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On foliations in Sikorski differential spaces with Brouwerian leaves

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Abstract. The class of locally connected and locally homeomorphically homogeneous topological spaces such that every one-to-one continuous mapping of an open subspace into the space is open has been considered. For a foliation F [3] on a Sikorski differential space M with leaves having the above properties it is proved that for some open sets U in M covering the set of all points of M the connected components of $U \cap \underline{L}$ in the topology of M coincide with the connected components in the topology of L for $L \in F$.

1. Brouwerian topological spaces. For a topological space X the set of all points of X will be denoted by $\underline{X}$. A continuous mapping $f : X \to Y$ is said to be *open* iff for any open set A in X the set $f(A)$ is open in Y. A topological space X is said to be *locally homeomorphically homogeneous* (l.h.h.) iff for any $p, q \in \underline{X}$ there exists a homeomorphism $h : U \to V$ such that $p \in U, q \in V, h(p) = q$, U and V are open subspaces of X. A set $\mathcal{T}$ of topological spaces will be called l.h.h. iff the disjoint union $\bigoplus \mathcal{T}$ of $\mathcal{T}$ is l.h.h. A locally connected l.h.h. topological space X such that every continuous 1-1 mapping $f : V \to X$ of an open subspace V of X into X is open will be called *Brouwerian*.

A set $\mathcal{T}$ of topological spaces such that the disjoint union $\bigoplus \mathcal{T}$ is Brouwerian will be called *Brouwerian*. By Brouwer's well-known theorem on open mappings in $\mathbb{R}^n$ every topological manifold is Brouwerian.

Example 1. $\underline{X} = \{0, 1\}$. The topology of X is of the form $\{\emptyset, \underline{X}\}$. X is Brouwerian but not a topological manifold.

The topological space induced by X in the set A is denoted by $X|A$. The set of all connected components of X will be denoted by $\mathrm{cc}(X)$.

By an easy verification we have

Proposition 1. *If X is l.h.h. and a non-empty open subspace of X is Brouwerian then X is Brouwerian.*

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As an immediate corollary of Proposition 1 we get

Proposition 2. *An l.h.h. set $\mathcal{T}$ of topological spaces is Brouwerian iff there exists a Brouwerian space belonging to $\mathcal{T}$.*

Proposition 2 together with the remark that the disjoint union of a set of mutually homeomorphic Brouwerian spaces is Brouwerian allows us to construct a Brouwerian space with an arbitrary infinite cardinal number of the set of points being not a topological manifold. Moreover, we construct a Sikorski differential structure [1] with the topology having the above features.

Example 2. Let I be any set and let I' be the set of all real functions α defined on $\{0,1\} \times I$ and such that $\alpha(0,i) = \alpha(1,i)$ for $i \in I$.

It is easy to check that the set I' is a Sikorski differential structure on $\{0,1\} \times I$. The topology of this structure, i.e. the weakest topology for which all the functions of I' are continuous, is the topology of $\bigoplus_{i \in I} I_i$, where I_i is the topological space with $\{0,1\} \times \{i\}$ as the set of all points and the topology $\{\emptyset, \{0,1\} \times \{i\}\}$, i.e. I_i is homeomorphic to the space in Example 1.

Proposition 3. *If $\mathcal{T}$ is a Brouwerian set of topological spaces such that $\underline{X} \cap \underline{X}' = \emptyset$ when $X \neq X'$, $X, X' \in \mathcal{T}$, T is a topological space satisfying*

$$\underline{T} = \bigcup_{X \in \mathcal{T}} \underline{X}\,, \tag{1}$$

$$\mathrm{id}_{\underline{X}} : X \to T \quad \text{for } X \in \mathcal{T}\,, \tag{2}$$

and there exists a homeomorphism

$$g : T \to Y \times S\,, \tag{3}$$

where Y is Brouwerian, S is a topological space and

$$\bigcup_{X \in \mathcal{T}} \mathrm{cc}(T|\underline{X}) = \{g^{-1}(\underline{Y} \times \{s\});\ s \in \underline{S}\}\,, \tag{4}$$

then

$$\bigcup_{X \in \mathcal{T}} \mathrm{cc}(T|\underline{X}) = \bigcup_{X \in \mathcal{T}} \mathrm{cc}(X)\,. \tag{5}$$

Proof. Let $A \in \mathrm{cc}(X)$, $X \in \mathcal{T}$. Because of the local connectedness of X we see that A is open in X and $X|A$ is connected.

So, by (2), $T|A$ is connected. Let $A \subset \widetilde{A} \in \mathrm{cc}(T|\underline{X})$. By (4) there is exactly one $w_A \in \underline{S}$ such that $\widetilde{A} = g^{-1}(\underline{Y} \times \{w_A\})$. Therefore,

$$A \subset g^{-1}(\underline{Y} \times \{w_A\})\,. \tag{6}$$

Take any $C \in \bigcup_{X\in\mathcal{T}} \mathrm{cc}(T|\underline{X})$. Setting

$$\widehat{C} = \Big\{A; A \in \bigcup_{X\in\mathcal{T}} \mathrm{cc}(X) \text{ and } w_A = s\Big\}, \tag{7}$$

where

$$C = g^{-1}(\underline{Y} \times \{s\}),\ s \in \underline{S}\,, \tag{8}$$

by (6)–(8) we get $A \subset C$ for $A \in \widehat{C}$. Then $\bigcup \widehat{C} \subset C$. On the other hand, taking any $c \in C$, by (1) we get $X \in \mathcal{T}$ with $c \in \underline{X}$. Thus there exists $A \in \mathrm{cc}(X)$ with $c \in A$. According to (6), $c \in g^{-1}(\underline{Y} \times \{w_A\})$. Hence, by (8), $w_A = s$. Therefore, $c \in A \in \widehat{C}$. Thus, $C \subset \bigcup \widehat{C}$. Hence,

$$C = \bigcup \widehat{C}\,. \tag{9}$$

From local connectedness of all topological spaces belonging to $\mathcal{T}$, by (7) and (9) it follows that C is open in X. The homeomorphism (3) induces the following one:

$$g|C:\ T|C \to Y \times S|\{s\}\,.$$

Taking continuous 1-1 mappings $\mathrm{id}_C : X|C \to T|C$ and $\mathrm{pr}_1 : Y \times S|\{s\} \to Y$ we get

$$\mathrm{pr}_1 \circ g|C \circ \mathrm{id}_C : X|C \to Y\,. \tag{10}$$

From Proposition 2 we find that the mapping (10) is open. Therefore (10) is a homeomorphism. Thus $X|C$ is connected. To prove that $C \in \mathrm{cc}(X)$ take any H connected in X with $C \subset H$. Then, by (2), H is connected in T. Therefore there is $C_0 \in \mathrm{cc}(T|\underline{X})$ with $H \subset C_0$. By (4), we get $C_0 = g^{-1}(\underline{Y} \times \{s_0\})\,, s_0 \in S$. From $\emptyset \neq C \subset C_0$ and (8) it follows that $s = s_0$. Thus $C_0 = C$. This yields $H \subset C$. Therefore $C \in \mathrm{cc}(X)$. Thus,

$$\bigcup_{X\in\mathcal{T}} \mathrm{cc}(T|\underline{X}) \subset \bigcup_{X\in\mathcal{T}} \mathrm{cc}(X)\,. \tag{11}$$

The families of sets on the left as well as on the right of the inclusion (11) are partitions of the same set $\underline{T}$. Hence it follows that the inverse inclusions is true. ∎

2. Connected components in distinguished sets of a foliation. For a Sikorski differential space (d.s.) M the set of all points of M and the differential structure of M are denoted by $\underline{M}$ and $F(M)$, respectively. For any set $A \subset \underline{M}$ the d.s. induced by M on A, i.e. the d.s. $(A, F(M)_A)$, is denoted by M_A. We recall the concept of foliation in the category of d.s. [3].

Let M be a d.s. and let $\mathcal{F}$ be a set of disjoint d.s. such that $\underline{M} = \bigcup_{L\in\mathcal{F}} \underline{L}$. $\mathcal{F}$ is assumed to be locally homogeneous (l.h.), i.e. for any $K, L \in \mathcal{F}, p \in \underline{K}$ and $q \in \underline{L}$ there exists a diffeomorphism $h : K_A \to L_B$, where $p \in A \in \mathrm{top}\, K$,

$q \in B \in \operatorname{top} L$ and $h(p) = q$. A set $U \in \operatorname{top} M$ will be called *distinguished* by $\mathcal{F}$ iff there exist $K \in \mathcal{F}$, $V \in \operatorname{top} K$, a d.s. N and a diffeomorphism

$$\Phi : M_U \to K_V \times N \tag{12}$$

such that

$$\bigcup_{L \in \mathcal{F}} \operatorname{cc}(\operatorname{top} M | U \cap \underline{L}) = \{\Phi^{-1}(V \times \{b\}); b \in \underline{N}\}\,. \tag{13}$$

The set $\mathcal{F}$ is said to be a *foliation* on M iff

(i) L is connected and regularly lying [2] in M for $L \in \mathcal{F}$, i.e. $\operatorname{id}_{\underline{L}} : L \to M$ is regular;

(ii) for any $p \in \underline{M}$ there exist $K \in \mathcal{F}$, $V \in \operatorname{top} K$ with $p \in V$ and a diffeomorphism (12) satisfying (13).

From (ii) it follows that M is covered by open sets distinguished by $\mathcal{F}$.

THEOREM. *If $\mathcal{F}$ is a Brouwerian foliation on M then for any open set U in M distinguished by $\mathcal{F}$ we have*

$$\bigcup_{L \in \mathcal{F}} \operatorname{cc}(\operatorname{top} M | U \cap \underline{L}) = \bigcup_{L \in \mathcal{F}} \operatorname{cc}(\operatorname{top} L | U \cap \underline{L})\,. \tag{14}$$

Proof. For a set U distinguished by $\mathcal{F}$ we have a diffeomorphism (12) with (13). Setting, in Proposition 3, $T = \operatorname{top} M_U$, $Y = \operatorname{top} K_V$, $S = \operatorname{top} N$, $\mathcal{T} = \{\operatorname{top} L | U \cap \underline{L}; L \in \mathcal{F}\}$ and the homeomorphism (3) as the one induced by the diffeomorphism (12) we get (4) and, consequently, (5). ■

Remark. In the proof of the Theorem the regularity of $\operatorname{id}_{\underline{L}} : L \to M$ for $L \in F$ has not been essential.

References

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