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On minimal periods of functional-differential equations and difference inclusions

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Abstract. We prove several results on lower bounds for the periods of periodic solutions of some classes of functional-differential equations in Hilbert and Banach spaces and difference inclusions in Hilbert spaces.

Introduction. In this paper we give a simple method for finding lower bounds for the periods of periodic solutions of some classes of functional-differential equations and difference inclusions in Hilbert spaces. First results on lower bounds of differential and functional-differential equations were proved by J. A. Yorke [8], A. Lasota and J. A. Yorke [4] and T. Y. Li [5]. Recently W. Słomczyński [7] gave a generalization of Theorem 4 of Lasota and Yorke [4] to delay differential equations of the form

$$\dot{x}(t) = F(x(t), x(\tau(t)), t)\,. \tag{1}$$

He also studied this problem for difference equations of the form

$$x_{n+1} - x_n = f(x_n, x_{n-1}, n)\,. \tag{2}$$

His method differs from that used in the above mentioned papers. The problem of the existence of a lower bound for the periods of periodic solutions of difference equations has also been studied by S. Busenberg, M. Martelli and D. Fisher (see [1]–[3]).

We give a new approach to the problem of finding a lower bound for the periods of periodic solutions of functional-differential equations which we apply to equations of the form

$$\dot{x}(t) = f_1(x(\tau_1(t))) + \ldots + f_m(x(\tau_m(t)))\,. \tag{3}$$

If we take the equation (3) with $\tau_1(t) \equiv t$, $f_3, \ldots, f_m \equiv 0$ we obtain an equation of the form (1). Our result concerning the latter equation is weaker than that proved in [7]. Probably it is possible to prove a stronger result concerning (3) with several delays by another method; however, our method is simple and we shall show that it is also suitable for a class of difference inclusions.

We also prove some results on functional-differential equations by the method developed in [1]–[3].

1. Bounds for periods of functional-differential equations

THEOREM 1. *Let H be a Hilbert space, let $f_i : H \to H$, $i = 1, \ldots, m$, be Lipschitz mappings with Lipschitz constant $L > 0$ and let $\tau_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, be continuously differentiable, strictly monotone functions. If the equation* (3) *has a nonconstant, T-periodic solution then*

$$T \geq 2/KLm\,, \tag{4}$$

where $K = K_1K_2$, $K_1 = \max\{|\dot\tau_i(t)|^{-1} : 0 \leq t \leq T,\ 1 \leq i \leq m\}$, $K_2 = \max\{|\dot\tau_i(s)| : 0 \leq s \leq T,\ 1 \leq i \leq m\}$.

Proof. Let $\varphi(t)$ be a T-periodic, nonconstant solution of (3) and $d = \max\{\|\varphi(t) - \varphi(s)\| : 0 \leq t,\ s \leq T\}$, where $\|u\| = (u,u)^{1/2}$, $(\cdot,\cdot)$ is the scalar product in H. Then there exist $x_0, y_0 \in \gamma := \{\varphi(t) :\ 0 \leq t \leq T\}$ such that $d = \|x_0 - y_0\|$. Obviously, there are $t_1, t_2 \in [0,T]$ such that $\varphi(t_1) = x_0$, $\varphi(t_2) = y_0$ (we assume $t_1 < t_2$), and so

$$\begin{aligned} d^2 &= (y_0 - x_0, y_0 - x_0) = (y_0 - x_0, \varphi(t_2) - \varphi(t_1)) \\ &= \Big(y_0 - x_0,\ \int_{t_1}^{t_2} (f_1(\varphi(\tau_1(s))) + \ldots + f_m(\varphi(\tau_m(s))))ds\Big). \end{aligned}$$

Thus

$$d^2 = \sum_{i=1}^{m} \int_{t_1}^{t_2} (y - x_0, f_i(\varphi(\tau_i(s))))ds\,. \tag{5}$$

Since $\tau_i \in C^1$ and it is strictly monotone, we have

$$\begin{aligned} \int_{t_1}^{t_2} (y_0 - x_0, f_i(\varphi(\tau_i(s))))\,ds &= \int_{\tau_i(t_1)}^{\tau_i(t_2)} (y_0 - x_0, f_i(\varphi(r))(\dot\tau_i(\tau_i^{-1}(r)))^{-1}\,dr \\ &\leq \max_{x\in\gamma}(y_0 - x_0, f_i(x))\Big|\int_{\tau_i(t_1)}^{\tau_i(t_2)} (\dot\tau_i(\tau_i^{-1}(r)))^{-1}\,dr\Big| \\ &\leq \max_{x\in\gamma}(y_0 - x_0, f_i(x)) \cdot \max_{0\leq t\leq T}(|\dot\tau_i(t)|^{-1})|\tau_i(t_2) - \tau_i(t_1)| \\ &\leq \max_{x\in\gamma}(y_0 - x_0, f_i(x)) \cdot \max_{0\leq t\leq T}(|\dot\tau_i(t)|^{-1}) \cdot \max_{0\leq s\leq T}|\dot\tau_i(s)||t_2 - t_1|\,. \end{aligned}$$

Thus we have proved that

$$d^2 \leq KT \sum_{i=1}^{m} \max_{x\in\gamma}(y_0 - x_0, f_i(x))\,, \tag{6}$$

where K is defined as in the theorem.

Let $\max_{x\in\gamma}(y_0 - x_0, f_i(x)) = (y_0 - x_0, f_i(x_i))$ for some $x_i \in \gamma$. Then we can write (6) in the form

$$d^2 \le KT\sum_{i=1}^{m}(y_0 - x_0, f_i(x_i))\,. \tag{7}$$

If we change the roles of x_0 and y_0 we obtain

$$d^2 \le KT\sum_{i=1}^{m}(x_0 - y_0, f_i(y_i)) \tag{8}$$

for some $y_i \in \gamma$. From (7), (8) we get

$$\begin{aligned} 2d^2 &\le KT(y_0 - x_0, \sum_{i=1}^{m}(f_i(x_i) - f_i(y_i))) \le KT\|y_0 - x_0\|\sum_{i=1}^{m} L\|x_i - y_i\| \\ &\le KLTm\|y_0 - x_0\|^2 = KLTmd^2\,. \end{aligned}$$

This inequality immediately yields (4).

Now consider the equation

$$\frac{d^n x(t)}{dt^n} = F(x(\tau_1(t)), \ldots, x(\tau_m(t)))\,, \qquad x \in B\,, \tag{9}$$

where B is a Banach space. As a direct consequence of [3, Lemma 3.1] we obtain

Lemma. *Let B be a Banach space and let $y : \mathbb{R} \to B$ be a T-periodic mapping of class C^{n-1} with $\|y^{(n)}(t)\|$ integrable. Then*

$$\int_0^T\int_0^T \|y(t) - y(s)\|\,dsdt \le (T/6)^n \int_0^T\int_0^T \|y^{(n)}(t) - y^{(n)}(s)\|\,dsdt$$

$(y^{(n)} := d^n y/dt^n)$.

Theorem 2. *Let B be a Banach space, let $F : B \times \ldots \times B \to B$ satisfy the Lipschitz condition*

$$\|F(x_1, \ldots, x_m) - F(y_1, \ldots, y_m)\| \le L\sum_{i=1}^{m}\|x_i - y_i\| \tag{10}$$

for all $x_i, y_i \in B$, and let $\tau_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, be continuously differentiable, strictly monotone functions with $|\dot{\tau}_i(t)| \le 1$ for all $t \in \mathbb{R}$. If the equation (9) has a nonconstant, T-periodic solution $x(t)$ then

$$T \ge 6(Lm)^{-1/n}\,. \tag{11}$$

P r o o f. The Lemma and (10) yield

$$\begin{aligned}\int_0^T\int_0^T \|x(t)-x(s)\|\,ds\,dt &\le L(T/6)^n\sum_{i=1}^m\int_0^T\int_0^T \|x(\tau_i(t))-x(\tau_i(s))\|\,ds\,dt\\ &\le L(T/6)^n\sum_{i=1}^m\int_{\tau_i(0)}^{\tau_i(T)}\int_{\tau_i(0)}^{\tau_i(T)} \|x(p)-x(q)|\,dp\,dq\\ &\le mL(T/6)^n\int_0^T\int_0^T \|x(p)-x(q)\|\,dp\,dq\,.\end{aligned}$$

This implies that $mL(T/6)^n \ge 1$ and (11) follows.

Now we use the above lemma to solve the problem of finding a lower estimate for the periods of periodic solutions of an equation of the form

$$\dot{x}(t) = G(x(t), x^2(t), \ldots, x^m(t))\,, \tag{12}$$

where $G:\mathbb{R}^m \to \mathbb{R}$, $x:\mathbb{R}\to\mathbb{R}$, and $x^i(t) = (x\circ\ldots\circ x)(t)$ is the ith iteration of x.

THEOREM 3. *Let $G:\mathbb{R}^m\to\mathbb{R}$ satisfy the Lipschitz condition*

$$|G(x_1,\ldots,x_m) - G(y_1,\ldots,y_m)| \le L\sum_{i=1}^m |x_i - y_i| \tag{13}$$

for all $x_i, y_i \in \mathbb{R}$ and suppose there is a constant $M > 0$ such that $|G(u)| \le M$ for all $u \in \mathbb{R}^m$. If the equation (12) *has a nonconstant*, *T-periodic solution $x(t)$ then*

$$\begin{aligned} &T \ge 6(M^{m-1}-1)((M-1)L)^{-1} \qquad \text{if } M \ne 1\,,\\ &T \ge 6(Lm)^{-1} \qquad \text{if } M = 1\,.\end{aligned} \tag{14}$$

We shall formulate and prove a more general theorem concerning functional-differential equations of the form appearing in ecological models (see e.g. [6]).

Consider the functional-differential equation

$$\dot{x}(t) = g(J^{k_1}(G_1\circ x)(t), \ldots, J^{k_m}(G_m\circ x)(t))\,, \tag{15}$$

where $g:\mathbb{R}^{k_1+1}\times\ldots\times\mathbb{R}^{k_m+1}\to B$, B is a Banach space, $G_i : B\to\mathbb{R}$, $G_i\circ x$ is the composition of G_i and x ($i = 1,\ldots,m$) and $(G_i\circ x)^j$ is the jth iteration of $G_i\circ x$, $J^p y(t) := (y(t), y^2(t), \ldots, y^p(t))$, $y^i(t)$ is the ith iteration of $y(t)$.

THEOREM 4. *Let B be a Banach space*, $X = \mathbb{R}^{k_1+1}\times\ldots\times\mathbb{R}^{k_m+1}$, *and let $g : X\to B$ be a mapping satisfying the Lipschitz condition*

$$\|g(x) - g(y)\|_B \le L\|x-y\| \qquad \text{for all } x, y\in X\,, \tag{16}$$

where $L > 0$, $\|\cdot\|$ *is the norm on* X, $\|\cdot\|_B$ *is the norm on* B *and there is a constant* $M > 0$ *such that* $\|g(x)\|_B \le M$ *for all* $x \in X$. *Let* $G_i : B \to R$, $i = 1, \ldots, m$, *be continuously differentiable functions with* $|DG_i(u)v| \le M_i\|v\|$ *for all* $u, v \in B$, $i = 1, \ldots, m$, *where* $DG_i(u) \in L(B, \mathbb{R})$ *is the Fréchet derivative of* G_i *at* u *and* $M_i > 0$. *If the equation* (15) *has a nonconstant*, T*-periodic solution* $x(t)$ *then*

$$T \ge 6(LS)^{-1}, \tag{17}$$

where

$$S = L\sum_{i=1}^{m}[M_i(M_iM - 1)][(M_iM)^{k_i} - 1]^{-1} \qquad \text{if } M_iM \ne 1,$$

$$S = \sum_{i=1}^{m} M_ik_i \qquad \text{if } M_iM = 1.$$

Proof. The Lemma, the condition (16), the boundedness of g and DG_i and the mean value theorem for mappings of Banach spaces imply

$$\begin{aligned}\int_0^T\int_0^T \|x(t) - x(s)\|_B\,dt\,ds &\le L(T/6)\int_0^T\int_0^T \|x(t) - x(s)\|_B\,dt\,ds\\
&\le L(T/6)\int_0^T\int_0^T [|(G_1\circ x)(t) - (G_1\circ x)(s)| + \ldots\\
&\quad + |(G_1\circ x)^{k_1}(t) - (G_1\circ x)^{k_1}(s)| + \ldots\\
&\quad + |(G_m\circ x)(t) - (G_m\circ x)(s)| + \ldots\\
&\quad + |(G_m\circ x)^{k_m}(t) - (G_m\circ x)^{k_m}(s)|]\,dt\,ds\\
&\le L(T/6)S\int_0^T\int_0^T \|x(t) - x(s)\|_B\,dt\,ds.\end{aligned}$$

From this inequality we obtain (17).

2. Bounds for periods of difference inclusions. Consider the difference inclusion

$$z_{i+1} - z_i \in F(z_i), \tag{18}$$

where $F : U \to H^c$, $U \subset H$, H is a Hilbert space and H^c is the set of all compact subsets of H. We shall use the Hausdorff metric on H^c defined as follows:

$$h(A, B) = \max\{r(A, B), r(B, A)\}, \qquad A, B \in H^c,$$

where $r(A, B) = \max_{x\in A} d(x, B)$, $d(x, B) = \inf\{\|x - y\| : y \in B\}$, $\|u\| = (u, u)^{1/2}$, $u \in H$.

THEOREM 5. *Let H be a Hilbert space, $U \subset H$, and let $F : U \to H^c$ be a multivalued mapping satisfying the following hypotheses*:

(H1) *If $x, y \in U$, $x \neq y$ and $F(x) \cap F(y) \neq \emptyset$ then* $\operatorname{diam}(F(x) \cap F(y)) \leq h(F(x), F(y))$, *where* $\operatorname{diam} X$ *is the diameter of the set X.*

(H2) $h(F(x), F(y)) \leq L\|x-y\|$ *for all $x, y \in U$, where $L > 0$ is a constant.*

Let $\gamma = \{x_0, x_1, \ldots, x_{N-1}\}$ be any N-periodic orbit of the inclusion (18) *satisfying*

(H3) $$\max_{x \in \gamma} \operatorname{diam} F(x) \leq 3L \operatorname{diam} \gamma .$$

Then

(19) $$N \geq 2/3L .$$

Proof. Let $d = \operatorname{diam} \gamma$. Then there exist $i, j \in \{0, 1, \ldots, N-1\}$ such that $d = \|x_j - x_i\|$. Assume $j > i$. Then

(20) $$d^2 = (x_j - x_i, (x_j - x_{j-1}) + (x_{j-1} - x_{j-2}) + \ldots + (x_{i+1} - x_i)) .$$

The mapping F is compact valued and therefore there exist $u_1 \in \gamma$ and $y_1 \in F(u_1)$ such that

(21) $$(x_j - x_i, y_1) = \max_{z \in F(u_1)} (x_j - x_i, z) = \max_{x \in \gamma} \max_{y \in F(x)} (x_j - x_i, y) .$$

Since $x_{k+1} - x_k \in F(x_k)$ for $k = i, i+1, \ldots, j-1$ we obtain from (20), (21)

(22) $$\begin{aligned} d^2 &\leq \max_{y \in F(x_{j-1})} (x_j - x_i, y) + \ldots + \max_{y \in F(x_i)} (x_j - x_i, y) \\ &\leq N(x_j - x_i, y_1) . \end{aligned}$$

Obviously

(23) $$x_i - x_j = \sum_{m=0}^{i-1} (x_{m+1} - x_m) + \sum_{n=j}^{N-2} (x_{n+1} - x_n) + x_0 - x_{N-1} .$$

There exist $u_2 \in \gamma$ and $y_2 \in F(u_2)$ such that

(24) $$(x_i - x_j, y_2) = \max_{z \in F(u_2)} (x_i - x_j, z) = \max_{x \in \gamma} \max_{y \in F(x)} (x_i - x_j, y) .$$

From (23), (24) we obtain

(25) $$\begin{aligned} d^2 &\leq \sum_{m=0}^{i-1} \max_{y \in F(x_m)} (x_i - x_j, y) + \sum_{n=j}^{N-2} \max_{y \in F(x_n)} (x_i - x_j, y) \\ &\quad + \max_{y \in F(x_{N-1})} (x_i - x_j, y) \\ &\leq N(x_i - x_j, y_2) = N(x_j - x_i, -y_2) . \end{aligned}$$

The inequalities (22), (25) imply

(26) $$2d^2 \leq N\|x_j - x_i\| \|y_1 - y_2\| .$$

We shall prove that

$$\|y_1 - y_2\| \le 3L\|x_j - x_i\|\,. \tag{27}$$

If $F(u_1) = F(u_2)$ then $\|y_1 - y_2\| \le \operatorname{diam} F(u_1)$. By the hypothesis (H3) we have $\operatorname{diam} F(u_1) \le 3L \operatorname{diam}\gamma = 3L\|x_j - x_i\|$, i.e. (27) holds. Let $F(u_1) \neq F(u_2)$. There exist $z_1, z_2 \in F(u_1)$, $v_1, v_2 \in F(u_2)$ such that $h(F(u_1), F(u_2)) = \max\{d_1, d_2\}$, where $d_1 = \|z_1 - v_1\| = r(F(u_2), F(u_1))$, $d_2 = \|z_2 - v_2\| = r(F(u_1), F(u_2))$.

First we assume that $F(u_1) \cap F(u_2) = \emptyset$. Then obviously

$$\|y_1 - v_1\| \le \|z_1 - v_1\| = d_1 \le h(F(u_1), F(u_2))\,, \tag{28}$$

$$\|y_2 - z_2\| \le \|z_2 - v_2\| = d_2 \le h(F(u_1), F(u_2))\,, \tag{29}$$

$$\|v_1 - z_2\| \le d_2 \le h(F(u_1), F(u_2))\,. \tag{30}$$

From these inequalities and the hypothesis (H2) we obtain

$$\|y_1 - y_2\| \le 3h(F(u_1), F(u_2)) \le 3L\|u_1 - u_2\| \le 3L\|x_j - x_i\|\,. \tag{31}$$

Let now $F(u_1) \cap F(u_2) \neq \emptyset$. If $y_1, y_2 \in F(u_1) \cap F(u_2)$ then from (H1), (H2) it follows that

$$\begin{aligned}\|y_1 - y_2\| &\le \operatorname{diam}(F(u_1) \cap F(u_2)) \le h(F(u_1), F(u_2))\\ &\le 3L\|u_1 - u_2\| \le 3L\|x_j - x_i\|\,.\end{aligned}$$

If $y_1 \in F(u_1) \setminus F(u_2)$ and $y_2 \in F(u_2) \setminus F(u_1)$ then (28) and (29) obviously hold and (H1) implies that

$$\|v_1 - z_2\| \le \operatorname{diam}(F(u_1) \cap F(u_2)) \le h(F(u_1), F(u_2))\,.$$

Therefore (31), and hence (27) holds. If $y_1 \in F(u_1) \setminus F(u_2)$ and $y_2 \in F(u_1) \cap F(u_2)$ then (28) obviously holds and (H1) yields

$$\|y_2 - z_2\| \le \operatorname{diam}(F(u_1) \cap F(u_2)) \le h(F(u_1), F(u_2))\,,$$
$$\|v_1 - z_2\| \le \operatorname{diam}(F(u_1) \cap F(u_2)) \le h(F(u_1), F(u_2))\,.$$

Therefore (27) again holds. The inequalities (26), (27) yield

$$2d^2 \le 3LN\|x_j - x_i\|^2 = 3LNd^2$$

and this implies (19).

Example 1. Let $f : [a, b] \to \mathbb{R}$ be a countinuously differentiable function, $a < b$, $c \in (a, b)$, $A = [0, c - \delta] \cup [c + \delta, b]$, $U = A \cup \{c\}$. Define a multivalued mapping $F : U \to \mathbb{R}^c$ ($\mathbb{R}^c$ is the set of all compact subsets of $\mathbb{R}$) as follows: $F(x) = \{f(x) - x\}$ if $x \in A$ and $F(c) = I_\varepsilon := [\alpha - \varepsilon, \alpha]$, where $\alpha = f(c) - c$ and $0 < \varepsilon$. If $x, y \in A$ then the mean value theorem implies that $h(F(x), F(y)) = |f(x) - x - (f(y) - y)| \le p|x - y|$, where $p = \max_{x\in A} |f'(x)| + 1$. If $x \in A$ then there exists $u \in I_\varepsilon$ such that $h(F(x), F(c)) = |f(x) - x - u|$. Define $k(x, y) = |(f(x) - x - u)(x - c)^{-1}|$ for $(x, y) \in A \times I_\varepsilon$. Then k has a maximum $q \ge 0$ and therefore $|f(x) - x - u| \le q|x - c|$ for all $x \in A$. Thus we have proved that $h(F(x),F(c)) \le q|x-c|$ for all $x \in A$. If $L = \max(p,q)$ then $h(F(x),F(y)) \le L|x-y|$ for all $x,y \in U$, i.e. F satisfies (H2). Since F is single-valued on A the hypothesis (H1) is trivially satisfied. By Theorem 5 if γ is an N-periodic trajectory of (18) and $\operatorname{diam}\gamma \ge \varepsilon/3L$ then $N \ge 2/3L$. We remark that if $x_i \ne c$ and $x_i \notin (c-\delta, c+\delta)$ then $x_{i+1} - x_i \in F(x_i)$ if and only if $x_{i+1} = f(x_i)$.

EXAMPLE 2. Let $f : [a,b] \to \mathbb{R}$ be a continuously differentiable function, $a < b$, $c_1, c_2 \in (a,b)$, $c_1 < c_2$, $f(c_1) - c_1 = f(c_2) - c_2$, $A = [a, c_1 - \delta] \cup [c_1 + \delta, c_2 - \delta] \cup [c_2 + \delta, b]$, $0 < \delta < c_1$, $\delta < c_2 - c_1$, $\delta < b - c_2$, $U = A \cup \{c_1\} \cup \{c_2\}$. Define $F : U \to \mathbb{R}^c$ by $F(x) = \{f(x) - x\}$ if $x \in A$, $F(c_2) = I_\varepsilon = [\beta - \varepsilon, \beta]$, $F(c_1) = I_{3\varepsilon} = [\beta - 3\varepsilon, \beta]$, $\beta = f(c_1) - c_1$. As above one can show that (H2) is satisfied, where $L = \max_{x \in A} |f'(x)| + 1$. Obviously, if $x \in U$, $x \ne c_1, c_2$, $F(x) \cap F(c_i) \ne \emptyset$ $(i = 1,2)$ then $\operatorname{diam}(F(x) \cap F(c_i)) = 0$ and $\operatorname{diam}(F(c_1) \cap F(c_2)) = \varepsilon < h(F(c_1), F(c_2)) = 2\varepsilon$, i. e. (H1) is satisfied. Theorem 5 implies that if γ is an N-periodic trajectory and $\varepsilon \le L \operatorname{diam}\gamma$ then $N \ge 2/3L$.

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