On minimal periods of functional-differential equations and difference inclusions

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Abstract. We prove several results on lower bounds for the periods of periodic solutions of some classes of functional-differential equations in Hilbert and Banach spaces and difference inclusions in Hilbert spaces.


\[ \dot{x}(t) = F(x(t), x(\tau(t)), t). \]

He also studied this problem for difference equations of the form

\[ x_{n+1} - x_n = f(x_n, x_{n-1}, n). \]

His method differs from that used in the above mentioned papers. The problem of the existence of a lower bound for the periods of periodic solutions of difference equations has also been studied by S. Busenberg, M. Martelli and D. Fisher (see [1]–[3]).

We give a new approach to the problem of finding a lower bound for the periods of periodic solutions of functional-differential equations which we apply to equations of the form

\[ \dot{x}(t) = f_1(x(\tau_1(t))) + \ldots + f_m(x(\tau_m(t))). \]

If we take the equation (3) with \( \tau_1(t) \equiv t, f_3, \ldots, f_m \equiv 0 \) we obtain an equation of the form (1). Our result concerning the latter equation is weaker than that proved in [7]. Probably it is possible to prove a stronger result concerning (3) with several delays by another method; however, our method is simple and we shall show that it is also suitable for a class of difference inclusions.
We also prove some results on functional-differential equations by the method developed in [1]–[3].

1. Bounds for periods of functional-differential equations

Theorem 1. Let $H$ be a Hilbert space, let $f_i : H \rightarrow H$, $i = 1, \ldots, m$, be Lipschitz mappings with Lipschitz constant $L > 0$ and let $\tau_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \ldots, m$, be continuously differentiable, strictly monotone functions. If the equation (3) has a nonconstant, $T$-periodic solution then

$$T \geq 2/KLm,$$

where $K = K_1K_2$, $K_1 = \max\{|\dot{\tau}_i(t)|^{-1} : 0 \leq t \leq T, 1 \leq i \leq m\}$, $K_2 = \max\{|\dot{\tau}_i(s)| : 0 \leq s \leq T, 1 \leq i \leq m\}$.

Proof. Let $\varphi(t)$ be a $T$-periodic, nonconstant solution of (3) and $d = \max\{\|\varphi(t) - \varphi(s)\| : 0 \leq t, s \leq T\}$, where $\|u\| = (u, u)^{1/2}$, $(\cdot, \cdot)$ is the scalar product in $H$. Then there exist $x_0, y_0 \in \gamma := \{\varphi(t) : 0 \leq t \leq T\}$ such that $d = \|x_0 - y_0\|$. Obviously, there are $t_1, t_2 \in [0, T]$ such that $\varphi(t_1) = x_0$, $\varphi(t_2) = y_0$ (we assume $t_1 < t_2$), and so

$$d^2 = (y_0 - x_0, y_0 - x_0) = (y_0 - x_0, \varphi(t_2) - \varphi(t_1)) = \left(y_0 - x_0, \int_{t_1}^{t_2} (f_1(\varphi(\tau_1(s))) + \ldots + f_m(\varphi(\tau_m(s))))ds\right).$$

Thus

$$d^2 = \sum_{i=1}^{m} \int_{t_1}^{t_2} (y - x_0, f_i(\varphi(\tau_i(s))))ds.$$

Since $\tau_i \in C^1$ and it is strictly monotone, we have

$$\int_{t_1}^{t_2} (y_0 - x_0, f_i(\varphi(\tau_i(s))))ds = \int_{\tau_i(t_1)}^{\tau_i(t_2)} (y_0 - x_0, f_i(\varphi(r)))(\dot{\tau}_i^{-1}(r))^{-1}dr$$

$$\leq \max_{x \in \gamma} (y_0 - x_0, f_i(x)) \int_{\tau_i(t_1)}^{\tau_i(t_2)} (\dot{\tau}_i^{-1}(r))^{-1}dr$$

$$\leq \max_{x \in \gamma} (y_0 - x_0, f_i(x)) \cdot \max_{0 \leq t \leq T} (|\dot{\tau}_i(t)|^{-1})|\tau_i(t_2) - \tau_i(t_1)|$$

$$\leq \max_{x \in \gamma} (y_0 - x_0, f_i(x)) \cdot \max_{0 \leq s \leq T} (|\dot{\tau}_i(s)|^{-1}) \cdot \max_{0 \leq t \leq T} |\tau_i(t_2) - \tau_i(t_1)|.$$

Thus we have proved that

$$d^2 \leq KT \sum_{i=1}^{m} \max_{x \in \gamma} (y_0 - x_0, f_i(x)).$$
where $K$ is defined as in the theorem.

Let $\max_{x \in \gamma}(y_0 - x_0, f_i(x)) = (y_0 - x_0, f_i(x_i))$ for some $x_i \in \gamma$. Then we can write (6) in the form

$$d^2 \leq KT \sum_{i=1}^{m} (y_0 - x_0, f_i(x_i)).$$  

If we change the roles of $x_0$ and $y_0$ we obtain

$$d^2 \leq KT \sum_{i=1}^{m} (x_0 - y_0, f_i(y_i))$$

for some $y_i \in \gamma$. From (7), (8) we get

$$2d^2 \leq KT(y_0 - x_0, \sum_{i=1}^{m} (f_i(x_i) - f_i(y_i))) \leq KT\|y_0 - x_0\| \sum_{i=1}^{m} L\|x_i - y_i\|$$

$$\leq KLTm\|y_0 - x_0\|^2 = KLTmd^2.$$ 

This inequality immediately yields (4).

Now consider the equation

$$\frac{d^n x(t)}{dt^n} = F(x(\tau_1(t)), \ldots, x(\tau_m(t))), \quad x \in B,$$

where $B$ is a Banach space. As a direct consequence of [3, Lemma 3.1] we obtain

**Lemma.** Let $B$ be a Banach space and let $y : \mathbb{R} \to B$ be a $T$-periodic mapping of class $C^{n-1}$ with $\|y^{(n)}(t)\|$ integrable. Then

$$\int_{0}^{T} \int_{0}^{T} \|y(t) - y(s)\| \, ds \, dt \leq \frac{T}{6} \int_{0}^{T} \int_{0}^{T} \|y^{(n)}(t) - y^{(n)}(s)\| \, ds \, dt$$

$(y^{(n)} := \frac{d^n y}{dt^n})$.

**Theorem 2.** Let $B$ be a Banach space, let $F : B \times \ldots \times B \to B$ satisfy the Lipschitz condition

$$\|F(x_1, \ldots, x_m) - F(y_1, \ldots, y_m)\| \leq L \sum_{i=1}^{m} \|x_i - y_i\|$$

for all $x_i, y_i \in B$, and let $\tau_i : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, m$, be continuously differentiable, strictly monotone functions with $|\dot{\tau}_i(t)| \leq 1$ for all $t \in \mathbb{R}$. If the equation (9) has a nonconstant, $T$-periodic solution $x(t)$ then

$$T \geq 6(Lm)^{-1/n}.$$
Proof. The Lemma and (10) yield
\[
\int_0^T \int_0^T \|x(t) - x(s)\| \, ds \, dt \leq L (T/6)^n \sum_{i=1}^m \int_0^T \int_0^T \|x(\tau_i(t)) - x(\tau_i(s))\| \, ds \, dt
\]
\[
\leq L (T/6)^n \sum_{i=1}^m \int_0^T \int_0^T \|x(p) - x(q)\| \, dp \, dq
\]
\[
\leq mL (T/6)^n \int_0^T \int_0^T \|x(p) - x(q)\| \, dp \, dq.
\]
This implies that \( mL(T/6)^n \geq 1 \) and (11) follows.

Now we use the above lemma to solve the problem of finding a lower estimate for the periods of periodic solutions of an equation of the form
\[
\dot{x}(t) = G(x(t), x^2(t), \ldots, x^m(t)),
\]
where \( G : \mathbb{R}^m \to \mathbb{R} \), \( x : \mathbb{R} \to \mathbb{R} \), and \( x^i(t) = (x \circ \ldots \circ x)(t) \) is the \( i \)th iteration of \( x \).

**Theorem 3.** Let \( G : \mathbb{R}^m \to \mathbb{R} \) satisfy the Lipschitz condition
\[
|G(x_1, \ldots, x_m) - G(y_1, \ldots, y_m)| \leq L \sum_{i=1}^m |x_i - y_i|
\]
for all \( x_i, y_i \in \mathbb{R} \) and suppose there is a constant \( M > 0 \) such that \( |G(u)| \leq M \) for all \( u \in \mathbb{R}^m \). If the equation (12) has a nonconstant, \( T \)-periodic solution \( x(t) \) then
\[
T \geq 6(M^{m-1} - 1)((M - 1)L)^{-1} \quad \text{if } M \neq 1,
\]
\[
T \geq 6(Lm)^{-1} \quad \text{if } M = 1.
\]

We shall formulate and prove a more general theorem concerning functional-differential equations of the form appearing in ecological models (see e.g. [6]).

Consider the functional-differential equation
\[
\dot{x}(t) = g(J^{k_1}(G_1 \circ x)(t), \ldots, J^{k_m}(G_m \circ x)(t)),
\]
where \( g : \mathbb{R}^{k_1+1} \times \ldots \times \mathbb{R}^{k_m+1} \to B \), \( B \) is a Banach space, \( G_i : B \to \mathbb{R} \), \( G_i \circ x \) is the composition of \( G_i \) and \( x \) (\( i = 1, \ldots, m \)) and \( (G_i \circ x)^j \) is the \( j \)th iteration of \( G_i \circ x \), \( J^p y(t) := (y(t), y^2(t), \ldots, y^p(t)) \), \( y^i(t) \) is the \( i \)th iteration of \( y(t) \).

**Theorem 4.** Let \( B \) be a Banach space, \( X = \mathbb{R}^{k_1+1} \times \ldots \times \mathbb{R}^{k_m+1} \), and let \( g : X \to B \) be a mapping satisfying the Lipschitz condition
\[
\|g(x) - g(y)\|_B \leq L \|x - y\| \quad \text{for all } x, y \in X,
\]

where $L > 0$, $\| \cdot \|$ is the norm on $X$, $\| \cdot \|_B$ is the norm on $B$ and there is a constant $M > 0$ such that $\| g(x) \|_B \leq M$ for all $x \in X$. Let $G_i : B \to R$, $i = 1, \ldots, m$, be continuously differentiable functions with $|DG_i(u)v| \leq M_i \| v \|$ for all $u, v \in B$, $i = 1, \ldots, m$, where $DG_i(u) \in L(B, R)$ is the Fréchet derivative of $G_i$ at $u$ and $M_i > 0$. If the equation (15) has a nonconstant, $T$-periodic solution $x(t)$ then

$$T \geq 6(LS)^{-1},$$

where

$$S = L \sum_{i=1}^{m} [M_i(M_iM - 1)][(M_iM)^{k_i} - 1]^{-1} \quad \text{if } M_iM \neq 1,$$

$$S = \sum_{i=1}^{m} M_i k_i \quad \text{if } M_iM = 1.$$

**Proof.** The Lemma, the condition (16), the boundedness of $g$ and $DG_i$ and the mean value theorem for mappings of Banach spaces imply

$$\int_0^T \int_0^T \| x(t) - x(s) \|_B \, dt \, ds \leq L(T/6) \int_0^T \int_0^T \| x(t) - x(s) \|_B \, dt \, ds$$

$$\leq L(T/6) \int_0^T \int_0^T [|((G_1 \circ x)(t) - (G_1 \circ x)(s)) + \ldots$$

$$+ |((G_1 \circ x)^{k_1}(t) - (G_1 \circ x)^{k_1}(s))| + \ldots$$

$$+ |(G_m \circ x)(t) - (G_m \circ x)(s))| + \ldots$$

$$+ |(G_m \circ x)^{k_m}(t) - (G_m \circ x)^{k_m}(s)|] \, dt \, ds$$

$$\leq L(T/6)S \int_0^T \int_0^T \| x(t) - x(s) \|_B \, dt \, ds.$$

From this inequality we obtain (17).

**2. Bounds for periods of difference inclusions.** Consider the difference inclusion

$$(18) \quad z_{i+1} - z_i \in F(z_i),$$

where $F : U \to H^c$, $U \subset H$, $H$ is a Hilbert space and $H^c$ is the set of all compact subsets of $H$. We shall use the Hausdorff metric on $H^c$ defined as follows:

$$h(A, B) = \max \{ r(A, B), r(B, A) \}, \quad A, B \in H^c,$$

where $r(A, B) = \max_{x \in A} d(x, B)$, $d(x, B) = \inf \{ \| x - y \| : y \in B \}$, $\| u \| = (u, u)^{1/2}$, $u \in H$. 

Theorem 5. Let \( H \) be a Hilbert space, \( U \subset H \), and let \( F : U \to H^c \) be a multivalued mapping satisfying the following hypotheses:

(H1) If \( x, y \in U \), \( x \neq y \) and \( F(x) \cap F(y) \neq \emptyset \) then \( \text{diam}(F(x) \cap F(y)) \leq h(F(x), F(y)) \), where \( \text{diam} X \) is the diameter of the set \( X \).

(H2) \( h(F(x), F(y)) \leq L \| x - y \| \) for all \( x, y \in U \), where \( L > 0 \) is a constant.

Let \( \gamma = \{ x_0, x_1, \ldots, x_{N-1} \} \) be any \( N \)-periodic orbit of the inclusion (18) satisfying

\[
\max_{x \in \gamma} \text{diam} F(x) \leq 3L \text{diam} \gamma.
\]

Then

\[ N \geq 2/3L. \]

Proof. Let \( d = \text{diam} \gamma \). Then there exist \( i, j \in \{ 0, 1, \ldots, N-1 \} \) such that \( d = \| x_j - x_i \| \). Assume \( j > i \). Then

\[ d^2 = (x_j - x_i, (x_j - x_{j-1}) + (x_{j-1} - x_{j-2}) + \ldots + (x_{i+1} - x_i)). \]

The mapping \( F \) is compact valued and therefore there exist \( u_1 \in \gamma \) and \( y_1 \in F(u_1) \) such that

\[ (x_j - x_i, y_1) = \max_{z \in F(u_1)} (x_j - x_i, z) = \max_{x \in \gamma} \max_{y \in F(x)} (x_j - x_i, y). \]

Since \( x_{k+1} - x_k \in F(x_k) \) for \( k = i, i + 1, \ldots, j - 1 \) we obtain from (20), (21)

\[ d^2 \leq \max_{y \in F(x_{j-1})} (x_j - x_i, y) + \ldots + \max_{y \in F(x_i)} (x_j - x_i, y) \]
\[ \leq N(x_j - x_i, y_1). \]

Obviously

\[ x_i - x_j = \sum_{m=0}^{j-1} (x_{m+1} - x_m) + \sum_{n=j}^{N-2} (x_{n+1} - x_n) + x_0 - x_{N-1}. \]

There exist \( u_2 \in \gamma \) and \( y_2 \in F(u_2) \) such that

\[ (x_i - x_j, y_2) = \max_{z \in F(u_2)} (x_i - x_j, z) = \max_{x \in \gamma} \max_{y \in F(x)} (x_i - x_j, y). \]

From (23), (24) we obtain

\[ d^2 \leq \sum_{m=0}^{i-1} \max_{y \in F(x_m)} (x_i - x_j, y) + \sum_{n=j}^{N-2} \max_{y \in F(x_n)} (x_i - x_j, y) \]
\[ + \max_{y \in F(x_{N-1})} (x_i - x_j, y) \]
\[ \leq N(x_j - x_i, y_1) = N(x_j - x_i, y_2). \]

The inequalities (22), (25) imply

\[ 2d^2 \leq N \| x_j - x_i \| \| y_1 - y_2 \|. \]
We shall prove that
\[(27) \quad \|y_1 - y_2\| \leq 3L\|x_j - x_i\|.
\]
If \(F(u_1) = F(u_2)\) then \(\|y_1 - y_2\| \leq \text{diam} F(u_1)\). By the hypothesis (H3) we have \(\text{diam} F(u_1) \leq 3L\|x_j - x_i\|\), i.e. (27) holds. Let \(F(u_1) \neq F(u_2)\). There exist \(z_1, z_2 \in F(u_1), v_1, v_2 \in F(u_2)\) such that \(h(F(u_1), F(u_2)) = \max\{d_1, d_2\}\), where \(d_1 = \|z_1 - v_1\| = r(F(u_2), F(u_1)), d_2 = \|z_2 - v_2\| = r(F(u_1), F(u_2))\).

First we assume that \(F(u_1) \cap F(u_2) = \emptyset\). Then obviously
\[(28) \quad \|y_1 - v_1\| \leq \|z_1 - v_1\| = d_1 \leq h(F(u_1), F(u_2)),
\]
\[(29) \quad \|y_2 - z_2\| \leq \|z_2 - v_2\| = d_2 \leq h(F(u_1), F(u_2)),
\]
\[(30) \quad \|v_1 - v_2\| \leq d_2 \leq h(F(u_1), F(u_2)).
\]
From these inequalities and the hypothesis (H2) we obtain
\[(31) \quad \|y_1 - y_2\| \leq 3h(F(u_1), F(u_2)) \leq 3L\|u_1 - u_2\| \leq 3L\|x_j - x_i\|.
\]
Let now \(F(u_1) \cap F(u_2) \neq \emptyset\). If \(y_1, y_2 \in F(u_1) \cap F(u_2)\) then from (H1), (H2) it follows that
\(\|y_1 - y_2\| \leq \text{diam}(F(u_1) \cap F(u_2)) \leq h(F(u_1), F(u_2)) \leq 3L\|u_1 - u_2\| \leq 3L\|x_j - x_i\|.
\]
If \(y_1 \in F(u_1) \setminus F(u_2)\) and \(y_2 \in F(u_2) \setminus F(u_1)\) then (28) and (29) obviously hold and (H1) implies that
\(\|v_1 - z_2\| \leq \text{diam}(F(u_1) \cap F(u_2)) \leq h(F(u_1), F(u_2)).
\]
Therefore (31), and hence (27) holds. If \(y_1 \in F(u_1) \setminus F(u_2)\) and \(y_2 \in F(u_1) \cap F(u_2)\) then (28) obviously holds and (H1) yields
\(\|y_2 - z_2\| \leq \text{diam}(F(u_1) \cap F(u_2)) \leq h(F(u_1), F(u_2)),
\]
\(\|v_1 - v_2\| \leq \text{diam}(F(u_1) \cap F(u_2)) \leq h(F(u_1), F(u_2)).
\]
Therefore (27) again holds. The inequalities (26), (27) yield
\[2d^2 \leq 3LN\|x_j - x_i\|^2 = 3LN^2d^2
\]
and this implies (19).

**Example 1.** Let \(f: [a, b] \to \mathbb{R}\) be a continuously differentiable function, \(a < b\), \(c \in (a, b), A = [b, c - \delta] \cup [c + \delta, b], U = A \cup \{c\}\). Define a multivalued mapping \(F: U \to \mathbb{R}^c\) (\(\mathbb{R}^c\) is the set of all compact subsets of \(\mathbb{R}\)) as follows:
\(F(x) = \{f(x) - x\}\) if \(x \in A\) and \(F(c) = I_c := [\alpha - \varepsilon, \alpha], \alpha = f(c) - c\) and \(0 < \varepsilon\). If \(x, y \in A\) then the mean value theorem implies that \(h(F(x), F(y)) = |f(x) - x - (f(y) - y)| \leq p|x - y|\), where \(p = \max_{x \in A} |f'(x)| + 1\). If \(x \in A\) then there exists \(u \in I_c\) such that \(h(F(x), F(c)) = |f(x) - x - u|\). Define \(k(x, y) = |f(x) - x - u|\), \(q = 0\) and therefore \(|f(x) - x - u| \leq q|x - c|\) for all \(x \in A\). Thus
we have proved that $h(F(x), F(c)) \leq q|x - c|$ for all $x \in A$. If $L = \max(p, q)$ then $h(F(x), F(y)) \leq L|x - y|$ for all $x, y \in U$, i.e. $F$ satisfies (H2). Since $F$ is single-valued on $A$ the hypothesis (H1) is trivially satisfied. By Theorem 5 if $\gamma$ is an $N$-periodic trajectory of (18) and $\text{diam}\gamma \geq \varepsilon/3L$ then $N \geq 2/3L$. We remark that if $x_1 \neq c$ and $x_1 \notin (c - \delta, c + \delta)$ then $x_{i+1} - x_i \in F(x_i)$ if and only if $x_{i+1} = f(x_i)$.

**Example 2.** Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function, $a < b$, $c_1, c_2 \in (a, b)$, $c_1 < c_2$, $f(c_1) - c_1 = f(c_2) - c_2$, $A = [a, c_1 - \delta] \cup [c_1 + \delta, c_2 - \delta] \cup [c_2 + \delta, b]$, $0 < \delta < c_1$, $\delta < c_2 - c_1$, $\delta < b - c_2$, $U = A \cup \{c_1\} \cup \{c_2\}$. Define $F : U \to \mathbb{R}^c$ by $F(x) = \{f(x) - x\}$ if $x \in A$, $F(c_1) = L = [\beta - \varepsilon, \beta]$, $F(c_2) = I_{3\varepsilon} = [\beta - 3\varepsilon, \beta]$, $\beta = f(c_1) - c_1$. As above one can show that (H2) is satisfied, where $L = \max_{x \in A} |f'(x)+1$. Obviously, if $x \in U$, $x \notin c_1, c_2$, $F(x) \cap F(c_i) \neq \emptyset$ ($i = 1, 2$) then $\text{diam}(F(x) \cap F(c_i)) = 0$ and $\text{diam}(F(c_1) \cap F(c_2)) = \varepsilon < h(F(c_1), F(c_2)) = 2\varepsilon$, i.e. (H1) is satisfied. Theorem 5 implies that if $\gamma$ is an $N$-periodic trajectory and $\varepsilon \leq L \text{diam}\gamma$ then $N \geq 2/3L$.

**References**


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