \mathfrak{g}' \le 1$; $\mathfrak{g} = 1 \Rightarrow \mathfrak{g}' \le 1$.

(iii) *If $\mathfrak{g}$ is finite but $n = \infty$, then* $\mathfrak{g}' \le 2$, *more precisely* $g = 0 \Rightarrow \mathfrak{g}' \le 1$ *and* $\mathfrak{g} = 1 \Rightarrow \mathfrak{g}' \le 1$.

(iv) *If* $\mathfrak{g} = n = \infty$ *but* $\overline{\lim}\, \mathfrak{g}_i/n_i = L < \infty$, *then* $\mathfrak{g}' \le 2 + L$.

All these results contain necessary conditions in order that $(X, T, T(X))$ be normally exhaustible. A similar discussion may be done by supposing $r \ge 1$ and $n_i \ge 2$, since we are interested in totally ramified discs, but it would not bring essentially new aspects.

§ 3. Disc theorem for normally exhaustible Klein coverings without borders. Let (X, T, Y) be a normally exhaustible Klein covering such that $BX = BY = \emptyset$ and $\{D_i\}_{i \in \mathbb{N}}$ a normal exhaustion sequence as before.

We suppose that there are h mutually disjoint, totally ramified discs $\varDelta_l$, $l = 1, \dots, h$ on $T(X)$. Starting from a certain index i_0, $T(D_i) \supset \bigcup \overline{\varDelta}_l$.

3.1. In order to simplify the notation we now drop the index i and designate by D one of the regions D_i with $i \geq i_0$. The Hurwitz formula (1) becomes for $(D, T|, T(D))$:

$$r = (c-2) - n(c'-2). \tag{1'}$$

Every region $\varDelta_l$ will be totally covered with n_k (≥ 2) sheets by normal regions δ_k with $\overline{\delta}_k \subset D$. Let ν be the number of these regions for all $\varDelta_l$, r_k the ramification order of the covering $(\delta_k, T|, \varDelta_l)$, c_k, g_k or $\mathfrak{g}_k$, and μ_k the connectivity, the genus and the number of the boundary components of δ_k. By applying again Hurwitz' formula [9], this time for the total covering $(\delta_k, T|, \varDelta_l)$, we can write

$$r_k = (c_k - 1) - 1 + n_k\,. \tag{4}$$

As in [4] we use the inequality $r \geq \sum_{k=1}^{\nu} r_k$ and deduce

$$(c-2) - n(c'-2) \geq \sum_{k=1}^{\nu} (c_k - 1) - \nu + \sum_{k=1}^{\nu} n_k\,. \tag{5}$$

Since $nh = \sum_{k=1}^{\nu} n_k \geq 2\nu$ and $c_k \geq 1$, it follows that

$$h \leq 2(2-c') + \frac{2}{n}(c-2) = \frac{2}{n} r\,.$$

This inequality can be written for each D_i, $i \geq i_0$, and implies the following general

First Disc Theorem. *Let (X, T, Y) be a normally exhaustible unbordered Klein covering and $\{D_i\}$ a normal exhaustion sequence. The maximal number h of mutually disjoint totally ramified discs $\varDelta_l$, $l = 1, \dots, h$, $\overline{\varDelta}_l \subset T(X)$, satisfies the inequality*

$$h \leq 2(2-c_i') + \frac{2}{n_i}(c_i - 2) = \frac{2}{n_i} r_i \tag{I}$$

for i sufficiently large.

Combined with Proposition 1 inequality (I) implies different formulations of the disc theorem, where c, c' and n refer to X, $T(X)$ and the covering $(X, T, T(X))$.

Case of total coverings: $n < \infty$

(i) $c = 0$, i.e. X is a sphere; then $T(X) = Y$ is a compact surface with $c' \leq 1$, the sphere or the projective plane.
— $c = 0$, $c' = 0 \Rightarrow h \leq 2$ for $n = 2, 3$ and $h \leq 3$ for $n \geq 4$.
— $c = 0$, $c' = 1 \Rightarrow h = 0$ for $n = 2, 3$ and $h \leq 1$ for $n \geq 4$.

(ii) $c = 1$, i.e. X is $\mathbb{C}$ or the projective plane, hence $c' = 1 \Rightarrow h \leq 1$.

(iii) $c=2$, $c'=0 \Rightarrow h \le 4$; $c=2$, $c'=1 \Rightarrow h \le 2$; $c=2$, $c'=2 \Rightarrow h=0$.
(iv) c finite $\Rightarrow c' \le 2+(1/n)(c-2)$ and $h \le 2(2-c')+(2/n)(c-2)$.

Case of effective normal exhaustibility: $n=\infty$

(i) c finite $\Rightarrow c' \le 2$ and $h \le 2(2-c')$. More precisely: $c'=0$ cannot arrive; $c=1$, $c'=1 \Rightarrow h \le 1$ [12]; $c \ge 2$, $c'=1 \Rightarrow h \le 2$ [4], [5]; $c \ge 2$, $c'=2 \Rightarrow h=0$.
(ii) $c=\infty$ but $\overline{\lim}\, c_i/n_i = L < \infty \Rightarrow c' \le 2+L$, $h \le 2(2-c')+2L$.

In *both cases*: $n<\infty$ and $n=\infty$, in order to have $h=\infty$ it is necessary that $c=\infty$.

3.2. As in §2, taking into account that $\mu_i \le n_i\mu_i'$, the inequality (I) leads to the

Second Disc Theorem. *Under the hypotheses of the First Disc Theorem,*

$$h \le 2(2-\mathfrak{g}_i') + \frac{2}{n_i}(\mathfrak{g}_i - 2), \tag{II}$$

in particular

$$h \le 4(1-g_i') + \frac{4}{n_i}(g_i - 1) \tag{II$'$}$$

for X and $T(X)$ orientable surfaces and

$$h \le 2(2-\mathfrak{g}_i') + \frac{4}{n_i}(g_i - 1) \tag{II$''$}$$

for X an orientable and $T(X)$ a non-orientable surface.

One obtains from (II), (II$'$) or (II$''$) and Proposition 2 three series of variants of the disc theorem. However, since they are similar we only present here the results for X and $T(X)$ orientable:

Case of total coverings: $n<\infty$

(i) $g=0 \Rightarrow g'=0$, $h \le 2$ for $n=2,3$ and $h \le 3$ for $n \ge 4$.
(ii) $g=1 \Rightarrow$ either $g'=0$ and $h \le 4$, or $g'=1$ and $h=0$.
(iii) $g=2 \Rightarrow$ either $g'=0$ and $h \le 6$ if $n=2$, $h \le 5$ if $n=3,4$, $h \le 4$ if $n \ge 5$, or $g'=1$ and $h \le 2$ if $n=2$, $h \le 1$ if $n=3,4$, $h=0$ if $n \ge 5$, or $g'=2, n=1, h=0$.
(iv) g finite $\Rightarrow g' \le 1+(1/n)(g-1)$ and $h \le 4(1-g')+(4/n)(g-1)$.

Case of effective normal exhaustibility: $n=\infty$

(i) $g=0 \Rightarrow g'=0$ and $h \le 3$ [4], [5].
(ii) g finite $\ge 1 \Rightarrow$ either $g'=0$ and $h \le 4$, or $g'=1$ and $h=0$.
(iii) If $g=\infty$, the existence of a finite $\overline{\lim}\, g_i/n_i = L$ implies $g' \le 1+L$ and $h \le 4(1-g')+4L$.

We can now make precise the last statement from 3.1: In both cases: $n < \infty$ and $n = \infty$, in order to have $h = \infty$ it is necessary that $\mathfrak{g} = \infty$.

§ 4. Disc theorem for normally exhaustible Klein coverings with borders. Now suppose that $BY \neq \emptyset$ and $T^{-1}(BY) \neq \emptyset$, so that folds, a feature of Klein coverings, can appear.

The covering $(D_i, T|, T(D_i))$ is again total and formula (1) holds [9] together with its consequences in §2, but we now consider two kinds of totally ramified discs: Δ'_l, $l = 1, \ldots, h'$, for which $\overline{\Delta}'_l \subset T(D_i) \setminus BY$, and Δ''_l, $l = 1, \ldots, h''$, for which $\overline{\Delta}''_l \subset T(D_i)$ but $\Delta''_l \cap BT(D_i)$ is an open Jordan arc $a_l b_l$ on BY, while $\partial \Delta''_l$ a Jordan arc γ_l ending at a_l and b_l, and contained except for the end points in $T(D_i) \setminus BY$, $i \geq i_0$ sufficiently large. Such a Δ''_l will be called a *bordered disc*.

The discs Δ'_l are totally ramified in the sense of the definition of §1. However, a normal region δ''_k over a bordered disc Δ''_l can be two-sheeted over Δ''_l without having any ramification point projected in Δ''_l, as a consequence of the existence of a fold. This is for instance the case of the covering $(\mathbb{C}, \varphi, \mathbb{C}^+)$, when we can choose infinitely many mutually disjoint bordered discs $\{z \in \mathbb{C}^+ : |z - x_0| < R,\ x_0 \in \mathbb{R},\ R > 0\}$ which are not covered by any one-sheeted island, the disc theorem thus loosing its sense. Therefore we call a bordered disc Δ''_l *totally ramified* if every relatively compact component δ''_k of $T^{-1}(\Delta''_l)$ has at least one ramification point of T over Δ''_l. Evidently, we set the condition that δ''_k be relatively compact only in order to have a general definition, since in the case of normal exhaustibility each component of $T^{-1}(\Delta''_l)$ is relatively compact.

4.1. Proceeding as in §3, we drop for the moment the index $i \geq i_0$, denote by δ'_k, $k = 1, \ldots, \nu'$, and δ''_k, $k = 1, \ldots, \nu''$, the components of $T^{-1}(\Delta'_l) \cap D$ and $T^{-1}(\Delta''_l) \cap D$ respectively, and use similar notations r'_k, n'_k, $c'_k = \mathfrak{g}'_k + \mu'_k$ for the covering $(\delta'_k, T|, \Delta'_l)$ and r''_k, n''_k, $c''_k = \mathfrak{g}''_k + \mu''_k$ for $(\delta''_k, T|, \Delta''_l)$.

As before, since $c'_k \geq 1$ and $nh' = \sum_{k=1}^{\nu'} n'_k \geq 2\nu'$, we have

$$r'_k = c'_k - 2 + n'_k \geq n'_k - 1, \qquad k = 1, \ldots, \nu',$$

and

$$\sum_{k=1}^{\nu'} r'_k \geq \frac{n}{2} h'. \tag{6}$$

Further, the generalization of the Hurwitz formula in [9] implies

$$r''_k = c''_k - 2 + n''_k - \frac{1}{2}(f^k_{a_l} + f^k_{b_l}), \qquad k = 1, \ldots, \nu'', \tag{7}$$

where f_p^k, $p = a_l$ and b_l, is the number of folds of the covering $(\delta_k'', T|, \varDelta_l'')$ ending at p, i.e. covering a neighborhood of p in $a_l b_l$ without covering p itself.

Since $f_p^k \leq \kappa_k$, where $n_k'' = 2\kappa_k$ or $2\kappa_k + 1$ according as n_k'' is even or odd, it follows from (7) and $c_k'' \geq 1$ that

$$r_k'' \geq n_k'' - 1 - \kappa_k = \begin{cases} n_k''/2 - 1 & \text{for } n_k'' \text{ even,} \\ (n_k'' - 1)/2 & \text{for } n_k'' \text{ odd.} \end{cases} \tag{8}$$

However, for $n_k'' = 2$ this inequality reduces to $r_k'' \geq 0$, thus we use now the hypothesis of the existence of at least one ramification point according to which $r_k'' \geq (n_k'' - 1)/2$ for $n_k'' = 2$.

Denote by ν_1'', ν_2'' and ν_3'' the number of the coverings $(\delta_k'', T\,|, \varDelta_l'')$ with $n_k'' = 2$, n_k'' odd ≥ 3, and n_k'' even ≥ 4 respectively. Then

$$\sum_{k=1}^{\nu''} r_k'' \geq \frac{1}{2} \sum_{k=1}^{\nu''} n_k'' - \frac{1}{2}\nu_1'' - \frac{1}{2}\nu_2'' - \nu_3''.$$

On the other hand,

$$nh'' = \sum_{k=1}^{\nu''} n_k'' \geq 2\nu_1'' + 3\nu_2'' + 4\nu_3'',$$

hence

$$\sum_{k=1}^{\nu''} r_k'' \geq \frac{n}{4} h''. \tag{9}$$

Consequently, from the inequalities

$$r \geq \sum_{k=1}^{\nu'} r_k' + \sum_{k=1}^{\nu''} r_k'',$$

(6) and (9), and from (1′) we deduce for $h = h' + h''$

$$h \leq \frac{4}{n} r \leq 4(2 - c') + \frac{4}{n}(c - 2),$$

or introducing again the index $i \geq i_0$, the

First Disc Theorem. *Let (X, T, Y) be a normally exhaustible bordered Klein covering, $BT(X) \neq \emptyset$, and $\{D_i\}$ a normal exhaustion sequence. Then the maximal number $h = h' + h''$ of mutually disjoint totally ramified interior discs $\varDelta_l'$ and bordered discs $\varDelta_l''$ satisfies the inequality*

$$h \leq \frac{4}{n_i} r_i \leq 4(2 - c_i') + \frac{4}{n_i}(c_i - 2) \tag{III}$$

for i sufficiently large.

4.2. Further, since we have again $\mu_i \leq n\mu_i'$, the inequality (III) also implies the

Second Disc Theorem for the bordered case. *Under the hypotheses of the First Disc Theorem from* 4.1,

$$h \leq 4(2 - \mathfrak{g}_i') + \frac{4}{n_i}(\mathfrak{g}_i - 2); \tag{IV}$$

in particular, h *satisfies two inequalities similar to* (II$'$) *and* (II$''$).

Just as the inequalities (I) or (II) in the unbordered case, the inequalities (III) and (IV) include various forms of the disc theorem for total and for effective normally exhaustible bordered Klein coverings. One obtains them as in §3 so that we omit their formulation.

Remark 1. The inequalities (III) and (IV) remain valid under a weaker definition of totally ramified bordered discs. Namely, it is sufficient to require that every δ_k'' over Δ_l'' covers Δ_l'' with at least two sheets and that $\overline{\delta}_k''$ contains a ramification point over a_l or b_l, any pair of closed discs $\overline{\Delta}_l''$ being mutually disjoint. Indeed, for any $D_i = D$, $r \geq \sum_{k=1}^{\nu'} r_k' + \sum_{k=1}^{\nu''} \widetilde{r}_k$, where $\widetilde{r}_k = r_k''$ if $n_k'' > 2$ but $\widetilde{r}_k = r_k'' + \frac{1}{2}(\widetilde{f}_{a_l}^k + \widetilde{f}_{b_l}^k)$ if $n_k'' = 2$ and $\widetilde{f}_p^k$, $p = a_l$ or b_l, is defined as follows: $\widetilde{f}_p^k = 1$ when $f_p^k = 1$ and the corresponding fold ends at a ramification point P of the covering $(D, T|, T(D))$, $T(P) = p$, and $\widetilde{f}_p^k = 0$ otherwise. A simple analysis of the ramification in the three possible cases: $f_{a_l}^k + f_{b_l}^k = 0$, 1 or 2 shows that $\widetilde{r}_k \geq 1/2 = (n_k'' - 1)/2$ for $n_k'' = 2$ while $\widetilde{r}_k = r_k''$ otherwise, and the device from 4.1 applies, leading again to (III).

Remark 2. The example of the total covering (X, T, Y) with $X = \{z \in \mathbb{C} : |z| \leq 1\}$, $Y = \{w \in \mathbb{C}^+ : |w| \leq 1\}$ and $T : w = \varphi(z^m)$, m an integer ≥ 2, shows that the inequality (III) is sharp. Indeed, $h'' = 3$ since there are three totally ramified mutually disjoint bordered discs containing the points $w = -1$, 0 and 1 respectively. On the other hand, $D_i = X$, $n = 2m$, $c = c' = 1$ and (III) gives $h = h'' \leq 4 - 4/(2m)$, hence $h'' \leq 3$.

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