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Nonlinear boundary value problems for differential inclusions $y'' \in F(t, y, y')$

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Abstract. Applying the topological transversality method of Granas and the *a priori* bounds technique we prove some existence results for systems of differential inclusions of the form $y'' \in F(t, y, y')$, where F is a Carathéodory multifunction and y satisfies some nonlinear boundary conditions.

§1. Introduction. In this paper we study the existence of solutions to differential inclusions of the form

$$(*)\qquad \begin{cases} y'' \in F(t, y, y')\,, \\ y \in \mathcal{B}\,, \end{cases}$$

where $F : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction and $\mathcal{B}$ denotes the (in general, nonlinear) boundary conditions. Our approach applies the topological transversality method of Granas and the *a priori* bounds technique for a class of multifunctions F with compact convex values satisfying Carathéodory conditions (cf. [35], [36]).

Differential inclusions have been studied by many authors, for example [1], [9], [6], [7], [5], [17], [28], [35], [36], [39] (see [36] for a historical outline and extensive list of references).

The topological transversality method of Granas and the *a priori* bounds technique have been used before in the study of boundary value problems in [27], [23], [24], [25], [26] (for scalar second order equations with continuous function F), in [17] (for scalar second order equations with multivalued function F) and in [22] (for scalar equations and Carathéodory function F).

In this paper we apply the method of Granas to the study of second order systems of differential inclusions. The results obtained in this way may be viewed as improvements, even in the case where F is a single-valued

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Carathéodory multifunction (cf. [27], [2], [11], [13], [31], [32], [30]). In fact, the classical methods are not always applicable in some situations treated here.

We also show the existence of solutions $y(t)$ to $(*)$ such that $y(t)\cdot P(t)y(t) \le r^2$, where $P(t)$ is a given symmetric positive definite matrix. Such problems were considered in [31], [32], [11] and [14] (see also [16] and [15]). In addition in §6 we apply results obtained in the previous sections to study differential inclusions on the infinite interval $[0,\infty)$.

The problems studied in §6 are related to [27] and [17] where differential equations of second order on an infinite interval were studied by similar methods. (See also [6], [7], [39]).

§2. Preliminaries. Suppose that $\mathbb{E}$ and $\mathbb{F}$ are spaces and $X \subset \mathbb{E}$ and $Y \subset \mathbb{F}$ are subsets. By $\mathcal{K}(Y)$ we denote the family of all nonempty convex and compact subsets of Y. A multivalued map $\varGamma : X \to \mathcal{K}(Y)$ is called *upper semi-continuous* (u.s.c.) if $\{x \in X : \varGamma(x) \subset U\}$ is an open subset of X for any open U in Y. $\varGamma$ is said to be *compact* if $\varGamma(X) = \bigcup\{\varGamma(x) : x \in X\}$ is relatively compact in Y. We denote by $\mathcal{C}$ the class of all multivalued, compact and upper semi-continuous maps $\varGamma$ with *nonempty, closed and convex values.* Let K be a convex subset of the Banach space $\mathbb{E}$. For any bounded closed subset A and B of K, such that $B \subset A$, we denote by $\mathcal{C}_K(A,B)$ the set of all multivalued maps $\varGamma : A \to \mathcal{K}(Y)$ such that (i) $\varGamma \in \mathcal{C}$ and (ii) $x \notin \varGamma(x)$ for all $x \in B$.

Let us present a short outline of the topological transversality method of Granas.

(2.1) Definition. A multivalued map $\varGamma \in \mathcal{C}_K(A,B)$ is called *essential* if for every multivalued map $G \in \mathcal{C}_K(A,B)$ such that $G\,|_B \equiv \varGamma\,|_B$ there exists a fixed point $x \in A$ of G, i.e. $x \in G(x)$.

It is well known (see [10]) that if U is a bounded open subset of K and $p \in U$ then the map $G(x) \equiv \{p\},\ x \in \overline{U}$, is essential in $\mathcal{C}_K(\overline{U}, \partial U)$.

(2.2) Definition. Two multivalued maps $\varGamma, G \in \mathcal{C}_K(A,B)$ are said to be *homotopic* (notation $\varGamma \sim G$) if there is a compact homotopy $H : A \times [0,1] \to \mathcal{K}(K)$ such that $H \in \mathcal{C}$ and (i) $\varGamma \equiv H_0$, $G \equiv H_1$, (ii) $H_\lambda \in \mathcal{C}_K(A,B)$ for all $\lambda \in [0,1]$, where $H_\lambda(x) := H(x,\lambda)$, $x \in A$, $\lambda \in [0,1]$.

(2.3) Theorem (Topological Transversality Theorem). *Let $\varGamma, G \in \mathcal{C}_K(A,B)$ be two homotopic multivalued maps. Then one of these maps is essential if and only if the other is.*

The Topological Transversality Theorem can be reformulated as the following alternative:

(2.4) Corollary. *Suppose that $\Gamma \in \mathcal{C}_K(A,B)$ and let $H : A \times [0,1] \to \mathcal{K}(K)$ be a multivalued homotopy such that $H \in \mathcal{C}$, $H_0 \equiv \Gamma$ and put $G \equiv H_1$. If Γ is essential, then either*

(i) *G is essential, or*
(ii) *there exist $x \in B$ and $\lambda \in [0,1]$ such that $x \in H(x,\lambda)$.*

Suppose that U is a bounded open subset of K. As a consequence of (2.3) we have the following corollaries:

(2.5) Corollary (Nonlinear Alternative). *Suppose that $\Gamma : \overline{U} \to \mathcal{K}(K)$ is a multivalued map such that $\Gamma \in \mathcal{C}$ and let $p \in U$. Then either*

(i) *Γ is essential in $\mathcal{C}_K(\overline{U}, \partial U)$, or*
(ii) *there exist $x \in \partial U$ and $\lambda \in (0,1]$ such that $x \in \lambda\Gamma(x) + (1-\lambda)p$.*

We say that $\Gamma : K \to \mathcal{K}(K)$ is *completely continuous* if $\Gamma|_X \in \mathcal{C}$ for every bounded subset X of K.

(2.6) Corollary (Leray–Schauder Alternative). *Suppose that $0 \in K$ and let $\Gamma : K \to \mathcal{K}(K)$ be a completely continuous multivalued map. Then either*

(i) *Γ has a fixed point in K, i.e. there is $x \in K$ such that $x \in \Gamma(x)$, or*
(ii) *the set $\{x \in K : \exists \lambda \in (0,1)$ with $x \in \lambda\Gamma(x)\}$ is unbounded.*

For more facts concerning *the topological transversality method* for multivalued maps and the proofs of the above results we refer to [10], [17], [36], [21], [30].

In what follows we will consider the following Banach function spaces:

$$C([a,b];\mathbb{R}^m) = \{u : [a,b] \to \mathbb{R}^m : u \text{ is continuous on } [a,b]\}$$

with the norm $\|u\|_\infty = \sup_{t\in[a,b]} \|u(t)\|$ where $\|\cdot\|$ denotes the usual euclidean norm in $\mathbb{R}^m$;

$$L^2([a,b];\mathbb{R}^m) = \{u : [a,b] \to \mathbb{R}^m : \|u(t)\| \text{ is } L^2\text{-integrable}\}$$

with the norm

$$\|u\|_2 = \Big(\int_a^b \|u(t)\|^2\,dt \Big)^{1/2};$$

$$H^k([a,b];\ \mathbb{R}^m) = \{u : [a,b] \to \mathbb{R}^m : u \text{ has weak derivatives } u^{(i)} \in L^2([a,b];\ \mathbb{R}^m) \text{ for } 0 \le i \le k\}$$

with the norm

$$\|u\|_{2;k} = \max\{\|u^{(i)}\|_2 : 0 \le i \le k\}.$$

The spaces $H^k([a,b];\mathbb{R}^m)$ are usual Sobolev spaces of vector functions denoted also by $W^{k,2}([a,b];\mathbb{R}^m)$ (for more details see [3]).

We now introduce the notion of a Carathéodory map or multifunction which will be central in the results to follow.

(2.7) Definition. A multifunction $F : [a,b] \times \mathbb{R}^m \to \mathcal{K}(\mathbb{R}^n)$ is said to be a *Carathéodory multifunction* in case it satisfies the following conditions:

(i) the map $t \to F(t,u)$ is Lebesgue measurable for each $u \in \mathbb{R}^m$;
(ii) the map $u \to F(t,u)$ is u.s.c. for each $t \in [a,b]$;
(iii) for any $r \geq 0$ there is a function $\psi_r \in L^2[a,b]$ such that for all $t \in [a,b]$, $u \in \mathbb{R}^m$ with $\|u\| \leq r$ and $y \in F(t,u)$ we have $\|y\| \leq \psi_r(t)$.

We note that as a consequence of conditions (i) and (ii) it follows (cf. [5]) that for each measurable $u : [a,b] \to \mathbb{R}^m$ the map $t \to F(t,u(t))$ has measurable single-valued selections.

We recall that if $F : [a,b] \times \mathbb{R}^m \to \mathcal{K}(\mathbb{R}^n)$ then the associated Nemytskiĭ operator $N_F : C([a,b];\mathbb{R}^m) \to L^2([a,b];\mathbb{R}^n)$ is given by

$$N_F(u) := \{w \in L^2([a,b];\mathbb{R}^n) : w(t) \in F(t,u(t)) \text{ for a.e. } t \in [a,b]\}.$$

As a consequence of results of [35], [36], if F is a Carathéodory map, then the Nemytskiĭ operator is well defined with nonempty closed convex values and is such that the composed multivalued map $(J \circ N_F)(u) := J(N_F(u))$ is completely continuous for any completely continuous linear operator $J : L^2([a,b];\mathbb{R}^n) \to C([a,b];\mathbb{R}^m)$ (cf. Prop. 1.7 in [35]).

Consider now a function $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^n$ such that $f = (f_1, \ldots, f_n)$, where $f_i : [a,b] \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n$. We define

$$\underline{f}(t,u) := (\underline{f}_1(t,u), \ldots, \underline{f}_n(t,u)) \quad \text{and} \quad \bar{f}(t,u) = (\bar{f}_1(t,u), \ldots, \bar{f}_n(t,u)),$$

where

$$\underline{f}_i(t,u) = \liminf_{y \to u} f_i(t,y) \quad \text{and} \quad \bar{f}_i(t,u) = \limsup_{y \to u} f_i(t,y)$$

for $i = 1, \ldots, n$. Assume that $\underline{f}_i$ and $\bar{f}_i$ are well defined finite-valued functions. We introduce the multivalued function $F : [a,b] \times \mathbb{R}^m \to \mathcal{K}(\mathbb{R}^n)$ defined by

$$F(t,u) := [\underline{f}(t,u), \bar{f}(t,u)] := [\underline{f}_1(t,u), \bar{f}_1(t,u)] \times \cdots \times [\underline{f}_n(t,u), \bar{f}_n(t,u)].$$

(2.8) Definition. We say that the multifunction F is of *type* $\mathcal{M}$ if both $\underline{f}_i(t,u(t))$ and $\bar{f}_i(t,u(t))$ are measurable for every measurable function $u : [a,b] \to \mathbb{R}^m$, $i = 1, \ldots, n$ (cf. [17]).

We note that if the function $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies (i) and (ii) of Definition (2.7), then $\underline{f}_i = \bar{f}_i = f$ and f is evidently of type $\mathcal{M}$. Moreover, by the definition of the functions $\underline{f}_i$ and $\bar{f}_i$ we find that $\underline{f}_i$ is lower semi-continuous and $\bar{f}_i$ is upper semi-continuous with respect to u. Therefore F is upper semi-continuous with respect to the variable u. This implies that if F is of type $\mathcal{M}$, and satisfies the growth condition (iii) of Definition (2.7) then F is a Carathéodory map.

§3. Abstract existence theorems for differential inclusions. Let $a_0 < a_1$ be real numbers and assume that $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a Carathéodory multifunction. Suppose that $g_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1$, are continuous functions and let

$$w_i(\widetilde{y}) := A_{i0}y_0 + A_{i1}y_0' + B_{i0}y_1 + B_{i1}y_1', \qquad i = 0, 1,$$

where $A_{ij}, B_{ij} \in \mathbb{R}^{n^2}$, $i, j = 0, 1$, are matrices and $\widetilde{y} = (y_0, y_0', y_1, y_1') \in \mathbb{R}^{4n}$. Thus, w_i, $i = 0, 1$, are linear operators from $\mathbb{R}^{4n}$ to $\mathbb{R}^n$.

In this section we study the existence problem for solutions $y \in H^2([a_0, a_1];\ \mathbb{R}^n)$ to the differential inclusion

$$Ly \in F(t, y, y') \quad \text{for a.e. } t \in [a_0, a_1],$$

where $Ly = y'' + b(t)y' + c(t)y$, and $b, c : [a_0, a_1] \to \mathbb{R}^{n^2}$ are L^2-functions which satisfy the boundary conditions

$$w_i(\widetilde{y}) = g_i(\widetilde{y}), \qquad i = 0, 1,$$

where $\widetilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1))$. It is clear that L maps continuously $H^2([a_0, a_1]; \mathbb{R}^n)$ into $L^2([a_0, a_1]; \mathbb{R}^n)$.

Let $\mathcal{B}_0$ be the set of all functions $y : [a_0, a_1] \to \mathbb{R}^n$ satisfying the homogeneous boundary conditions: $w_i(\widetilde{y}) = 0$, $i = 0, 1$. We set

$$\begin{aligned} H^2_{\mathcal{B}_0} &:= \{y \in H^2([a_0, a_1]; \mathbb{R}^n) : y \in \mathcal{B}_0\}, \\ L^2 &:= L^2([a_0, a_1]; \mathbb{R}^n)\,. \end{aligned}$$

We shall make the following assumption in some of the results to follow:

(A) The operator $L|_{H^2_{\mathcal{B}_0}} : H^2_{\mathcal{B}_0} \to L^2$ is one-to-one.

The main result of this section is the following existence theorem. It may also be proved using the results of [36]. However, the proof given below is different.

(3.1) Theorem. *Suppose that $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ and $Ly = y'' + b(t)y' + c(t)y$ are such that F is a Carathéodory multifunction and L satisfies the assumption* (A). *If there is a constant $M < \infty$ such that $\|(y, y')\|_0 := \max\{\|y\|_\infty,\ \|y'\|_\infty\} < M$ for all solutions y to the differential inclusion*

$$(2_\lambda) \qquad \begin{cases} Ly \in \lambda F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = \lambda g_i(\widetilde{y}), & i = 0, 1, \end{cases}$$

for $\lambda \in [0, 1]$, then the differential inclusion

$$(2) \qquad \begin{cases} Ly \in F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = g_i(\widetilde{y}), & i = 0, 1, \end{cases}$$

has at least one solution in $H^2([a_0, a_1]; \mathbb{R}^n)$.

Proof. Put

$$C := C([a_0, a_1]; \mathbb{R}^n \times \mathbb{R}^n), \quad H^2 := H^2([a_0, a_1]; \mathbb{R}^n),$$
$$\widetilde{L} : H^2 \to L^2 \times \mathbb{R}^n \times \mathbb{R}^n, \quad \widetilde{L}u = (Lu, w_0(\widetilde{u}), w_1(\widetilde{u})),$$

$j : H^2 \to C$, $j(u) = (u, u')$, where $u \in H^2$ and $\widetilde{u} = (u(a_0), u'(a_0), u(a_1), u'(a_1)) \in \mathbb{R}^{4n}$. By the Ascoli theorem j is a completely continuous linear operator.

Next, we define a multivalued map $\Gamma : C \to L^2 \times \mathbb{R}^n \times \mathbb{R}^n$ by

$$\Gamma(u, v) := \{w \in L^2 : w(t) \in F(t, u(t), v(t)) \text{ a.e.}\} \times \{g_1(\overline{u}, \overline{v})\} \times \{g_2(\overline{u}, \overline{v})\},$$

where $(u, v) \in C$ and $(\overline{u}, \overline{v}) = (u(a_0), v(a_0), u(a_1), v(a_1)) \in \mathbb{R}^{4n}$.

We consider the following diagram:

$$\begin{tikzcd} C \arrow[r, "\Gamma"] & L^2 \times \mathbb{R}^n \times \mathbb{R}^n \\ & H^2 \arrow[ul, "j"] \arrow[u, "\widetilde{L}"'] \end{tikzcd}$$

By the assumption (A) the operator $L|_{H^2_{\mathcal{B}}} : H^2_{\mathcal{B}_0} \to L^2$ is one-to-one, and therefore the linear operator $\widetilde{L} : H^2 \to L^2 \times \mathbb{R}^n \times \mathbb{R}^n$ is an isomorphism. Indeed, by the open mapping theorem it is sufficient to verify that $\widetilde{L}$ is surjective. Let $r_0, r_1 \in \mathbb{R}^n$ and let $y_0 \in H^2$ be the unique solution to the equation $Ly_0 = 0$ which satisfies $w_i(\widetilde{y_0}) = r_i$, $i = 0, 1$. Then $\widetilde{L}^{-1}(x, r_0, r_1) = L^{-1}x + y_0$, where L^{-1} is the inverse of $L : H^2_{\mathcal{B}_0} \to L^2$.

Since the operator $\widetilde{L}$ is invertible, we can define a multivalued map $\mathcal{F} : C \to C$ by $\mathcal{F}(w) := j \circ \widetilde{L}^{-1} \circ \Gamma(w)$, $w \in C$. Since F is a Carathéodory map, $\mathcal{F}$ is a completely continuous multivalued map with nonempty compact convex values, i.e. $\mathcal{F}\,|_X \in \mathcal{C}$ for all bounded sets $X \subset C$.

We wish to solve (2), i.e. we are looking for $y \in H^2$ such that $\widetilde{L}y \in \Gamma(j(y))$, that is, $y \in \widetilde{L}^{-1} \circ \Gamma(j(y))$. It follows that $j(y) \in j \circ \widetilde{L}^{-1} \circ \Gamma(j(y))$ and we see that the problem (2) is equivalent to the fixed point problem

$$w \in \mathcal{F}(w), \quad w = (u, v) \in C.$$

Put $U := \{(u, v) \in C : \|(u, v)\|_0 < M\}$ and let $H : \overline{U} \times [0, 1] \to C$ be the homotopy defined by $H(w, \lambda) = \lambda\mathcal{F}(w)$, $w \in \overline{U}$, $\lambda \in [0, 1]$. The homotopy H is a well defined multivalued homotopy such that $H \in \mathcal{C}$. Suppose that $w \in H(w, \lambda)$ for some $\lambda \in [0, 1]$. Then by definition of H, it follows that $w \in \mathrm{Im}(j)$, thus $w = (u, u')$, and therefore u satisfies the differential inclusion (2_λ). By hypothesis, $\|w\|_0 < M$, thus $w \in U$. Now we can apply Corollary (2.5) or Theorem (2.3) to obtain the existence of a fixed point of $\mathcal{F}$ in the set U. Indeed, for every $\lambda \in (0, 1]$ there is no $w \in \partial U$ such that $w \in \lambda F(w)$, therefore the condition (ii) of Corollary (2.5) cannot be satisfied, and it follows that $\mathcal{F}_0 := \mathcal{F}|_{\overline{U}} : \overline{U} \to C$ is essential in $\mathcal{C}_C(\overline{U}; \partial U)$. ■

We should remark that Theorem (3.1) may be stated in the following, slightly more general form, admitting a broader class of deformations of the boundary conditions. The proof is essentially the same and therefore is omitted.

(3.2) THEOREM. *Let L satisfy* (A) *and let $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction such that $\mathcal{F}_0 : \overline{U} \to C$ (defined in the proof of* (3.1)) *is an essential map in $\mathcal{C}_C(\overline{U}; \partial\overline{U})$. Suppose that $h_i : \mathbb{R}^{4n} \times [0,1] \to \mathbb{R}^n$, $i = 0, 1$, are continuous maps such that $h_i(\cdot, 0) \equiv g_i$ for $i = 0, 1$. If $\|(y, y')\|_0 < M$ for all solutions y to the differential inclusion*

$$(3_\lambda) \qquad \begin{cases} Ly \in F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = h_i(\widetilde{y}, \lambda), & i = 0, 1, \end{cases}$$

for $\lambda \in [0,1]$, then the differential inclusion

$$(3) \qquad \begin{cases} Ly \in F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = h_i(\widetilde{y}, 1), & i = 0, 1, \end{cases}$$

has at least one solution in $H^2([a_0, a_1]; \mathbb{R}^n)$.

Finally, we can apply the Leray–Schauder Alternative (Corollary (2.6)) to obtain the following theorem.

(3.3) THEOREM. *Suppose that $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a Carathéodory multifunction and $Ly = y'' + b(t)y' + c(t)y$ satisfies the assumption* (A). *Let $U \subset C([a_0, a_1]; \mathbb{R}^n \times \mathbb{R}^n)$ be an open and bounded neighborhood of zero. Suppose that for all $\lambda \in (0,1)$ the differential inclusion*

$$(2_\lambda) \qquad \begin{cases} Ly \in \lambda F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = \lambda g_i(\widetilde{y}), & i = 0, 1, \end{cases}$$

has no solution $y \in H^2([a_0, a_1]; \mathbb{R}^n)$ such that $(y, y') \in \partial U$. Then the differential inclusion

$$(2) \qquad \begin{cases} Ly \in F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = g_i(\widetilde{y}), & i = 0, 1, \end{cases}$$

has a solution $y \in H^2([a_0, a_1]; \mathbb{R}^n)$ such that $(y, y') \in \overline{U}$.

P r o o f. We use the same notation as in the proof of (3.1). We have the diagram

$$\begin{tikzcd} C \arrow[r, "\Gamma"] & L^2 \times \mathbb{R}^n \times \mathbb{R}^n \\ & H^2 \arrow[ul, "j"] \arrow[u, "\widetilde{L}"'] \end{tikzcd}$$

and the problem (2) is equivalent to the fixed point problem

$$w = \mathcal{F}(w) := j \circ \widetilde{L}^{-1} \circ \Gamma(w), \qquad w \in C.$$

We have $\mathcal{F}_0 := \mathcal{F}|_{\overline{U}} \in \mathcal{C}$, thus we can apply Corollary (2.6). Since the inclusion $w \in \lambda \mathcal{F}_0(w)$, $\lambda \in (0,1)$, is equivalent to (2_λ), we find that the condition (ii) of (2.6) cannot be satisfied. This implies that $\mathcal{F}_0$ has a fixed point in $\overline{U}$. ∎

§4. Existence results for nonlinear differential inclusions. The proofs for the abstract theorems, presented in §3, depend on finding bounds for the solutions and their derivatives of first order. This section deals with *a priori* bounds which are applied to prove the Main Theorem characterizing the existence of solutions to the differential inclusion $y'' \in F(t,y,y')$ with the nonlinear boundary conditions.

In this section, unless otherwise specified, we shall assume that $F : [a_0,a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$, $a_0 < a_1$, is a Carathéodory multifunction and w_i, $i = 0,1$, denotes one of the following sets of linear boundary operators:

(α) $w_i(\widetilde{y}) = y_i$,

(β) $w_i(\widetilde{y}) = y_i'$,

(γ) $w_0(\widetilde{y}) = Ay_1 - y_0$, or $w_1(\widetilde{y}) = By_1' - y_0'$, where $A \in O(n)$, $B \in GL(n)$, and $x \cdot AB^{-1}y \le 0$ provided $x \cdot y \le 0$,

where $\widetilde{y} = (y_0, y_0', y_1, y_1') \in \mathbb{R}^{4n}$. As in §3, let $\mathcal{B}_0$ denote the set of functions satisfying the homogeneous boundary conditions $w_i(\widetilde{y}) = 0$, $i = 0,1$, and $H^2_{\mathcal{B}_0} := \{y \in H^2([a_0,a_1];\ \mathbb{R}^n) : y \in \mathcal{B}_0\}$, $L^2 := L^2([a_0,a_1];\mathbb{R}^n)$. It is well known that the operator $L_0 : H^2_{\mathcal{B}_0} \subset L^2 \to L^2$, $L_0 y := y''$, has a discrete denumerable spectrum $\sigma(L_0) \subset \mathbb{R}$ (cf. [4]), thus for all small $\varepsilon > 0$ the operator $Ly := y'' - \varepsilon y$ satisfies the assumption (A).

We shall state the following hypothesis, which is a generalization of analogous conditions from [27], [23], [24], [25], [26], [17], [13].

(H1) There exists a constant $R > 0$ such that if $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$ then there is a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[a_0,a_1]} \inf\{y \cdot w + \|y'\|^2 : w \in F(t,y,y'),\ (y,y') \in \mathcal{D}_\delta\} > 0,$$

where $\mathcal{D}_\delta := \{(y,y') \in \mathbb{R}^{2n} : \|y - y_0\| + \|y' - y_0'\| < \delta\}$.

In the case where F is a continuous function (H1) reduces to the classical Nagumo–Hartman condition (cf. [29]) and becomes simply $y \cdot F(t,y,y') + \|y'\|^2 > 0$ if $\|y\| > R$ and $y \cdot y' = 0$. In the scalar case (H1) in a modified form was used in, for example, [27], [23], [24], [25], [26], [22]. In the latter, the authors considered an even more general condition for the case when F is a Carathéodory function. Another example of a multivalued function satisfying (H1) was considered in [12].

The next two lemmas will be used to obtain the necessary *a priori* bounds required in Theorems (3.1) and (3.2).

(4.1) LEMMA. *Let $\varepsilon > 0$ be such that $Ly = y'' - \varepsilon y$ satisfies* (A) *and suppose that F satisfies* (H1). *Let Y be a solution to the differential inclusion $y'' - \varepsilon y \in \lambda[F(t, y, y') - \varepsilon y]$, $\lambda \in [0, 1]$, such that the function $r(t) = \|y(t)\|^2$ achieves its maximum at a point $t_0 \in [a_0, a_1]$ with $r'(t_0) = 0$. Then $\|y(t)\| \leq R$ for all $t \in [a_0, a_1]$.*

Proof. Notice that the multifunction $F_\lambda(t, y, y') := \lambda F(t, y, y') + (1 - \lambda)\varepsilon y$ also satisfies (H1). For, let $w_1 \in F_\lambda(t, y, y')$ and $\lambda w = w_1 - (1 - \lambda)\varepsilon y$, where $w \in F(t, y, y')$. We can suppose that $\lambda > 0$. If $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$ then for $(y, y') \in \mathcal{D}_\delta$, where $\delta > 0$ is given by (H1), we have

$$\begin{aligned} y \cdot w_1 + \|y'\|^2 &= \lambda y \cdot w + (1 - \lambda)\varepsilon\|y\|^2 + \|y'\|^2 \\ &= \lambda(y \cdot w + \|y'\|^2) + (1 - \lambda)[\varepsilon\|y\|^2 + \|y'\|^2] \\ &\geq \lambda[y \cdot w + \|y'\|^2], \end{aligned}$$

thus

$$\begin{aligned} &\operatorname*{ess\,inf}_{t \in [a_0, a_1]} \inf\{y \cdot w_1 + \|y'\|^2 : w_1 \in F_\lambda(t, y, y'),\ (y, y') \in \mathcal{D}_\delta\} \\ &\qquad \geq \lambda \operatorname*{ess\,inf}_{t \in [a_0, a_1]} \inf\{y \cdot w + \|y'\|^2 : w \in F(t, y, y'),\ (y, y') \in \mathcal{D}_\delta\} > 0. \end{aligned}$$

Suppose now that $r(t_0) = \max r(t) > R^2$ and $r'(t_0) = 2y(t_0) \cdot y'(t_0) = 0$. Since $(y(t), y'(t)) \to (y(t_0), y'(t_0))$ as $t \to t_0$, there exists an $\alpha > 0$ and an $\eta > 0$ such that for almost every $t \in A_\eta := \{t \in [a_0, a_1] : |t_0 - t| < \eta\}$

$$\inf\{y(t) \cdot w + \|y'\|^2 : w \in F_\lambda(t, y(t), y'(t))\} > \alpha > 0$$

and therefore

$$\tfrac{1}{2} r''(t) = y(t) \cdot y''(t) + \|y'(t)\|^2 > 0$$

for almost every $t \in A_\eta$. But this contradicts the maximum principle (see [3]). ∎

(4.2) LEMMA. *Let $\phi : (0, \infty) \to [0, \infty)$ be a function such that $s/\phi(s) \in L^\infty_{\text{loc}}[0, \infty)$ and let α, K, R, τ be nonnegative constants such that*

$$\int_{M_1}^{\infty} \frac{s\,ds}{\phi(s)} > \frac{T}{2} M_1 + 2\alpha R^2, \tag{4}$$

$$\text{where } M_1 := \frac{4R(1 + \alpha R)}{T} + \frac{KT}{4}\,,\ T = a_1 - a_0.$$

Then there exists a constant M (depending only on $\phi(s)$, α, R, τ, K) with the following property. Suppose $x \in H^2([a_0, a_1]; \mathbb{R}^n)$ satisfies

$$\|x''(t)\| \leq \phi(\|x'(t)\|) \quad \text{for a.e. } t \in [a_0, a_1],\ a_1 - a_0 \geq \tau\,, \tag{5}$$

$$\|x(t)\| \le R \quad \textit{for all } t \in [a_0, a_1],$$
$$(6) \qquad \|x''(t)\| \le \alpha r''(t) + K \quad \textit{for a.e. } t \in [a_0, a_1],$$

where $r(t) = \|x(t)\|^2$. *Then* $\|x'(t)\| \le M$ *for all* $t \in [a_0, a_1]$.

Proof. The proof is standard and is similar to the proof of this result when $x \in C^2$ (cf. [29]). However, we have to apply here the following lemma (cf. [17]).

LEMMA. *Let* $f \in W^{1,1}[a,b]$ *be a function such that* $k_1 < f(t) < k_2$ *for all* $t \in [a,b]$ *and let* $g : [k_1, k_2] \to \mathbb{R}$ *be measurable and bounded. Then*

$$\int\limits_{f(a)}^{f(b)} g(x)\,dx = \int\limits_a^b g(f(t))f'(t)\,dt\,. \ \blacksquare$$

Before we present the existence results, we shall state some additional hypotheses on the mutlifunction $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$:

(H2) There is a function $\phi : [0,\infty) \to (0,\infty)$ such that $s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$, and $\|F(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [a_0, a_1]$ and all $(y, y') \in \mathcal{D} := \{(x, x') \in \mathbb{R}^n \times \mathbb{R}^n : \|x\| \le R\}$, where R is the same as in (H1).

(H3) There exist constants $k, \alpha > 0$ such that

$$\|F(t,y,y')\| \le 2\alpha(y \cdot w + \|y'\|^2) + k$$

for a.e. $t \in [a_0, a_1]$, all $(y, y') \in \mathcal{D}$ and $w \in F(t, y, y')$.

The conditions (H2) and (H3) are related to the usual Bernstein–Nagumo growth conditions (cf. [29], [25], [17], etc.) and for a continuous single-valued function F they coincide with these conditions.

Now we introduce the hypotheses which are related to the classes of boundary conditions which we shall study.

Let $G_i : \mathbb{R}^{4n} \to \mathbb{R}^n$, $i = 0, 1$, be continuous functions. For a fixed function G_i, $i = 0, 1$, we introduce the following conditions:

(N1) One of the following inequalities is satisfied for all $u_0, u_0', u_1, u_1' \in \mathbb{R}^n$:

$$(-1)^i[u_i \cdot u_i' - G_i(u_0, u_0', u_1, u_1') \cdot u_i'] \ge 0.$$

(N2) One of the following inequalities is satisfied for all $u_0, u_0', u_1, u_1' \in \mathbb{R}^n$:

$$(-1)^i[u_i \cdot u_i' - G_i(u_0, u_0', u_1, u_1') \cdot u_i] \ge 0.$$

(N3) One of the following relations is satisfied for all $\lambda \in [0,1]$ and all $u_0, u_0', u_1, u_1' \in \mathbb{R}^n$ such that $\|u_i\| > R$:

$$u_i \neq \lambda[u_i - G_i(u_0, u_0', u_1, u_1')].$$

The conditions (N1) and (N2) are associated with the conditions $(-1)^i g_i(\widetilde{y}) \cdot y_i \geq 0$ and $(-1)^i g_i(\widetilde{y}) \cdot y_i' \geq 0$ resp., where $i = 0,1$, $\widetilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1))$ and $G_i(\widetilde{y}) = y_i - g_i(\widetilde{y})$ and $G_i(\widetilde{y}) = y_i' - g_i(\widetilde{y})$ respectively. Special cases of these have been considered by many authors, see e.g. [25], [36], [32].

Theorem 4.3 (Main Theorem). *Suppose that* $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that the hypotheses* (H1)–(H3) *are satisfied, and let* $G_i : \mathbb{R}^{4n} \to \mathbb{R}^n$, $i = 0,1$, *be continuous functions satisfying one of the conditions* (N1)–(N3). *Then the nonlinear boundary value problem*

$$\text{(P)} \qquad \begin{cases} y'' \in F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ G_i(y(a_0), y'(a_0), y(a_1), y'(a_1)) = 0, & i = 0,1, \end{cases}$$

has at least one solution in $H^2([a_0, a_1]; \mathbb{R}^n)$.

Proof. If G_i satisfies (N1) or (N3) we consider the nonlinear boundary condition

$$y(a_i) = \lambda[y(a_i) - G_i(\widetilde{y})] := \lambda g_i(\widetilde{y}), \tag{7}$$

and if G_i satisfies (N2) then

$$y'(a_i) = \lambda[y'(a_i) - G_i(\widetilde{y})] := \lambda g_i(\widetilde{y}), \tag{8}$$

where $i = 0,1$ and $\widetilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1))$.

We denote the linear boundary conditions given by the left-hand side of (7) or (8) by $w_i(\widetilde{y})$, $i = 0,1$, they are of the type (α), (β), or (γ) mentioned before. Consider the following family of nonlinear boundary value problems:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} y'' - \varepsilon y \in \lambda\{F(t, y, y') - \varepsilon y\} & \text{for a.e. } t \in [a_0, a_1], \\ w_i(\widetilde{y}) = \lambda g_i(\widetilde{y}), \quad i = 0,1, \ \lambda \in [0,1]. \end{cases}$$

We note that for sufficiently small $\varepsilon > 0$, the operator $Ly := y'' - \varepsilon y$, defined on the space $H^2_{\mathcal{B}_0}$, satisfies the assumption (A) and we also observe that we can choose this ε such that

$$\int_{M_1}^{\infty} \frac{s\,ds}{\widetilde{\phi}(s)} > \frac{T\widetilde{M_1}}{2} + 2\alpha R^2,$$

where $\widetilde{k} := k + \varepsilon R$, $\widetilde{M_1} := 4R(1 + \alpha R)/T + \widetilde{k}T/4$, $T = a_1 - a_0$, and $\widetilde{\phi}(s) = \phi(s) + \varepsilon R$. Thus we can apply Lemma (4.2) with the constants α, $\widetilde{K}$, R, T and function $\widetilde{\phi}$, to obtain the existence of a constant M (depending only on

α, R, K, ε and $\phi(s)$) such that every solution $y(t)$ to (P_λ) with $\|y(t)\| \le R$ satisfies $\|y'(t)\| \le M$.

In order to apply Theorem (3.1) we need *a priori* bounds on $\|y(t)\|$ for all solutions y to (P_λ). Let $y(t)$ be a solution to (P_λ), and put $r(t) = \|y(t)\|^2$. Suppose that $r(t)$ achieves its maximum at $t_0 \in (a_0, a_1)$. Then the *a priori* bound $\|y(t_0)\| \le R$ follows from Lemma (4.1). Suppose then that $r(t)$ achieves its maximum at $t_0 = a_i$, $i = 0$ or 1. We consider the following cases:

G_i *satisfies* (N1): We have

$$\begin{aligned} 0 &\ge (-1)^i r'(a_i) = (-1)^i 2y(a_i) \cdot y'(a_i) \\ &= (-1)^i 2\lambda[y(a_i) \cdot y'(a_i) - G_i(\widetilde{y}) \cdot y'(a_i)] \ge 0 \end{aligned}$$

and thus $r'(a_i) = 0$. Therefore we can apply again Lemma (4.1) to obtain the *a priori* bound $\|y(t)\| \le R$.

G_i *satisfies* (N2): We have

$$\begin{aligned} 0 &\ge (-1)^i r'(a_i) = (-1)^i 2y(a_i) \cdot y'(a_i) \\ &= (-1)^i 2\lambda[y(a_i) \cdot y'(a_i) - G_i(\widetilde{y}) \cdot y(a_i)] \ge 0 \end{aligned}$$

and thus $r'(a_i) = 0$. By (4.1) we find that $\|y(t)\| \le R$.

G_i *satisfies* (N3): We can suppose that $\lambda > 0$ and thus we have the equality

$$y(a_i) = \lambda[y(a_i) - G_i(\widetilde{y})]$$

and hence $\|y(a_i)\| \le R$. Thus the conclusion follows from Theorem (3.1). ∎

Let $\mathcal{B}$ denote the set of all functions $y(t)$ satisfying one of the following sets of boundary conditions:

(I) $\quad y(a_0) = r, \quad y(a_1) = s\,,$
(II) $\quad y'(a_0) = 0, \quad y'(a_1) = 0\,,$
(III) $\quad -Ay(a_0) + By'(a_0) = r\,, \quad Cy(a_1) + Dy'(a_1) = s\,,$
(IVa) $\quad y(a_0) = r\,, \quad Cy(a_1) + Dy'(a_1) = s\,,$
(IVb) $\quad -Ay(a_0) + By'(a_0) = r\,, \quad y(a_1) = s\,,$
(Va) $\quad y'(a_0) = 0\,, \quad Cy(a_1) + Dy'(a_1) = s\,,$
(Vb) $\quad -Ay(a_0) + By'(a_0) = r\,, \quad y'(a_1) = 0\,,$

where A, B, C, D are nonnegative definite symmetric $n \times n$-matrices and $r, s \in \mathbb{R}^n$. We suppose that if $y(a_0) = r$ (resp. $y(a_1) = s$), then $\|r\| \le R$ (resp. $\|s\| \le R$), and if $-Ay(a_0)+By'(a_0) = r$ (resp. $Cy(a_1)+Dy'(a_1) = s$), then A, B (resp. C, D) are nonsingular and $\|B^{-1}\|\,\|A^{-1}B\|\,\|r\| \le R$ (resp. $\|C^{-1}\|\,\|D^{-1}C\|\,\|s\| \le R$) but if $r = 0$ (resp. $s = 0$), we suppose that only one of the matrices A, B (resp. C, D) is nonsingular.

The boundary conditions (I)–(V) in the scalar case were studied in [27], [23], [24], [25], [26], [22]. For the case of a Carathéodory function or a multivalued scalar operator F these conditions were considered in [17]. For second order systems similar problems were considered in [14]. We also refer the reader to [35], [36].

Under the above hypotheses we have the following corollary:

(4.4) COROLLARY. *Suppose that* $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that the hypotheses* (H1)–(H3) *are satisfied. Then the differential inclusion*

$$\begin{cases} y'' \in F(t, y, y') & \textit{for a.e. } t \in [a_0, a_1], \\ y \in \mathcal{B}, \end{cases} \tag{9}$$

has at least one solution in $H^2([a_0, a_1]; \mathbb{R}^n)$.

Proof. First, consider the cases of boundary conditions $\mathcal{B}$, where the conditions $-Ay(a_0)+By'(a_0) = r$ and $Cy(a_1)+Dy'(a_1) = s$ are replaced by the homogeneous conditions $-Ay(a_0)+By'(a_0) = 0$ and $Cy(a_1)+Dy'(a_1) = 0$ respectively. We define for $i = 0, 1$

$$G_i^1(u_0, u_0', u_1, u_1') = u_i - r, \qquad G_i^2(u_0, u_0', u_1, u_1') = u_i',$$
$$G_0^3(u_0, u_0', u_1, u_1') = -Au_0 + Bu_0', \qquad G_1^3(u_0, u_0', u_1, u_1') = Cu_1 + Du_1'.$$

Note that G_i^1 satisfies (N3) and G_i^2 satisfies (N2). We shall verify that G_0^3 and G_1^3 satisfy one of the conditions (N1) or (N2). Assume first that A is nonsingular. Then $A^{-1}B$ is a nonnegative definite matrix and thus

$$u_0 \cdot u_0' + A^{-1}G_0^3(\widetilde{u}) \cdot u_0' = A^{-1}Bu_0' \cdot u_0' \geq 0, \qquad \text{where } \widetilde{u} = (u_0, u_0', u_1, u_1'),$$

and therefore (N2) is satisfied. Suppose now that B is nonsingular. Then

$$u_0 \cdot u_0' - B^{-1}G_0^3(\widetilde{u}) \cdot u_0 = B^{-1}Au_0 \cdot u_0 \geq 0$$

and thus (N1) is satisfied.

It can be verified in a similar way that G_1^3 satisfies (N1) or (N2). By Theorem (4.3), it follows that the differential inclusion

$$\begin{cases} y'' \in F(t, y, y') & \text{for a.e. } t \in [a_0, a_1], \\ y \in \mathcal{B}, \end{cases} \tag{9}$$

has a solution.

Before we prove Corollary (4.4) in the general case, we need to recall some notation used in the proof of Theorem (3.1). Put

$$U = \{(u, v) \in C : \|(u, v)\| < \max(M, R) + 1\},$$

where M is given by Lemma (4.2), and let $\mathcal{F}_0 : \overline{U} \to C$ be defined by

$$\mathcal{F}_0(w) = j \circ \widetilde{L}^{-1} \circ \Gamma(w), \qquad w \in C$$

(see the proof of (3.1) for notation), where $Ly = y'' - \varepsilon y$. Then it is known that $\mathcal{F}_0$ is essential in $\mathcal{C}_C(\overline{U}, \partial U)$ (Theorem (4.3)). We shall apply Theorem

(3.2). We consider the family of differential inclusions

$$(9_\lambda)\qquad \begin{cases} y'' \in F(t, y, y'), \\ y \in \mathcal{B}_\lambda, \end{cases}$$

where $\mathcal{B}_\lambda$ denotes one of the boundary conditions of $\mathcal{B}$ with $-Ay(a_0) + By'(a_0) = r$ and $Cy(a_1) + Dy'(a_1) = s$ replaced by $-Ay(a_0) + By'(a_0) = \lambda r$ and $Cy(a_1) + Dy'(a_1) = \lambda s$ respectively, for $\lambda \in [0, 1]$. By the same argument as in the proof of (4.3), we need only prove that if $r(t) = \|y(t)\|^2$, where $y(t)$ is a solution to (9_λ), achieves its maximum at $t_0 = a_0$ or a_1 then $\|y(t_0)\| \le R$. Suppose first that $t_0 = a_0$ and $-Ay(a_0) + By'(a_0) = \lambda r$. Then we have

$$\begin{aligned} 0 \ge r'(a_0) &= 2y(a_0) \cdot y'(a_0) = 2y(a_0) \cdot [B^{-1}\lambda r + B^{-1}Ay(a_0)] \\ &= 2y(a_0) \cdot B^{-1}\lambda r + 2y(a_0) \cdot B^{-1}Ay(a_0). \end{aligned}$$

Evidently, if $r = 0$, then $r'(a_0) = 0$. If $r \ne 0$, then, by assumption, $A > 0$ and $B > 0$. Since $B^{-1}A > 0$, we have $x \cdot B^{-1}Ax \ge \frac{1}{\|A^{-1}B\|}\|x\|^2$ and thus

$$\begin{aligned} 0 \ge r'(a_0) &\ge 2\left[\frac{1}{\|A^{-1}B\|}\|y(a_0)\|^2 - \|y(a_0)\|\,\|B^{-1}\|\,\|\lambda r\|\right] \\ &\ge 2\|y(a_0)\|\left[\frac{\|y(a_0)\|}{\|A^{-1}B\|} - \|B^{-1}\|\,\|r\|\right] \end{aligned}$$

and this implies

$$\|y(a_0)\| \le \|A^{-1}B\|\,\|B^{-1}\|\,\|r\| \le R\,.$$

Suppose now that $r(a_1) = \max r(t)$ and $Cy(a_1) + Dy'(a_1) = \lambda s$. Thus $r'(a_1) \ge 0$ and we have

$$\begin{aligned} 0 \le r'(a_1) &= 2y(a_1) \cdot y'(a_1) = 2y(a_1) \cdot [D^{-1}\lambda s - D^{-1}Cy(a_1)] \\ &= 2y(a_1) \cdot D^{-1}\lambda s - 2y(a_1) \cdot D^{-1}Cy(a_1) \\ &\le -2\|y(a_1)\|^2 \frac{1}{\|C^{-1}D\|} + 2\|y(a_1)\|\,\|D^{-1}\|\,\|s\| \\ &= 2\|y(a_1)\|\left[-\frac{\|y(a_1)\|}{\|C^{-1}D\|} + \|D^{-1}\|\,\|s\|\right] \end{aligned}$$

and thus $\|y(a_1)\| \le \|C^{-1}D\|\,\|D^{-1}\|\,\|s\| \le R$.

Now we can apply Theorem (3.2) and the existence of a solution to the differential inclusion (9) follows. ∎

We remark that if we replace r and s by any bounded functions $r(u_0, u_0', u_1, u_1')$ and $s(u_0, u_0', u_1, u_1')$ such that the required inequalities, such as: $\|r(\widetilde{u})\| \le R$, $\|s(\widetilde{u})\| \le R$, $\|B^{-1}\|\,\|A^{-1}B\|\,\|r(\widetilde{u})\| \le R$ or $\|C^{-1}\|\,\|D^{-1}C\| \times \|s(\widetilde{u})\| \le R$ (depending on the boundary conditions), are satisfied, then Corollary (4.4) is still true.

(4.5) DEFINITION. We say that two matrices $A \in O(n)$ and $B \in GL(n)$ have the *property* (P) if $u \cdot v \leq 0$ implies $u \cdot AB^{-1}v \leq 0$ for all $u, v \in \mathbb{R}^n$. If $A : X \to O(n)$ and $B : X \to GL(n)$ are two maps, then we say that A and B have the *property* (P) if $A(x)$ and $B(x)$ have (P) for all $x \in X$.

(4.6) THEOREM. *Suppose that* $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that the hypotheses* (H1)–(H3) *are satisfied and let* $A : \mathbb{R}^{4n} \to O(n)$ *and* $B : \mathbb{R}^{4n} \to GL(n)$ *be continuous maps having the property* (P). *Then the boundary value problem*

$$\begin{cases} y'' \in F(t, y, y') & \textit{for a.e. } t \in [a_0, a_1], \\ y(a_0) = A(\widetilde{y})y(a_1), \\ y'(a_0) = B(\widetilde{y})y'(a_1), \end{cases} \tag{10}$$

where $\widetilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1)) \in \mathbb{R}^{4n}$, *has at least one solution in* $H^2([a_0, a_1]; \mathbb{R}^n)$.

Proof. Since the space $\mathbb{R}^{4n}$ is contractible, there exists a homotopy $h : \mathbb{R}^{4n} \times [0, 1] \to \mathbb{R}^{4n}$ between the identity map and a constant map, say $h_0 \equiv x_0 \in \mathbb{R}^{4n}$ and $h_1 \equiv \mathrm{Id} : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$. Denote by $\widetilde{A}, \widetilde{B} : \mathbb{R}^{4n} \times [0, 1] \to GL(n)$ the composed maps $\widetilde{A} = A \circ h$ and $\widetilde{B} = B \circ h$. It is clear that $\widetilde{A}$ and $\widetilde{B}$ still have the property (P), and $A_\lambda := \widetilde{A}(\cdot, \lambda)$ and $B_\lambda := \widetilde{B}(\cdot, \lambda)$ are two homotopies such that

$$A_0 \equiv A(x_0)\,, \quad B_0 \equiv B(x_0)\,, \quad A_1 = A\,, \quad B_1 = B\,.$$

We consider the diagram

$$\begin{tikzcd} C \arrow[r, "\Gamma"] & L^2 \times \mathbb{R}^n \times \mathbb{R}^n \\ & H^2 \arrow[ul, "j"] \arrow[u, "\widetilde{L}"'] \end{tikzcd}$$

where

$$\begin{aligned} &C := C([a_0, a_1]; \mathbb{R}^n \times \mathbb{R}^n), \\ &H^2 := H^2([a_0, a_1]; \mathbb{R}^n), \\ &L^2 := L^2([a_0, a_1]; \mathbb{R}^n), \\ &\Gamma(u, v) = \{w \in L^2 : w(t) \in \mathcal{F}_\varepsilon(t, u(t), v(t)) \text{ a.e.}\} \\ &\qquad\qquad \times \{A(\widetilde{w})u(a_1)\} \times \{B(\widetilde{w})v(a_1)\}\,, \\ &F_\varepsilon(t, u, v) = F(t, u, v) - \varepsilon u, \quad w = (u, v), \\ &\widetilde{w} = (u(a_0), v(a_0), u(a_1), v(a_1)), \\ &j(u) = (u, u'), \quad \widetilde{L}u = (u'' - \varepsilon u, u(a_0), u'(a_0))\,. \end{aligned}$$

The operator $\widetilde{L}$ is a continuous, linear and one-to-one map of H^2 onto $L^2 \times \mathbb{R}^n \times \mathbb{R}^n$ and thus it has a continuous inverse map $\widetilde{L}^{-1}$. The problem

(10) is equivalent to the fixed point problem

$$(u,v) \in j \circ \widetilde{L}^{-1} \circ \Gamma(u,v), \qquad (u,v) \in C\,.$$

We define a multivalued homotopy $H : C \times [0,1] \to C$ by $H(w,\lambda) = j \circ \widetilde{L}^{-1} \circ \Gamma_\lambda(w)$, $w = (u,v) \in C$, $\lambda \in [0,1]$, where

$$\begin{aligned}\Gamma_\lambda(w) = \Gamma_\lambda(u,v) = \{w \in L^2 : w(t) \in \lambda F_\varepsilon(t,u(t),v(t)) \text{ a.e.}\}\\ \times \{A_\lambda(\widetilde{w})u(a_1)\} \times \{B_\lambda(\widetilde{w})v(a_1)\}\,.\end{aligned}$$

H is a well defined multivalued homotopy such that $H \in \mathcal{C}$, and if $w \in H(w,\lambda)$ then $w = (u,u')$ and u satisfies the differential inclusion

$$(10_\lambda) \qquad \begin{cases} u'' - \varepsilon u \in \lambda\{F(t,u(t),u'(t)) - \varepsilon u\} & \text{for a.e. } t \in [a_0,a_1]\,,\\ u(a_0) = A_\lambda(\widetilde{u})u(a_1)\,, \\ u'(a_0) = B_\lambda(\widetilde{u})u'(a_1)\,, \quad \lambda \in [0,1]. \end{cases}$$

Observe that for $\lambda = 0$, $H(w,0) = j \circ \widetilde{L}^{-1} \circ \Gamma_0(w) \equiv \{0\}$, since the equation $u'' - \varepsilon u = 0$ with the boundary conditions $u(a_0) = A(x_0)u(a_1)$, $u'(a_0) = B(x_0)u'(a_1)$ has only the trivial solution. Therefore, the constant map $H_0|_{\overline{U}}$ is essential in $\mathcal{C}_C(\overline{U};\partial U)$ for every neighborhood U of zero in C, and it is sufficient, by (2.2), to prove that there exists a constant $M < \infty$ such that for any solution $y(t)$ to the inclusion (10_λ) we have

$$(y,y') \in U := \{(u,v) : \|(u,v)\| < M\}.$$

Exactly in the same way as in the proof of (4.3) we observe that it is enough to show that if $r(t) = \|y(t)\|^2$ achieves its maximum at $t_0 = a_i$, $i = 0$ or 1, then $r(t_0) \le R^2$. First we notice that

$$\begin{aligned} r(a_0) = y(a_0)\cdot y(a_0) &= \widetilde{A}_\lambda y(a_1)\cdot \widetilde{A}_\lambda y(a_1)\\ &= y(a_1)\cdot \widetilde{A}_\lambda^T \widetilde{A}_\lambda y(a_1) = y(a_1)\cdot y(a_1) = r(a_1)\,, \end{aligned}$$

where $\widetilde{A}_\lambda := A_\lambda(\widetilde{y})$. Hence, $r(a_0) = r(a_1) = \max r(t)$, and thus $r'(a_0) \le 0 \le r'(a_1)$. Since $\widetilde{A}_\lambda$ and $\widetilde{B}_\lambda$ have the property (P) and $r'(a_0) = 2y(a_0)\cdot y'(a_0) \le 0$, it follows that

$$\begin{aligned} 0 \le r'(a_1) = 2y(a_1)\cdot y'(a_1) &= 2\widetilde{A}_\lambda^T y(a_0)\cdot \widetilde{B}_\lambda^{-1} y'(a_0)\\ &= 2y(a_0)\cdot \widetilde{A}_\lambda \widetilde{B}_\lambda^{-1} y'(a_0) \le 0 \end{aligned}$$

and we have $r'(a_1) = 0$. We can now apply Lemma (4.1) to obtain the *a priori* bound $\|y(t)\| \le R$ and the result follows. ∎

Suppose that $G_i : \mathbb{R}^{4n} \to \mathbb{R}^n$, $i = 0,1$, are continuous functions satisfying one of the conditions (N1)–(N3), and $A : \mathbb{R}^{4n} \to O(n)$ and $B : \mathbb{R}^{4n} \to GL(n)$ are continuous maps having the property (P). By Λ we denote one of the boundary conditions (I)–(V) or

$$G_i(y(a_0), y'(a_0), y(a_1), y'(a_1)) = 0, \qquad i = 0,1,$$

or

$$y(a_0) = A(\widetilde{y})y(a_1), \quad y'(a_0) = B(\widetilde{y})y'(a_1).$$

Then we have the following corollary:

(4.7) Corollary. *Let $f : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Suppose that there are nonnegative constants α, k, R and a function $\phi : [0,\infty) \to (0,\infty)$ such that the following conditions are satisfied*:

(a) *If $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$ then there is a $\delta > 0$ such that*

$$\operatorname*{ess\,inf}_{t\in[a_0,a_1]} \inf\{y \cdot f(t,y,y') + \|y'\|^2 : (y,y') \in \mathcal{D}_\delta\} > 0\,,$$

where $\mathcal{D}_\delta := \{(y,y') : \|y - y_0\| + \|y' - y_0'\| < \delta\}$.

(b) *$s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ and $\|f(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [a_0,a_1]$ and all $(y,y') \in \mathcal{D} := \{(x,x') : \|x\| \le R\}$.*

(c) *$\|f(t,y,y')\| \le 2\alpha(y \cdot f(t,y,y') + \|y'\|^2) + k$ for a.e. $t \in [a_0,a_1]$ and all $(y,y') \in \mathcal{D}$.*

Then the boundary value problem

$$\begin{cases} y'' = f(t,y,y') & \text{for a.e. } t \in [a_0,a_1]\,, \\ y \in \Lambda\,, \end{cases}$$

has at least one solution $y(t)$ in $H^2([a_0,a_1];\mathbb{R}^n)$.

The previous results may be further extended by introducing the generalized *height* function (cf. [12]) that replaces the usual euclidean norm in the space $\mathbb{R}^n$.

Suppose that there is given a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ of class C^2 such that Ψ is convex and coercive, i.e. $\Psi(x) \to \infty$ as $\|x\| \to \infty$. This implies that for every $R \in \mathbb{R}$ the set $\Psi^{-1}(-\infty, R]$ is bounded and we can put $\xi(R) := \sup\{\|y\| : y \in \Psi^{-1}(\infty, R]\}$. We will call the function Ψ a *height function*.

Now, let us introduce the following modified version of the hypotheses (H1)–(H3):

(A1) There exists a constant $R > 0$ such that if $\Psi(a)$, $\Psi(b) \le R$ and if $\Psi(x_0) > R$ then for every $x_0' \in \mathbb{R}^n$ such that $D\Psi(x_0) \cdot x_0' = 0$ there exists a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[a_0,a_1]} \inf\{D\psi(x) \cdot w + D^2\psi(x)x' \cdot x' : w \in F(t,x,x'),$$
$$(x,x') \in \mathcal{D}_\delta\} > 0\,,$$

where $\mathcal{D}_\delta := \{(x,x') \in \mathbb{R}^{2n} : \|x_0 - x\| + \|x_0' - x'\| < \delta\}$.

(A2) There is a function $\phi : [0,\infty) \to (0,\infty)$ such that $s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ and $\|F(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [a_0,a_1]$ and all $(y,y') \in \mathcal{D} := \{(x,x') : \|x\| \le \xi(R)\}$.

(A3) There exist constants K, $\alpha > 0$ such that

$$\|F(t,y,y')\| \le 2\alpha(y\cdot w + \|y'\|^2) + K$$

for a.e. $t\in[a_0,a_1]$ and all $(y,y')\in\mathcal{D}$, $w\in F(t,y,y')$.

It is very easy to construct examples, even in the case of single-valued mappings, that satisfy the conditions (A1), (A2) and (A3) for some height function Ψ, but do not satisfy the condition (H1).

The following two conditions are modifications of the conditions (N1) and (N2). Assume that $G_i:\mathbb{R}^{4n}\to\mathbb{R}^n$, $i=0,1$, are continuous functions such that for a fixed function G_i, $i=0,1$, one of the following conditions is satisfied:

(M1) $(-1)^i D\Psi(u_i)\cdot[u_i' - G_i(u_0,u_0',u_1,u_1')]\ge 0$, $i=0,1$ for all u_0,u_0', $u_1,u_1'\in\mathbb{R}^n$.

(M2) There exists a constant $M<\xi(R)$ such that

$$(-1)^i D\Psi[u_i - G_i(u_0,u_0',u_1,u_1')]\cdot u_i'\ge 0,\qquad i=0,1,$$

for all $u_0,u_0',u_1,u_1'\in\mathbb{R}^n$ and $u_i\ne\lambda[u-G_i(u_0,u_0',u_1,u_0')]$ for all $\lambda\in(0,1]$ and $u_0,u_0',u_1\in\mathbb{R}^n$ such that $\|u_i\|>M$, $i=0,1$.

Now we can state the following result which generalizes Theorem (4.3).

(4.8) Theorem. *Suppose that* $F:[a_0,a_1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that the hypotheses* (A1)–(A3) *are satisfied and suppose that* $G_i:\mathbb{R}^{4n}\to\mathbb{R}^n$, $i=0,1$, *are continuous functions satisfying one of the conditions* (M1) *or* (M2). *Then the nonlinear boundary value problem*

$$\begin{cases} y''(t)\in F(t,y(t),y'(t)) & \textit{for a.e. } t\in[a_0,a_1],\\ G_i(y(a_0),y'(a_0),y(a_1),y'(a_1))=0, & i=0,1,\end{cases}\tag{11}$$

has at least one solution in $H^2([a_0,a_1];\mathbb{R}^n)$.

Proof. First, we consider the case where G_i, $i=0,1$, satisfy the condition (M1). Let us introduce the following family of nonlinear boundary conditions:

$$\begin{aligned} y'(a_0) &= \lambda[y'(a_0) - G_0(y(a_0),y'(a_0),y(a_1),y'(a_1))]\\ &=: \lambda g_0(y(a_0),y'(a_0),y(a_1),y'(a_1)) = \lambda g_0(\widetilde{y}),\\ y'(a_1) &= \lambda[y'(a_1) - G_1(y(a_0),y'(a_0),y(a_1),y'(a_1))]\\ &=: \lambda g_1(y(a_0),y'(a_0),y(a_1),y'(a_1)) = \lambda g_1(\widetilde{y}),\end{aligned}$$

where $\lambda\in[0,1]$ and $\widetilde{y}=(y(a_0),y'(a_0),y(a_1),y'(a_1))$.

Using the topological transversality method, in a standard way we conclude that in order to prove the existence result for (11) it is sufficient to find *a priori* bounds on solutions (with respect to the C^1-norm) for the family of differential inclusions

$$(11_\lambda)\qquad \begin{cases} y'' - \varepsilon y \in \lambda\{F(t, y(t), y'(t)) - \varepsilon y(t)\} & \text{for a.e. } t \in [a_0, a_1], \\ y'(a_0) = \lambda g_0(\widetilde{y}),\ y'(a_1) = \lambda g_1(\widetilde{y}), \quad i = 0, 1, & \lambda \in [0, 1], \end{cases}$$

where $\varepsilon > 0$ is chosen to be a sufficiently small number. Suppose then that $y(t)$ is a solution to (11_λ). We can assume that $\lambda \in (0, 1]$. Consider the function $\gamma(t) := \Psi(y(t))$. If $\gamma(t)$ achieves its maximum at $t \in (a_0, a_1)$, then exactly as in the proof of Theorem (3.4) in [12], the hypothesis (A1) will imply that $\Psi(y(t)) \le R$ and thus $\|y(t)\| \le \xi(R)$. We have to emphasize that in that proof the only one essential point was that $\gamma'(t_0) = 0$. Therefore, if we can show that if $\gamma(t_0) = \max \Psi(y(t))$ then $\gamma'(t_0) = 0$, the conclusion $\|y(t)\| \le \xi(R)$ will follow.

Suppose now that $\gamma(t)$ achieves its maximum at a_0. We have

$$\begin{aligned} 0 \ge \gamma'(a_0) &= D\Psi(y(a_0)) \cdot y'(a_0) \\ &= D\Psi(y(a_0)) \cdot \lambda[y'(a_0) - G_i(y(a_0), y'(a_0), y(a_1), y'(a_1))] \ge 0. \end{aligned}$$

Thus $\gamma'(a_0) = 0$. In a similar way we deduce that if $t_0 = a_1$ then $\gamma'(a_1) = 0$. Therefore the hypothesis (A1) implies that $\|y(t)\| \le \xi(R)$. The hypotheses (A2), (A3) give the *a priori* bounds $\|y'(t)\| \le M_1$, and the result follows.

Consider now the case where the functions G_0, G_1 satisfy the condition (N2). In this case we consider the family of differential inclusions

$$(12_\lambda)\qquad \begin{cases} y'' - \varepsilon y \in \lambda\{F(t, y(t), y'(t)) - \varepsilon y(t)\} & \text{for a.e. } t \in [a_0, a_1], \\ y(a_0) = h_0(\widetilde{y})\,, \\ y(a_1) = h_1(\widetilde{y})\,, \quad \lambda \in [0, 1], \end{cases}$$

where $h_i(u_0, u_0', u_1, u_1') = u_i - G_i(u_0, u_0', u_1, u_1')$. For $\lambda = 0$, the system (12_λ) becomes

$$(12_0)\qquad \begin{cases} y''(t) - \varepsilon y(t) = 0 & \text{for a.e. } t \in [a_0, a_1], \\ y(a_0) = h_0(\widetilde{y})\,, \quad y(a_1) = h_1(\widetilde{y}). \end{cases}$$

Now, we introduce the family of problems

$$(13_\lambda)\qquad \begin{cases} y''(t) - \varepsilon y(t) = 0, \\ y(a_0) = \lambda \widetilde{h}_0(\widetilde{y})\,, \quad y(a_1) = \lambda \widetilde{h}_1(\widetilde{y}), \quad \lambda \in [0, 1], \end{cases}$$

where

$$\widetilde{h}_i(u_0, u_0', u_1, u_1') = u_i - G_i(u_0, u_0', u_1, u_1').$$

For $\lambda = 0$, the system (13_0) can be reformulated as a fixed point problem involving a constant map, thus the topological transversality implies that in order to obtain the existence result for (11) it is sufficient to find *a priori* bounds on solutions to (12_λ) and (13_λ) (with respect to the C_1-norm).

In the same way as in the previous case it is sufficient to show that if the function $\gamma(t) = \Psi(y(t))$, where $y(t)$ is a solution to (12_λ), achieves its maximum at an end-point t_0 of the interval $[a_0, a_1]$, then $\gamma'(t_0) = 0$. This will imply, by (A1), that $\|y(t)\| \le \xi(R)$. Suppose, for example, that $t_0 = a_0$. Then

$$0 \ge \gamma'(a_0) = D\Psi(y(a_0)) \cdot y'(a_0) = D\Psi[y(a_0) - G_0(\widetilde{y})] \cdot y'(a_0)] \ge 0.$$

Thus $\gamma'(a_0) = 0$. For $t_0 = a_1$, we see exactly in the same way that $\gamma'(a_1) = 0$. Now, we have to show that if $y(t)$ is a solution to (13_λ) then $\|y(t)\| \le \xi(R)$. But this follows from the hypothesis (A1) and the assumption (M2). Now, the hypotheses (A2) and (A3) imply the existence of *a priori* bounds $\|y'(t)\| \le M_1$ on all solutions to (12_λ) and (13_λ). Therefore, there exists a solution to (11) in the class $H^2([a_0, a_1]; \mathbb{R}^n)$. ∎

(4.9) Definition. We say that two matrices $A \in O(n)$ and $B \in GL(n, \mathbb{R})$ have the *property* (P′) if $u \cdot v \le 0$ implies $u \cdot A^{-1}B^{-1}v \le 0$ for all $u, v \in \mathbb{R}^n$. If $A : X \to O(n)$ and $B : X \to GL(n, \mathbb{R})$ are two maps, then we say that A and B have the *property* (P′) if $A(x)$ and $B(x)$ have (P′) for all $x \in X$.

Moreover, we will say that A is *Ψ-invariant* if $\Psi(A(x)y) = \Psi(y)$ for all $x \in X$ and $y \in \mathbb{R}^n$.

(4.10) Theorem. *Suppose that $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a Carathéodory multifunction such that the hypotheses* (A1)–(A3) *are satisfied and let $A : \mathbb{R}^{4n} \to O(n)$ and $B : \mathbb{R}^{4n} \to GL(n, \mathbb{R})$ be continuous maps having the property* (P') *such that A is Ψ-invariant. Then the boundary value problem*

$$\begin{cases} y'' \in F(t, y, y') \quad \text{for a.e. } t \in [a_0, a_1], \\ y(a_0) = A(\widetilde{y})y(a_1), \\ y'(a_0) = B(\widetilde{y})y'(a_1), \end{cases} \tag{14}$$

where $\widetilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1)) \in \mathbb{R}^{4n}$, has at least one solution in $H^2([a_0, a_1]; \mathbb{R}^n)$.

Proof. Suppose that $h : \mathbb{R}^{4n} \times [0, 1] \to \mathbb{R}^{4n}$ is a deformation of the identity map to a constant map, say $h_0 \equiv x_0$ and $h_1 \equiv \mathrm{Id} : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$. Define $\widetilde{A}$, $\widetilde{B} : \mathbb{R}^{4n} \times [0, 1] \to \mathbb{R}^{4n}$ by $\widetilde{A} = A \circ h$, $\widetilde{B} = B \circ h$. It is clear that $\widetilde{A}$ and $\widetilde{B}$ still have the property (P′). Put

$$A_\lambda := \widetilde{A}(\cdot, \lambda), \quad B_\lambda := \widetilde{B}(\cdot, \lambda), \quad A_0 := A(x_0), \quad B_0 := B(x_0).$$

We now consider the family of differential inclusions

$$\begin{cases} y'' - \varepsilon y \in \lambda\{F(t, y(t), y'(t)) - \varepsilon y(t)\} \quad \text{for a.e. } t \in [a_0, a_1], \\ y(a_0) = A_\lambda(\widetilde{y})y(a_1)\,, \\ y'(a_0) = B_\lambda(\widetilde{y})y'(a_1)\,, \quad \text{where } \lambda \in [0, 1]. \end{cases} \tag{14_λ}$$

Suppose that $y(t)$ is a solution to (14_λ). It is enough to show that if $\gamma(t) = \Psi(y(t))$ achieves its maximum at $t_0 = a_0$ or a_1 then $\gamma(t_0) = \gamma(a_1)$ and

$\gamma'(a_1) = 0$. Observe first that

$$\gamma(a_0) = \Psi(y(a_0)) = \Psi(A\lambda(\widetilde{y})y(a_1)) = \Psi(y(a_1)) = \gamma(a_1).$$

If $\gamma(t)$ achieves its maximum at $t_0 = a_0$ or a_1 then

$$\begin{aligned} 0 \le \gamma'(a_1) &= D\Psi(y(a_1)) \cdot y'(a_1) = D\Psi(A_\lambda^T(\widetilde{y})y(a_0)) \cdot B_\lambda^{-1}(\widetilde{y})y'(a_0) \\ &= A_\lambda^T(\widetilde{y})D\Psi(y(a_0)) \cdot B_\lambda^{-1}(\widetilde{y})y'(a_0) = D\Psi(y(a_0)) \cdot A_\lambda(\widetilde{y})B_\lambda^{-1}(\widetilde{y})y'(a_0). \end{aligned}$$

But $0 \ge \gamma'(a_0) = D\Psi(y(a_0)) \cdot y'(a_0)$, thus the property (P′) implies that $\gamma'(a_1) \le 0$, therefore $\gamma'(a_1) = 0$. ∎

§5. Other results for nonlinear differential inclusions. Let $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. In this section we consider the differential inclusion

$$(\dagger) \qquad y'' \in F(t, y, y') \quad \text{for a.e. } t \in [a_0, a_1]$$

and we investigate the following problem: under what conditions will there exist a solution $y(t)$ to (†), satisfying certain boundary conditions, which remains in

$$\mathcal{P} \equiv \{(t, x) : a_0 \le t \le a_1, \quad x \cdot P(t)x \le r^2\},$$

where $r > 0$ is a given real number and $P(t)$ is a given symmetric positive definite $n \times n$-matrix which is elementwise of class C^2 on $[a_0, a_1]$.

Problems of this type for differential systems have been considered by many authors, for example see [31], [33], [11] or [14]. For the Dirichlet problem Knobloch (cf. [31]) showed the existence and uniqueness results under certain positivity assumptions (see also [11]). In [14] some further existence results for such systems were obtained.

We introduce the following sets of boundary conditions:

($\mathrm{B_I}$) $y(a_0) = 0, \quad y(a_1) = 0,$

($\mathrm{B_{II}}$) $y(a_0) = Qy(a_1), \quad y'(a_0) = Qy'(a_1)$, where we suppose that Q is an $n \times n$-matrix satisfying

$$Q^T P(a_0) Q = P(a_1) \quad \text{and} \quad Q^T P'(a_0) Q \ge P'(a_1)\,,$$

($\mathrm{B_{III}}$) $y(a_0) = Ay'(a_0), \quad y(a_1) = By'(a_1),$

($\mathrm{B_{IV}}$)$_\mathrm{a}$ $y(a_0) = 0, \quad y(a_1) = By'(a_1),$

($\mathrm{B_{IV}}$)$_\mathrm{b}$ $y(a_0) = 0, \quad y(a_0) = Ay'(a_0),$

($\mathrm{B_V}$)$_\mathrm{a}$ $y'(a_0) = 0, \quad y(a_1) = By'(a_1)$ if $P'(a_0) \ge 0,$

($\mathrm{B_V}$)$_\mathrm{b}$ $y'(a_1) = 0, \quad y(a_0) = Ay'(a_0)$ if $P'(a_1) \le 0,$

where A and B are two $n \times n$-matrices satisfying

$$2P(a_0)A + A^T P'(a_0) A \ge 0 \quad \text{and} \quad 2P(a_1)B + B^T P'(a_1) B \le 0.$$

Define

$$\alpha(t,x,x') := 2x' \cdot P(t)x' + 4x' \cdot P'(t)x + x \cdot P''(t)x\,,$$
$$\beta(t,x,x') := 2x' \cdot P(t)x + x \cdot P'(t)x\,,$$

where $t \in [a_0,a_1]$, $x,x' \in \mathbb{R}^n$. Observe that for a fixed $t \in [a_0,a_1]$, the functions $\alpha(t,\cdot,\cdot)$ and $\beta(t,\cdot,\cdot)$ are quadratic forms on the phase space $\mathbb{R}^n \times \mathbb{R}^n$. We shall make the following assumption:

(P0) $$\alpha(t,x,x') \ge 0 \quad \text{for all } t \in [a_0,a_1],\ x,x' \in \mathbb{R}^n.$$

Notice that in the special case where the matrix $P(x)$ satisfies the commutativity conditions

$$P(t)P'(t) = P'(t)P(t) \quad \text{and} \quad P(t)P''(t) = P''(t)P(t) \quad \text{for all } t \in [a_0,a_1],$$

the condition (P0) is equivalent to

(P0′) $$P(t)P''(t) - 2[P'(t)]^2 \ge 0, \qquad P''(t) \ge 0 \quad \text{for all } t \in [a_0,a_1].$$

Therefore, if the matrix $P(t)$ is diagonal, i.e.

$$P(t) = \begin{bmatrix} p_1(t) & & 0 \\ & \ddots & \\ 0 & & p_n(t) \end{bmatrix}$$

then (P0) can be replaced by the following assumptions:

(i) $p_i(t) > 0$,
(ii) $p_i''(t) \ge 0$,
(iii) $p_i(t)p_i''(t) - 2[p_i'(t)]^2 \ge 0$,

for all $t \in [a_0,a_1]$, where $i = 1,\dots,n$.

Now we shall state some hypotheses, which are the modified versions of the hypotheses (H1)–(H3), on the multifunction $F : [a_0,a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$:

(P1) If $y_0 \cdot P(t_0)y_0 = r^2$ and $\beta(t_0,y_0,y_0') = 0$ then there exists a $\delta > 0$ and an open neighborhood V of t_0 in $[a_0,a_1]$ such that

$$\operatorname*{ess\,inf}_{t\in V} \inf\{2w \cdot P(t)y + \alpha(t,y,y') : w \in F(t,y,y'),\ (y,y') \in \mathcal{D}_\delta\} > 0,$$

where $\mathcal{D}_\delta := \{(y,y') : \|y-y_0\| + \|y'-y_0'\| < \delta\}$.

(P2) There is a function $\phi : [0,\infty) \to (0,\infty)$ such that $s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ and $\|F(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [a_0,a_1]$ and all $(y,y') \in \mathcal{D} := \{(x,x') : \|x\| \le R\}$, where $R = \sup\{\|y\| : (t,y) \in \mathcal{P}\}$.

(P3) There exist constants $K, \mathfrak{a} > 0$ such that

$$\|F(t,y,y')\| \le 2\mathfrak{a}(y \cdot w + \|y'\|^2) + K$$

for a.e. $t \in [a_0,a_1]$, all $(y,y') \in \mathcal{D}$ and $w \in F(t,y,y')$.

(5.1) Theorem. *Suppose that* $F : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that* $P(t)$ *and* F *satisfy the hypotheses* (P0)–(P3). *Then the differential inclusion*

$$y'' \in F(t, y, y') \quad \text{for a.e. } t \in [a_0, a_1] \tag{15}$$

for each of the boundary conditions $(\mathrm{B_I})$–$(\mathrm{B_V})$ *has at least one solution* $y \in H^2([a_0, a_1]; \mathbb{R}^n)$ *satisfying*

$$y(t) \cdot P(t)y(t) \le r^2.$$

Proof. Notice that for $\varepsilon > 0$ sufficiently small, the operator $Ly = y'' - \varepsilon y$ defined on the space $H^2_{\mathcal{B}_0}$, where $\mathcal{B}_0$ denotes one of the sets of boundary conditions $(\mathrm{B_I})$–$(\mathrm{B_V})$, satisfies the assumption (A1). Observe also that we can choose $\varepsilon > 0$ such that

$$\int_{\widetilde{M}_1}^{\infty} \frac{s\,ds}{\widetilde{\phi}(s)} > \frac{T\widetilde{M}_1}{2} + 2\mathfrak{a}R^2,$$

where $\widetilde{K} := K + \varepsilon R$, $\widetilde{M}_1 := 4R(1 + \mathfrak{a}R)/T + \widetilde{K}T/4$, $T = a_1 - a_0$, and $\widetilde{\phi}(s) = \phi(s) + \varepsilon R$. We can apply Lemma (4.2) to obtain the existence of a constant M_0 (depending only on $\mathfrak{a}$, R, K, ε and $\phi(s)$) such that every solution $y(t)$ to the differential inclusion

$$\begin{cases} y'' - \varepsilon y \in \lambda\{F(t, y, y') - \varepsilon y\}, \\ y \in \mathcal{B}_0, \end{cases} \tag{15_λ}$$

satisfying $y(t) \cdot P(t)y(t) \le r^2$, satisfies $\|y'\|_0 < M_0$. We put

$$U := \{(u, v) \in C : u(t) \cdot P(t)u(t) < r^2 \text{ and } \|v(t)\| < M_0\}.$$

We shall apply Theorem (3.3). We need to show that there is no solution $y(t)$ to the differential inclusion (15_λ) such that $(y, y') \in \partial U$, where $\lambda \in (0, 1)$. Set $h(t) = y(t) \cdot P(t)y(t)$. Then $(y, y') \in \partial U$ implies that there is a point $t_0 \in [a_0, a_1]$ such that $h(t_0) = r^2 = \max h(t)$. Suppose first that $t_0 = a_i$, $i = 0$ or 1. Since the case $y \in (\mathrm{B_I})$ is excluded, we have to consider only the following cases:

Case $y \in (\mathrm{B_{II}})$: In this case we have

$$\begin{aligned} h(a_0) &= y(a_0) \cdot P(a_0)y(a_0) = Qy(a_1) \cdot P(a_0)Qy(a_1) \\ &= y(a_1) \cdot Q^T P(a_0)Qy(a_1) = y(a_1) \cdot P(a_1)y(a_1) = h(a_1), \end{aligned}$$

thus $h(a_0) = h(a_1) = \max h(t)$, and hence $h'(a_0) \le 0 \le h'(a_1)$. But

$$\begin{aligned} h'(a_0) &= 2y'(a_0) \cdot P(a_0)y(a_0) + y(a_0) \cdot P'(a_0)y(a_0) \\ &= 2y'(a_1) \cdot Q^T P(a_0)Qy(a_1) + y(a_1) \cdot Q^T P'(a_0)Qy(a_1) \\ &\ge 2y'(a_1) \cdot P(a_1)y(a_1) + y(a_1) \cdot P'(a_1)y(a_1) = h'(a_1) \end{aligned}$$

and thus $h'(a_0) = h'(a_1) = 0$.

Case $y \in (\mathrm{B_{III}})$: We see that if $t_0 = a_0$, then $h'(a_0) \le 0$ and since

$$\begin{aligned} h'(a_0) &= 2y'(a_0) \cdot P(a_0)y(a_0) + y(a_0) \cdot P'(a_0)y(a_0) \\ &= 2y'(a_0) \cdot P(a_0)Ay'(a_0) + y'(a_0) \cdot A^T P'(a_0)Ay'(a_0) \\ &= y'(a_0) \cdot [2P(a_0)A + A^T P'(a_0)A]y'(a_0) \ge 0 \end{aligned}$$

we have $h'(a_0) = 0$. Similarly, if $t_0 = a_1$ we find that $h'(a_1) = 0$.

Cases $y \in (\mathrm{B_{IV}})$ or $(\mathrm{B_V})$ are analogous to $(\mathrm{B_{III}})$. We conclude that if $t_0 = a_0$ or a_1, then $h'(t_0) = 0$.

Consider now $h''(t_0) = 2y''(t_0) \cdot P(t_0)y(t_0) + \alpha(t_0, y(t_0), y'(t_0))$, where $h(t_0) = \max h(t) = r$, $t_0 \in [a_0, a_1]$. We already know that $\beta(t_0, y(t_0), y'(t_0)) = h'(t_0) = 0$, thus by the assumption (P1) there are $\delta > 0$ and an open neighborhood V of t_0 in $[a_0, a_1]$ such that

$$\operatorname*{ess\,inf}_{t\in V} \inf\{2w \cdot P(t)y + \alpha(t, y, y') : w \in F(t, y, y'),\ (y, y') \in \mathcal{D}_\delta\} > 0.$$

Since $y(t)$ satisfies (15_λ), we have

$$y''(t) \in \lambda F(t, y, y') + (1-\lambda)\varepsilon y \quad \text{for a.e. } t \in [a_0, a_1]$$

and if we define $w(t)$ by $y''(t) = \lambda w(t) + (1-\lambda)\varepsilon y(t)$, then $w(t) \in F(t, y, y')$ and

$$\begin{aligned} h''(t) &= 2\lambda w(t) \cdot P(t)y(t) + 2(1-\lambda)\varepsilon y(t) \cdot P(t)y(t) + \alpha(t, y(t), y'(t)) \\ &\ge \lambda\{w(t) \cdot P(t)y(t) + \alpha(t, y(t), y'(t))\}\,. \end{aligned}$$

But if t is sufficiently close to t_0, then $\|y(t) - y(t_0)\| + \|y'(t) - y'(t_0)\| < \delta$ and this implies that there is an open neighborhood $V_1 \subset V$ of t_0 such that

$$h''(t) \ge \lambda\{2w(t) \cdot P(t)y(t) + \alpha(t, y(t), y'(t))\} > 0 \quad \text{for a.e. } t \in V_1.$$

But this is impossible by the maximum principle (see [2]), and hence we have a contradiction. ∎

(5.2) Corollary. *Let $r > 0$ and let $f : [a_0, a_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Suppose that there are nonnegative constants $\mathfrak{a}$, K and a function $\phi : [0,\infty) \to (0,\infty)$ such that*

(a) $2f(t, y, y') \cdot P(t)y + \alpha(t, y, y') > 0$ *if* $\beta(t, y, y') = 0$ *and* $y \cdot P(t)y = r^2$.

(b) $s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ *and* $\|f(t, y, y')\| \le \phi(\|y'\|)$ *for* $(y, y') \in \mathcal{D} := \{(x, x') : \|x\| \le R\}$, *where* $R = \sup\{\|y\| : (t, y) \in \mathcal{P}\}$.

(c) $\|f(t, y, y')\| \le 2\mathfrak{a}(y \cdot f(t, y, y') + \|y'\|^2) + K$ *for* $(y, y') \in \mathcal{D}$ *and* $t \in [a_0, a_1]$.

Then the equation

$$y'' = f(t, y, y'), \quad t \in [a_0, a_1],$$

for each of the boundary conditions $(\mathrm{B_I})$–$(\mathrm{B_V})$ *has at least one solution* $y(t)$ *in* $C^2([a_0,a_1];\mathbb{R}^n)$ *satisfying*

$$y(t)\cdot P(t)y(t)\le r^2.$$

(5.3) Corollary. *Let* $r>0$ *and let* $f:[0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *be a Carathéodory function. Suppose that there are nonnegative constants* $\mathfrak{a}$, K *and a function* $\phi:[0,\infty)\to(0,\infty)$ *such that*:

(a) *If* $y_0\cdot P(t_0)y_0=r^2$ *and* $\beta(t_0,y_0,y_0')=0$ *then there exist a* $\delta>0$ *and an open neighborhood* V *of* t_0 *in* $[a_0,a_1]$ *such that*

$$\operatorname*{ess\,inf}_{t\in V}\inf\{2f(t,y,y')\cdot P(t)y+\alpha(t,y,y'):(y,y')\in\mathcal{D}_\delta\}>0\,,$$

where $\mathcal{D}_\delta=\{(y,y'):\|y-y_0\|+\|y'-y_0'\|<\delta\}$.

(b) $s/\phi(s)\in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty(s/\phi(s))\,ds=\infty$ *and* $\|f(t,y,y')\|\le\phi(\|y'\|)$ *for a.e.* $t\in[a_0,a_1]$ *and all* $(y,y')\in\mathcal{D}:=\{(x,x')\in\mathbb{R}^n\times\mathbb{R}^n:\|x\|\le R\}$, *where* $R=\sup\{\|y\|:(t,y)\in\mathcal{P}\}$.

(c) $\|f(t,y,y')\|\le 2\mathfrak{a}(y\cdot f(t,y,y')+\|y'\|^2)+K$ *for a.e.* $t\in[a_0,a_1]$ *and all* $(y,y')\in\mathcal{D}$.

Then the equation

$$y''=f(t,y,y')\quad \textit{for a.e. } t\in[a_0,a_1]$$

for each of the boundary conditions $(\mathrm{B_I})$–$(\mathrm{B_V})$ *has at least one solution* $y(t)$ *in* $H^2([a_0,a_1];\mathbb{R}^n)$ *satisfying*

$$y(t)\cdot P(t)y(t)\le r^2.$$

§6. Differential inclusions on the interval $[0,\infty)$**.** In this section we discuss boundary value problems for differential inclusions on the interval $[0,\infty)$. Such problems have been discussed in [23], [24] and [17] for the scalar case. We refer also to [6], [7], and [39].

Let $F:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{K}(\mathbb{R}^n)$ be a multifunction. We say that F is a *Carathéodory multifunction* if $F|_{[0,r]}$ is a Carathéodory multifunction for all $r>0$. In this section we study the problem of existence of solution to the differential inclusion

$$y''\in F(t,y,y')\quad \text{for a.e. } t\in[0,\infty),$$

where F is a Carathéodory multifunction and y satisfies a certain boundary condition $(\mathcal{A})$. More precisely, $(\mathcal{A})$ denotes the set of all functions satisfying one of the following boundary conditions:

$(\mathrm{C_I})$ $y(0)=r$.

$(\mathrm{C_{II}})$ $Ay(0)-By'(0)=r$, where A and B are symmetric nonnegative definite $n\times n$-matrices such that if $r=0$ then at least one of these matrices is nonsingular, otherwise both are nonsingular.

(C_{III}) $G(y(0), y'(0)) = 0$, where $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which satisfies either (N1) or (N3) for $i = 0$, where $a_0 = 0$.

(C_{IV}) $y'(0) = 0$.

(C_V) $G(y(0), y'(0)) = 0$, where $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which satisfies (N2) for $i = 0$, where $a_0 = 0$.

Note that (C_I) is a special case of (C_{III}), and (C_{IV}) is a special case of (C_V). Moreover, we can replace r by any bounded function $r(u_0, u_0')$. In order to use the results of the previous sections we need some assumptions on the multifunction F.

(I1) There exists a constant $R > 0$ such that if $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$ then for all compact subsets $[0, r] \subset [0, \infty)$, $r > 0$, there is a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[0,r]} \inf\{y \cdot w + \|y'\|^2 : w \in F(t, y, y'),\ (y, y') \in \mathcal{D}_\delta\} > 0,$$

where $\mathcal{D}_\delta := \{(y, y') : \|y - y_0\| + \|y' - y_0'\| < \delta\}$.

(I2) There exists a function $\phi : [0, \infty) \to [0, \infty)$ such that $s/\phi(s) \in L^\infty_{\text{loc}}[0, \infty)$, $\int_0^\infty (s/\phi(s))\, ds = \infty$ and $\|F(t, y, y')\| \le \phi(\|y'\|)$ for a.e. $t \in [0, \infty)$ for all $(y, y') \in \mathbb{R}^n \times \mathbb{R}^n$.

(I3) There exist constants $\mathfrak{a}$ and k such that $\|F(t, y, y')\| \le 2\mathfrak{a}(y \cdot w + \|y'\|^2) + k$ for a.e. $t \in [0, \infty)$ and for all $(y, y') \in \mathbb{R}^{2n}$ and $w \in F(t, y, y')$.

We would like to emphasize that if we suppose that $\|r\| \le R$ in (C_I) and $\|B^{-1}\|\,\|A^{-1}B\|\,\|r\| \le R$ in (C_{II}) then in the assumption (I2) (resp. (I3)) the inequality $\|F(t, y, y')\| \le \phi(\|y'\|)$ (resp. $\|F(t, y, y')\| \le 2\mathfrak{a}(y \cdot w + \|y'\|^2) + k$) can be assumed to be satisfied only for $(y, y') \in \mathcal{D} := \{(x, x') : \|x\| \le R\}$.

(6.1) Theorem. *Suppose that $F : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ is a Carathéodory multifunction such that the hypotheses* (I1)–(I3) *are satisfied. Then the differential inclusion*

$$\begin{cases} y'' \in F(t, y, y') & \text{for a.e. } t \in [0, \infty), \\ y \in \mathcal{A}, \end{cases} \tag{16}$$

has a solution $y \in H^2_{\text{loc}}([0, \infty), \mathbb{R}^n)$. Moreover, if $\phi \in L^\infty_{\text{loc}}[0, \infty)$, then $y \in W^{2,\infty}([0, \infty); \mathbb{R}^n)$.

Proof. Consider the family of differential inclusions

$$\begin{cases} y'' \in F(t, y, y') & \text{for a.e. } t \in [0, m], \\ y \in \mathcal{A}_m, \end{cases} \tag{16_m}$$

where $\mathcal{A}_m$ denotes the set of all functions satisfying either (C_I), (C_{II}) or (C_{III}) and the condition $y(m) = 0$, or (C_{IV}) or (C_V) and the condition

$y'(m) = 0$. We can suppose that there is an $\varepsilon > 0$ such that the operators $L_m y = y'' - \varepsilon y$, defined on $H^2([0,m];\mathbb{R}^n)$, satisfy the assumption (A1). Therefore, it follows from Theorem (4.3) and Corollary (4.4) that (16_m) has a solution $y_m \in H^2([0,m];\mathbb{R}^n)$. Moreover, we find that for all $m \in \mathbb{N}$ the sequence $\{y_{m+k}\}_{k=1}^{\infty}$ restricted to the space $H^2([0,m];\mathbb{R}^n)$ is bounded and thus it contains a subsequence convergent in C^1-norm. Using a "diagonal method" of choosing successively convergent subsequences of $\{y_{m+k}\}_k$ in $C^1([0,m];\mathbb{R}^n)$, as $m \to \infty$, we construct a subsequence $\{y_{m(k)}\}_k$ of $\{y_m\}_m$ such that there is a C^1-function $y : [0,\infty) \to \mathbb{R}^n$ such that $y_{m(k)}|_{[0,m]} \to y|_{[0,m]}$ in C_1-norm for all $m \in \mathbb{N}$. We have to show that for all $m \in \mathbb{N}$, $y|_{[0,m]} \in H^2([0,m];\mathbb{R}^n)$ and that y satisfies (16). Set $C_m := C([0,m];\mathbb{R}^n \times \mathbb{R}^n)$, $L^2_m := L^2([0,m];\mathbb{R}^n)$ and $H^2_m := H^2([0,m];\mathbb{R}^n)$ and consider the diagram

$$\begin{tikzcd} C_m \arrow[r, "\Gamma_m"] & L^2_m \times \mathbb{R}^n \times \mathbb{R}^n \\ & H^2_m \arrow[ul, "j"] \arrow[u, "L_m"'] \end{tikzcd}$$

where

$$\begin{aligned} \Gamma_m(u,v) &= \{w \in L^2_m : w(t) \in F_\varepsilon(t,u,v) \text{ a.e.}\} \times \{g(u,v)\} \times \{w_m(u,v)\}, \\ F_\varepsilon(t,u,v) &= F(t,u,v) - \varepsilon u, \qquad w_m(u,v) = u(m) \text{ or } v(m)\,, \\ L_m u &= (u'' - \varepsilon u, w_0(u,u'), w_m(u,u')), \end{aligned}$$

and $w_0(u,v) = g(u,v)$ denotes the condition $(\mathcal{A})$, where $w_0(u,v) = u(0)$ or $v(0)$ (see the proof of (4.3) for more details). The function $u \in H^2([0,m];\mathbb{R}^n)$ satisfies (16) if and only if

$$(u,u') \in j \circ L_m^{-1} \circ \Gamma_m(u,u') =: \mathcal{F}_m(u,u').$$

By the assumption on F, we see that $\mathcal{F}_m : C_m \to C_m$ is a completely continuous and also upper semi-continuous multivalued map. Therefore, for all closed subsets $A \subset C_m$ the set $\mathcal{F}_m^{-1}(A) = \{w \in C_m : \mathcal{F}_m(w) \cap A \neq \emptyset\}$, $w = (u,v)$, is closed. Take

$$A = \overline{\{(y_{m(k)}, y'_{m(k)})\}} \subset C_m\,.$$

It is clear that $(y_{m(k)}, y'_{m(k)}) \in \mathcal{F}_m^{-1}(A)$, thus $(y,y') \in \mathcal{F}_m^{-1}(A)$ and this means that $\mathcal{F}_m(y,y') \cap A \neq \emptyset$. We shall show that $(y,y') \in \mathcal{F}_m(y,y')$. Suppose for contradiction that $(y,y') \notin \mathcal{F}_m(y,y')$. Thus there exists a subsequence $\{(y_{m(k_l)}, y'_{m(k_l)})\}$ such that $(y_{m(k_l)}, y'_{m(k_l)}) \in \mathcal{F}_m(y,y')$ for $l = 1,2,\ldots$ Since $\mathcal{F}_m(y,y')$ is compact and $(y_{m(k_l)}, y'_{m(k_l)}) \to (y,y')$, we find that $(y,y') \in \mathcal{F}_m(y,y')$ and this is a contradiction. Therefore we have proved that $(y,y') \in j \circ L_m^{-1} \circ \Gamma_m(y,y')$ and this implies that $y \in H^2([0,m];\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [0,\infty)\,, \\ y \in \mathcal{A}\,. \end{cases}$$

Therefore y satisfies (19) and $y \in H^2_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)$. It is left to prove that if $\phi \in L^\infty_{\mathrm{loc}}[0,\infty)$ then $y \in W^{2,\infty}([0,\infty);\mathbb{R}^n)$. Since $\|F(t,y,y')\| \le \phi(\|y'\|)$ we have

$$\|y''(t)\| \le \phi(\|y'(t)\|) \le \operatorname{ess\,sup}\{\phi(s) : s \in [0, \|y'\|_0]\} < \infty$$

and this proves that $y \in W^{2,\infty}([0,\infty);\mathbb{R}^n)$. ∎

(6.2) Corollary. *Suppose that $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and let $\mathfrak{a}$, k, R be nonnegative constants and $\phi : [0,\infty) \to (0,\infty)$ a function such that*:

(a) *If $y \cdot y' = 0$ and $\|y\| > R$ then $y \cdot f(t,y,y') + \|y'\|^2 > 0$ for all $t \in [0,\infty)$.*

(b) *$s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ and $\|f(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [0,\infty)$ and all $(y,y') \in \mathbb{R}^{2n}$.*

(c) *$\|f(t,y,y')\| \le 2\mathfrak{a}(y \cdot f(t,y,y') + \|y'\|^2) + k$ for all $t \in [0,\infty)$ and all $(y,y') \in \mathbb{R}^{2n}$.*

Then the problem

$$\begin{cases} y'' \in F(t,y,y')\,, & t \in [0,\infty), \\ y \in \mathcal{A}\,, \end{cases}$$

has at least one solution $y(t)$ in $W^{2,\infty}([0,\infty);\mathbb{R}^n) \cap C^2([0,\infty);\mathbb{R}^n)$.

(6.3) Corollary. *Let $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Suppose that there are nonnegative constants $\mathfrak{a}$, k, R and a function $\phi : [0,\infty) \to (0,\infty)$ such that*:

(a) *If $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$ then for all compact subsets $[0,r] \subset [0,\infty)$, $r > 0$, there is a $\delta > 0$ such that*

$$\operatorname*{ess\,inf}_{t \in [0,r]} \inf\{y \cdot f(t,y,y') + \|y'\|^2 : (y,y') \in \mathcal{D}_\delta\} > 0\,,$$

where $\mathcal{D}_\delta := \{(y,y') : \|y - y_0\| + \|y' - y_0'\| < \delta\}$.

(b) *$s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ and $\|f(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [0,\infty)$ and all $(y,y') \in \mathbb{R}^{2n}$.*

(c) *$\|f(t,y,y')\| \le 2\mathfrak{a}(y \cdot f(t,y,y') + \|y'\|^2) + k$ for all $(y,y') \in \mathbb{R}^{2n}$ and for a.e. $t \in [0,\infty)$.*

Then the problem

$$\begin{cases} y'' \in f(t,y,y') & \text{for a.e. } t \in [0,\infty)\,, \\ y \in \mathcal{A}\,, \end{cases}$$

has at least one solution $y(t)$ in $W^{2,\infty}([0,\infty);\mathbb{R}^n)$.

Let us now consider the symmetric positive definite $n \times n$-matrix $P(t)$ which is elementwise of class C^2 on $[0,\infty)$. Let $r > 0$ be a given real number. We put (as in §5)

$$\alpha(t,x,x') := 2x' \cdot P(t)x' + 4x' \cdot P'(t)x + x \cdot P''(t)x\,,$$
$$\beta(t,x,x') := 2x' \cdot P(t)x + x \cdot P'(t)x$$

and we suppose that the following condition is satisfied:

(P0)′ $\alpha(t,x,x') \ge 0$ for all $t \in [0,\infty)$, $x, x' \in \mathbb{R}^n$.

Let $F : [0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction such that the following hypotheses are satisfied:

(P1)′ If $y_0 \cdot P(t_0)y_0 = r^2$ and $\beta(t_0,y_0,y_0') = 0$ then there is a $\delta > 0$ and an open neighborhood $\mathcal{V}$ of t_0 in $[0,\infty)$ such that

$$\operatorname*{ess\,inf}_{t\in\mathcal{V}} \inf\{2w \cdot P(t)y + \alpha(t,y,y') : w \in F(t,y,y'),\ (y,y') \in \mathcal{D}_\delta\} > 0,$$

where $\mathcal{D}_\delta := \{(y,y') : \|y - y_0\| + \|y' - y_0'\| < \delta\}$.

(P2)′ There is a function $\phi : [0,\infty) \to (0,\infty)$ such that $s/\phi(s) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $\int_0^\infty (s/\phi(s))\,ds = \infty$ and $\|F(t,y,y')\| \le \phi(\|y'\|)$ for a.e. $t \in [0,\infty)$ and all $(y,y') \in \mathcal{D} := \{(x,x') : \|x\| \le R\}$, where $R = \sup\{\|y\| : y \cdot P(t)y \le r^2,\ t \in [0,\infty)\}$.

(P3)′ There exist constants $k, \mathfrak{a} > 0$ such that $\|F(t,y,y')\| \le 2\mathfrak{a}(y \cdot w + \|y'\|^2) + k$ for a.e. $t \in [0,\infty)$, all $(y,y') \in \mathcal{D}$ and $w \in F(t,y,y')$.

(6.4) Theorem. *Suppose that* $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that* $P(t)$ *and* F *satisfy the hypotheses* (P0)′–(P3)′. *Then the differential inclusion*

$$\begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [0,\infty)\,, \\ y(0) = Ay'(0)\,, \end{cases} \tag{17}$$

where A is an $n \times n$-matrix satisfying $2P(0)A + A^T P'(0)A \ge 0$, *has at least one solution* $y \in H^2_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)$ *satisfying*

$$y(t) \cdot P(t)y(t) \le r^2\,.$$

P r o o f. Consider the family of differential inclusions

$$\begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [0,m], \\ y(0) = Ay'(0), \quad y(m) = 0. \end{cases} \tag{17_m}$$

It follows from Theorem (5.1) that (17_m) has a solution $y_m \in H^2([0,m];\mathbb{R}^n)$ such that $y_m(t) \cdot P(t)y_m(t) \le r^2$ and the sequence $\{y_{m+k}\}_{k=1}^\infty$ restricted to the space $H^2([0,m];\mathbb{R}^n)$ is bounded in H^2-norm. Therefore, it contains a subsequence which is convergent in C^1-norm. Using the "diagonal method" of choosing successively convergent subsequences of $\{y_{m+k}\}_k$ in

$C^1([0,m];\mathbb{R}^n)$, as $m\to\infty$, we construct a subsequence $\{y_{m(k)}\}_k$ of $\{y_m\}_m$ such that there is a C^1-function $y:[0,\infty]\to\mathbb{R}^n$ such that $y_{m(k)}|_{[0,m]}\to y|_{[0,m]}$ in C^1-norm for all $m\in\mathbb{N}$, and $y(t)\cdot P(t)y(t)\le r^2$. We show exactly in the same way as in the proof of Theorem (6.1) that $y|_{[0,m]}\in H^2([0,m];\mathbb{R}^n)$ and that y satisfies (17). ∎

(6.5) Corollary. *Suppose that* $F:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that* $P(t)$ *and* F *satisfy the hypotheses* (P0)′–(P3)′ *and suppose that* $\lim_{t\to\infty}\|P^{-1/2}(t)\|=0$. *Then the differential inclusion*

$$\begin{cases} y''\in F(t,y,y') & \text{for a.e. } t\in[0,\infty),\\ y(0)=Ay'(0), & \lim_{t\to\infty}y(t)=0,\end{cases}$$

where A *is an* $n\times n$*-matrix satisfying* $2P(0)A+A^TP'(0)A\ge 0$, *has at least one solution* $y\in H^2_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)$.

(6.6) Corollary. *Let*

$$P(t)=\begin{bmatrix} p_1(t) & & 0\\ & \ddots & \\ 0 & & p_n(t)\end{bmatrix}$$

be a diagonal matrix such that $p_i(t)>0$, $p_i''(t)\ge 0$ *and* $p_i(t)p_i''(t)-2[p_i'(t)]^2\ge 0$ *for* $i=1,\dots,n$ *and* $t\in[0,\infty)$. *Suppose that* $F:[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{K}(\mathbb{R}^n)$ *is a Carathéodory multifunction such that* $P(t)$ *and* F *satisfy the hypotheses* (P1)′–(P3)′ *and suppose that for* $i=1,\dots,k$, $\lim_{t\to\infty}p_i(t)=\infty$, $k\le n$. *Then the differential inclusion*

$$\begin{cases} y''\in F(t,y,y') & \text{for a.e. } t\in[0,\infty),\\ y(0)=Ay'(0), & \lim_{t\to\infty}y_i(t)=0,\ i=1,\dots,k,\end{cases}$$

where A *satisfies* $2P(0)A+A^TP'(0)A\ge 0$, *has at least one solution* $y\in H^2_{\mathrm{loc}}([0,\infty);\mathbb{R}^n)$.

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References

[1] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin 1984.

[2] J. W. Bebernes and K. Schmitt, *Periodic boundary value problems for systems of second order differential equations*, J. Differential Equations 13 (1973), 33–47.

[3] J.-M. Bony, *Principe du maximum dans les espaces de Sobolev*, C. R. Acad. Sci. Paris 265 (1967), 333–336.

[4] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Masson, Paris 1982.

[5] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, 1977.

[6] M. Cecchi, M. Marini and P. Zecca, *Asymptotic properties of the solutions of nonlinear equations with dichotomies and applications*, Boll. Un. Mat. Ital. (6) 1-C (1982), 209–234.

[7] —, —, —, *Linear boundary value problems for systems of ordinary differential equations on non-compact intervals*, *I*, *II*, Ann. Mat. Pura Appl. (4) 123 (1980), 267–285; 134 (1980), 367–379.

[8] A. Cellina and A. Lasota, *A new approach to the definition of topological degree for multivalued mappings*, Rend. Accad. Naz. Lincei 47 (1969), 434–440.

[9] K. C. Chang, *The obstacle problems and partial differential equations with discontinuous nonlinearities*, Comm. Pure Appl. Math. 33 (1980), 117–146.

[10] J. Dugundji and A. Granas, *Fixed Point Theory*, Vol. 1, PWN, Warszawa 1982.

[11] L. H. Erbe and H. W. Knobloch, *Boundary value problems for systems of second order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A 101 (1985), 61–76.

[12] L. H. Erbe, W. Krawcewicz and S. Chen, *Some existence results for solutions of differential inclusions with retardations*, this issue, 227–239.

[13] L. H. Erbe and P. K. Palamides, *Boundary value problems for second-order differential systems*, J. Math. Anal. Appl. 127 (1) (1987), 80–92.

[14] L. H. Erbe and K. Schmitt, *On solvability of boundary value problems for systems of differential equations*, J. Appl. Math. Phys. 38 (1987), 184–192.

[15] C. Fabry, *Nagumo conditions for systems of second order differential equations*, J. Math. Anal. Appl. 107 (1985), 132–143.

[16] C. Fabry and P. Habets, *The Picard boundary value problem for nonlinear second order vector differential equations*, J. Differential Equations 42 (2) (1987), 186–198.

[17] M. Frigon, *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations différentielles ordinaires*, Dissertationes Math. 296 (1990).

[18] R. E. Gaines and J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Math. 568, Springer, 1977.

[19] K. Gęba, A. Granas, T. Kaczyński et W. Krawcewicz, *Homotopie et équations non linéaires dans les espaces de Banach*, C. R. Acad. Sci. Paris Sér. I Math. 300 (10) (1985), 303–306.

[20] A. Granas, *Homotopy extension theorem in Banach spaces and some of its applications to the theory of non-linear equations*, Bull. Acad. Polon. Sci. 7 (1959), 387–394.

[21] —, *Sur la méthode de continuité de Poincaré*, C. R. Acad. Sci. Paris 287 (1976), 983–985.

[22] A. Granas et Zine el Abdine Guennoun, *Quelques résultats dans la théorie de Bernstein–Carathéodory de l'équation* $y'' = f(t, y, y')$, C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), 703–706.

[23] A. Granas, R. B. Guenther and J. W. Lee, *On a theorem of S. Bernstein*, Pacific. J. Math. 74 (1978), 78–82.

[24] —, —, —, *Nonlinear boundary value problems for some classes of ordinary differential equations*, Rocky Mountain J. Math. 10 (1980), 35–58.

[25] —, —, —, *Nonlinear boundary value problems for ordinary differential equations*, Dissertationes Math. 244 (1981).

[26] —, —, —, *Topological transversality II*; *Applications to the Neumann problem for* $y'' = f(t, y, y')$, Pacific J. Math. 104 (1983), 95–109.

[27] A. Granas, R. B. Guenther, J. W. Lee and D. O'Regan, *Topological transversality III*; *Applications to the boundary value problems on infinite intervals and semiconductor devices*, J. Math. Anal. Appl. 116 (1986), 335–348.

[28] J. Haddad and J. M. Lasry, *Periodic solutions of functional differential inclusions and fixed points of G-selectionable correspondences*, J. Math. Anal. Appl. 110 (1983), 295–312.
[29] P. Hartman, *Ordinary Differential Equations*, Wiley, New York 1964.
[30] T. Kaczyński, *Topological transversality and nonlinear equations in locally convex spaces*, 1987, preprint.
[31] H. W. Knobloch, *Boundary value problems for systems of nonlinear differential equations*, in: Proc. Equadiff IV 1977, Lecture Notes in Math. 703, Springer, 1978, 197–204.
[32] H. W. Knobloch and K. Schmitt, *Nonlinear boundary value problems for systems of differential equations*, Proc. Roy. Soc. Edinburgh Sect. A 78 (1977), 139–159.
[33] W. Krawcewicz, *Contribution à la théorie des équations non linéaires dans les espaces de Banach*, Dissertationes Math. 273 (1988).
[34] J. Mawhin and K. Schmitt, *Upper and lower solutions and semilinear second order elliptic equations with non-linear boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A 97 (1984), 199–207.
[35] T. Pruszko, *Topological degree methods in multivalued boundary value problems*, Nonlinear Anal. 5 (9) (1981), 953–973.
[36] —, *Some applications of the topological degree theory to multivalued boundary value problems*, Dissertationes Math. 229 (1984).
[37] K. Schmitt, *Boundary value problems for quasilinear second order elliptic equations*, Nonlinear Anal. 2 (1978), 263–309.
[38] C. A. Stuart, *Differential equations with discontinuous nonlinearities*, Arch. Rational Mech. Anal. 63 (1976), 59–75.
[39] P. Zecca and P. L. Zecca, *Nonlinear boundary value problems in Banach space for multivalued differential equations on a noncompact interval*, Nonlinear Anal. 3 (1979), 347–352.

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