A class of analytic functions defined by
Ruscheweyh derivative

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Abstract. The function $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ ($p \in \mathbb{N} = \{1, 2, 3, \ldots\}$) analytic in the unit disk $E$ is said to be in the class $K_{n,p}(h)$ if

$$\frac{D^{n+p} f}{D^{n+p-1} f} < h,$$

where

$$D^{n+p-1} f = \frac{z^p}{(1 - z)^{n+p}} * f$$

and $h$ is convex univalent in $E$ with $h(0) = 1$. We study the class $K_{n,p}(h)$ and investigate whether the inclusion relation $K_{n+1,p}(h) \subseteq K_{n,p}(h)$ holds for $p > 1$. Some coefficient estimates for the class are also obtained. The class $A_{n,p}(a,h)$ of functions satisfying the condition

$$a \frac{D^{n+p} f}{D^{n+p-1} f} + (1 - a) \frac{D^{n+p+1} f}{D^{n+p} f} < h$$

is also studied.

Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. We denote by $f * g(z)$ the Hadamard product of two functions $f(z)$ and $g(z)$ in $A(p)$.

Following Goel and Sohi [2] we put

$$D^{n+p-1} f(z) = \frac{z^p}{(1 - z)^{n+p}} * f(z) \quad (n > -p)$$

for the $(n + p - 1)$th order Ruscheweyh derivative of $f(z) \in A(p)$. Let $h$ be convex univalent in $E$, with $h(0) = 1$.

Definition 1. We say that a function $f(z) \in A(p)$ for which

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\[ D^{n+p-1}f(z) \neq 0, \ 0 < |z| < 1, \text{ is in } K_{n,p}(h) \text{ if and only if} \]
\[
\frac{D^{n+p}f}{D^{n+p-1}f} < h. \tag{3}
\]
If we take \( h(z) = 1/(1 + z) \), then (3) reduces to \( \text{Re}(D^{n+p}f/D^{n+p-1}f) > \frac{1}{2} \)
and the class \( K_{n,p}(1/(1 + z)) \) reduces to the class \( K_{n+p-1} \) in the notation employed in [2] for \( n + p \in \mathbb{N} \) and \( p \in \mathbb{N} \). Further, for \( p = 1 \) this class \( K_{n,1} \) reduces to the class \( K_n \) studied by Ruscheweyh [3] who proved that \( K_n \subset K_{n-1}, \ n \in \mathbb{N} \).

In [3] it is proved that \( K_{n+p} \subset K_{n+p-1} \). We are interested in investigating whether \( K_{n+1,p}(h) \subseteq K_{n,p}(h) \) for an arbitrary \( h \). We show that this is not true if \( p > 1 \), even for the choice of \( h(z) = (1 + Az)/(1 + z), \ 0 \leq A < 1 \).

**Definition 2** [1]. Let \( \beta \) and \( \gamma \) be complex constants and let \( h(z) = 1 + h_1(z) + \ldots \) be univalent in the unit disc \( E \). The univalent function \( q(z) = 1 + q_1(z) + \ldots \) analytic in \( E \) is said to be a dominant of the differential subordination
\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z) \tag{4}
\]
if and only if (4) implies that \( p(z) < q(z) \) for all \( p(z) = 1 + p_1 z + \ldots \) that are analytic in \( E \). If \( q(z) < \tilde{q}(z) \) for all dominants \( \tilde{q}(z) \) of (4), then \( q(z) \) is said to be the best dominant of (4).

We need the following theorems which provide a method for finding the best dominant for certain differential subordinations.

**Theorem A** [1]. Let \( \beta \) and \( \gamma \) be complex constants, and let \( h \) be convex (univalent) in \( E \), with \( h(0) = 1 \) and \( \text{Re}[\beta h(z) + \gamma] > 0 \). If \( p(z) = 1 + p_1 z + \ldots \) is analytic in \( E \), then
\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z) \Rightarrow p(z) < h(z). \tag{5}
\]

**Theorem B** [1]. Let \( \beta \) and \( \gamma \) be complex constants, and let \( h \) be convex in \( E \) with \( h(0) = 1 \) and \( \text{Re}[\beta h(z) + \gamma] > 0 \). Let \( p(z) = 1 + p_1 z + \ldots \) be analytic in \( E \), and let it satisfy the differential subordination
\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z). \tag{6}
\]
If the differential equation
\[
q(z) + \frac{zp'(z)}{\beta q(z) + \gamma} = h(z), \tag{7}
\]
with \( q(0) = 1 \), has a univalent solution \( q(z) \), then \( p(z) < q(z) < h(z) \), and \( q(z) \) is the best dominant of (6).
Remark 1 [1]. (i) The conclusion of Theorem B can be written in the form
\[ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \Rightarrow p(z) \prec q(z). \]

(ii) The differential equation (7) has a formal solution given by
\[ q(z) = \frac{zF'(z)}{F(z)} = \frac{\beta + \gamma}{\beta} \left[ \frac{H(z)}{F(z)} \right]^\beta - \frac{\gamma}{\beta}, \]
where
\[ F(z) = \left[ \frac{\beta + \gamma}{z^\gamma} \int_0^z H(\tau)\tau^{\gamma-1} d\tau \right]^{1/\beta}, \]
\[ H(z) = z \exp \int_0^z h(t) - \frac{1}{t} dt. \]

Corollary 1 [1]. Let \( p(z) \) be analytic in \( E \) and let it satisfy the differential subordination
\[ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 - (1 - 2\delta)z}{1 + z} \equiv h(z), \]
with \( \beta > 0 \) and \( -\text{Re}(\gamma/\beta) \leq \delta < 1 \). Then the differential equation
\[ q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = 1, \]
has a univalent solution \( q(z) \). In addition, \( p(z) \prec q(z) \prec h(z) \) and \( q(z) \) is the best dominant of (8).

Finally, we study the class \( A_{n,p}(a,h) \) of functions \( f(z) \in A(p) \) satisfying the condition
\[ a \frac{D^{n+p}f}{D^{n+p-1}f} + (1 - a) \frac{D^{n+p+1}f}{D^{n+p}f} \prec h \]
for \( h \) univalent convex.

1. The classes \( K_{n,p}(h) \)

Theorem 1.1. Let \( f \in K_{n+1,p}(h) \), that is, \( D^{n+p+1}f/D^{n+p}f \prec h \), \( n + p > 0 \). Then
\[ \frac{D^{n+p}f}{D^{n+p-1}f} \prec K \quad \text{where} \quad K = \frac{n + p + 1}{n + p} h - \frac{1}{n + p}, \]
and for \( h = (1 + Az)/(1 + z) \), \( 0 \leq A < 1 \), we have \( D^{n+p}f/D^{n+p-1}f \prec q \prec K_1 \).
and $q$ is the best dominant given by

$$q = \frac{z^{n+p}}{(n+p)(1+z)^{(1-A)(n+p+1)} \int_0^z \frac{t^{n+p-1}}{(1+t)^{(1-A)(n+p+1)}} dt},$$

where $K_1 = \frac{(n+p)(1+Az) - z(1-A)}{(n+p)(1+z)}$.

**Proof.** Set $g(z) = \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}$. Taking logarithmic derivatives and multiplying by $z$, we get

$$zg'(z) g(z) = z(D^{n+p}f(z))' - z(D^{n+p-1}f(z))'.$$

Using the fact that $z(D^{n+p}f)' = (n+p+1)D^{n+p}f - (n+1)D^{n+p}f$, we obtain

$$\frac{zg'(z)}{g(z)} + g(z) = \frac{n+p+1}{n+p} \cdot \frac{D^{n+p}f}{D^{n+p-1}f} - \frac{1}{n+p}.$$

This means that if $D^{n+p-1}f/D^{n+p}f \prec h$, then

$$\frac{zg'(z)}{g(z)} + g(z) \prec \frac{n+p+1}{n+p} \cdot h(z) - \frac{1}{n+p} = K(z).$$

Theorem A now implies that $g(z) \prec K(z)$ if $n+p > 0$ and $\text{Re} K(z) > 0$, which will be true if $\text{Re} h(z) > 1/(n+p+1)$. Next choose $h(z) = (1+Az)/(1+z)$, $0 \leq A < 1$. This choice of $A$ is consistent with the condition on $\text{Re} h$. Then the differential equation

$$\frac{zg'(z)}{g(z)} + g(z) = K(z)$$

has a univalent solution $g(z) = q(z)$ by Corollary 1 and $g(z) \prec q(z) \prec K(z)$.

In the notation of Theorem B and Remark 1, we have

$$H(z) = z \exp \int_0^z \{K(t) - 1\} t^{-1} dt,$$

which gives on substitution for $K(t)$ the following:

$$H(z) = z \exp \int_0^z \left\{ \frac{n+p+1}{n+p} \cdot \frac{1+At}{1+t} - \frac{1}{n+p} - 1 \right\} t^{-1} dt.$$

On simplification we get

$$H(z) = \frac{z}{(1+z)^{(1-A)(n+p+1)/(n+p)}},$$

(11)
From (11) and (12) we obtain \( q(z) = [H(z)/F(z)]^{(n+p)} \). This leads to (9).

\[ F(z) = \left[ (n + p) \int_0^z \frac{t^{n+p}}{(1 + t)^{(1-A)(n+p+1)}} \cdot \frac{1}{t} \, dt \right]^{1/(n+p)}. \]

\[ \text{Corollary 1.1. Let } f \in K_{n+1,p}(1/(1+z)), \text{ that is } D^{n+p+1}f/D^{n+p}f \prec 1/(1+z). \text{ Then } D^{n+p}f/D^{n+p-1}f \prec 1/(1+z) \text{ or } f \in K_{n,p}(1/(1+z)) \text{ so that} \]

\[ K_{n+1,p} \left( \frac{1}{1+z} \right) \subseteq K_{n,p} \left( \frac{1}{1+z} \right), \quad n + p \geq 0. \]

\[ \text{Proof. Now (11) becomes } H(z) = z/(1+z)^{(n+p+1)/(n+p)} \text{ and} \]

\[ F(z) = \left[ (n + p) \int_0^z \frac{t^{n+p}}{(1 + t)^{(n+p+1)}} \cdot \frac{dt}{t} \right]^{1/(n+p)} = \frac{z}{1+z}. \]

Hence \( D^{n+p}f/D^{n+p-1}f \prec 1/(1+z) \), that is, \( f \in K_{n,p}(1/(1+z)) \) or \( \text{Re} (D^{n+p}f/D^{n+p-1}f) > 1/2 \). This is the result obtained by Goel and Sohi [2].

In the above corollary put \( p = 1 \); we then obtain the following:

\[ \text{Corollary 1.2. Let } f \in K_{n+1} \text{ in Ruscheweyh’s notation, that is, } D^{n+2}f(z)/D^{n+1}f(z) \prec 1/(1+z). \text{ Then } D^{n+1}f/D^n f \prec 1/(1+z) \text{ or } f \in K_n \text{ or equivalently } \text{Re} (D^{n+1}f/D^n f) > 1/2. \]

This is the same as Ruscheweyh’s result [3], \( K_{n+1} \subseteq K_n \).

Since

\[ K_{n,p} \left( \frac{1}{1+z} \right) \subseteq K_{n-1,p} \left( \frac{1}{1+z} \right) \subseteq \ldots \subseteq K_{-(p-1),p} \left( \frac{1}{1+z} \right), \quad n + p \geq 0, \]

from Corollary 1.1 we obtain

\[ \text{Corollary 1.3. Let } f \in K_{n,p}(1/(1+z)), \quad n + p \geq 0. \text{ Then } f \in K_{-(p-1),p}(1/(1+z)), \text{ that is, } D^pf/D^0f = z^p/f \prec 1/(1+z), \text{ that is, } \text{Re} (z^p/f) > 1/2. \text{ Such functions } f \text{ of the form } f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \text{ are known to be p-valent [4].} \]

Now we proceed to investigate the case \( A \neq 0 \). In order that the best dominant \( q \) given by (9) may reduce to \((1 + Az)/(1 + z)\), we should have

\[ \left[ \frac{z}{(1+z)^{(1-A)(n+p+1)/(n+p)}} \right]^{n+p} = [F(z)]^{n+p} \frac{1 + Az}{1 + z}. \]
Taking derivative with respect to \( z \) we get

\[
F'(z) = \frac{(n + p)(1 + Az)(1 + z)^{n+p-1} - A(1 + z)^n}{(1 + Az)^2(1 + z)^{(n-p+1)}}
\]

From (12) we get

\[
F'(n+p) = \frac{(n + p)z^{n+p-1}}{(1 + z)^{(n+p+1)}},
\]

(13) and (14) must be identical; which on simplification gives the conditions 
\( A = 0 \) or \( A = 1 \). \( A = 1 \) forces \( h \) to be a constant. We rule out this possibility. Hence the best possible solution exists only when \( A = 0 \). Hence we conclude that \( K_{n+1,p}(h) \) is not contained in \( K_{n,p}(h) \) for \( p > 1 \), even for the choice of \( h(z) = (1 + Az)/(1 + z) \).

Let \( f \in K_{n,p}(h) \). Define

\[
G(z) = z^p \left( \frac{D^{n+p-1}f(z)}{z^p} \right)^{(n+p)/p}.
\]

Then \( zG'/G = p(D^{n+p}f/D^{n+p-1}f) \). We observe that \( f \in K_{n,p}(h) \) if and only if \( (1/p)zG'/G < h \).

We now prove the following

**Theorem 1.2.** Let \( m, n \in \mathbb{N}_0 \). Then \( f \in K_{n,p}(h) \) if and only if

\[
g(z) = (m + p - 1)!z^{1-m} \int_0^z \int_0^x \cdots \int_0^{x_{m-2}} \int_0^{x_{m-1}} \left( \frac{1}{(n+p-1)!} (x_1^{n-1}f(x_1))^{(n+p-1)} \right)^{(m+p)/(n+p)} dx_1 \cdots dx_{m+p-1}
\]

belongs to \( K_{m,p}(h) \).

**Proof.** We have

\[
g(z)z^{m-1} = \int_0^z \int_0^x \cdots \int_0^{x_{m-2}} \int_0^{x_{m-1}} \left( \frac{1}{(n+p-1)!} (x_1^{n-1}f(x_1))^{(n+p-1)} \right)^{(m+p)/(n+p)} dx_1 \cdots dx_{m+p-1}.
\]

Differentiating \( m + p - 1 \) times, we get

\[
\left[ \frac{g(z)}{(m+p-1)!} \right]^{(m+p-1)} = \left[ \frac{1}{(n+p-1)!} (z^{n-1}f(z))^{(n+p-1)} \right]^{(m+p)/(n+p)}.
\]
Since $D^{n+p-1}f = z^p(z^{n-1}f)^{(n+p-1)/(n+p-1)}!$, we get
\[
\frac{D^{n+p-1}g(z)}{z^p} = \left( \frac{D^{n+p-1}f}{z^p} \right)^{(m+p)/(n+p)}.
\]

Set
\[
G(z) = z^p \left( \frac{D^{m+p-1}g}{z^p} \right)^{p/(m+p)} = z^p \left( \frac{D^{n+p-1}f}{z^p} \right)^{p/(n+p)}.
\]

As we have already observed we then have
\[
\frac{zG'}{G} = p \left( \frac{D^{m+p}g}{D^{m+p-1}g} \right) = p \left( \frac{D^{n+p}f}{D^{n+p-1}f} \right),
\]
which implies that
\[
\frac{1}{p} \frac{zG'}{G} \prec h \iff g \in K_{m,p}(h) \iff f \in K_{n,p}(h).
\]

**Coefficient estimates**

**Theorem 1.3.** Let $f \in A(p)$ satisfy
\[
\text{Re} \left\{ \frac{zf'(z)}{pf(z)} \right\} > \frac{1}{2}, \quad z \in E.
\]

Then
\[
|a_{p+k}| \leq \frac{p(p+1) \ldots (p+k-1)}{k!}, \quad k = 1, 2, \ldots
\]

**Proof.** Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k}$ and
\[
g(z) = 2 \left( \frac{zf'(z)}{pf(z)} - \frac{1}{2} \right).
\]

Then $g(0) = 1$ and $\text{Re} \ g(z) > 0$.

Writing $g(z) = 1 + \sum_{k=1}^{\infty} g_k z^k$, we note that $|g_k| \leq 2, \ k = 1, 2, \ldots$

From (16) we get
\[
g(z) = \frac{2zf' - pf}{pf}.
\]

Substituting for $f, f'$ and $g_k$ and simplifying we obtain
\[
\left( 1 + \sum_{k=1}^{\infty} a_{p+k}z^k \right) \left( 1 + \sum_{k=1}^{\infty} g_k z^k \right) = \left\{ 2 + \sum_{k=1}^{\infty} \frac{(p+k)}{p} a_{p+k} z^k \right\} - \left\{ 1 + \sum_{k=1}^{\infty} a_{p+k} z^k \right\}.
\]

Comparing the coefficients of $z^n$, we obtain
\[
a_{p+n} + a_{p+n-1}g_1 + a_{p+n-2}g_2 + \ldots + g_n = \left( 1 + \frac{2n}{p} \right) a_{p+n},
\]
Then for 

\[ \mu_k \]

From (17) and (18) we get

\[ c_{18} \]

\[ c_{17} \]

Simplifying and equating like powers of \( z \) we get

\[ c_1 = -a_{p+1}, \]

\[ c_2 + a_{p+1}c_1(n + p + 1) + a_{p+2}(n + p + 1) = 0. \]

From (17) and (18) we get

\[ (n + p + 1)(a_{p+2} - a_{p+1}^2) = -c_2. \]

Using the well known fact \( |c_2| \leq 1 - |c_1|^2 \), we obtain

\[ |a_{p+2} - a_{p+1}^2| \leq (1 - |a_{p+1}|^2)/(n + p + 1). \]

For \( p = 1 \) this reduces to Theorem 3 of [3]. This fact increases the interest in estimates of the functional \( |a_{n+p-1} - a_{p+1}^{k+p-2}| \) over the functions in the class \( K_{n,\mu}(1/(1 + z)) \). Such functions, as already observed, are \( p \)-valent.

**Theorem 1.5.** Let \( f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \in K_{n,\mu}(1/(1 + z)) \) and

\[ \gamma(n, k, \mu) = \left( \frac{(n+p)/p}{k-1} \right)^{p-1} \left( \frac{n+p+k-2}{k-1} \right). \]

Then for \( \mu \leq \gamma(n, k, \mu) \), we have the sharp estimate

\[ |a_{p+k-1} - \mu a_{p+1}^{k-1}| \leq 1 - \mu, \quad k = 3, 4, \ldots \]

**Proof.** Let

\[ f(z) = (n + p + 1)!z^{1-n} \int_{0}^{z} x_{n+p-1}^{k} \int_{0}^{x} \cdots \]
Functions defined by Ruscheweyh derivative

\[
\int_0^{x_2} \left[ \frac{1}{(p-1)!} \left( \frac{g(x_1)}{x_1} \right)^{(p+1)/p} \right] dx_1 \ldots dx_{n+p-1},
\]

where \( g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \). Using Theorem (1.2), from the above integral we find that
\[
\frac{D^{n+p} f}{D^{n+p-1} f} = \frac{D^p g}{D^{p-1} g}.
\]
Therefore, \( \text{Re} \left( \frac{D^p g}{D^{p-1} g} \right) > 1/2 \) if and only if \( \text{Re} \left( \frac{D^p g}{D^{p-1} g} \right) > 1/2 \). Since
\[
\text{Re} \left( \frac{z(D^p-1)g}{pD^{p-1}g} \right) > \frac{1}{2},
\]
Applying Theorem 1.3 to the function \( D^{p-1}g \), we conclude that
\[
|b_{p+k}| \leq 1, \quad k = 1, 2, \ldots
\]
Further \( a_{p+1} = b_{p+1} \). Put
\[
\left[ \left( \frac{g(z)}{z(p-1)!} \right)^{(p-1)/(n+p)} \right] = \sum_{j=0}^{\infty} c_{j+1} z^j,
\]
so that
\[
(1 + pb_{p+1}z + \frac{p(p+1)}{2!} b_{p+2}z^2 + \ldots)^{(n+p)/p} = \sum_{j=0}^{\infty} c_{j+1} z^j.
\]
This yields
\[
(21) \quad c_k = \frac{(n+p)/p}{k-1} p^{k-1} b_{p+1} + F(b_{p+1}, b_{p+2}, \ldots, b_{p+k-1}).
\]
Also from (20) we get
\[
f(z)z^{n-1} = \frac{z^{n+p-1} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k+n-1}}{(n+p-1)!} = \int_0^x \int_0^{x_2} \cdots \int_0^{x_{n+p-1}} \sum_{j=0}^{\infty} c_{j+1} x_1^j \ dx_1 \ldots dx_{n+p-1}.
\]
This becomes on simplification
\[
\frac{z^{p+n-1} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k+n-1}}{(n+p-1)!} = \sum_{j=0}^{\infty} \frac{c_{j+1} z^{j+n+p-1}}{(j+1)(j+2) \ldots (j+n+p-1)}.
\]
Equating coefficients of like powers we get
\[
\frac{a_{p+k}}{(n+p-1)!} = \frac{c_{k+1}}{(k+1)(k+2) \ldots (k+n+p-1)}.
\]
We prove the following

\[ c_{k+1} = \binom{p+k+n-1}{n+p-1} a_{p+k} = \binom{p+k+n-1}{k} d_{p+k}. \]

Set \((1 - z)^{-(n+p)} = \sum_{j=0}^{\infty} d_j z^j\) so that \(d_k = \binom{n+p+k-2}{k-1}\). We now have from (21)

\[ c_k - \sigma b_{p+1}^{k-1} = F(b_{p+1}, b_{p+2}, \ldots, b_{p+k-1}) \]

Also it is easily seen that \(d_k = c_k\) if \(b_{p+1} = \ldots = b_{p+k-1} = 1\). So we write

\[ \left( \binom{n+p+k-2}{k-1} - \sigma \right) = d_k - \sigma \]

\[ = F(1, 1, \ldots, 1) + \left( \binom{n+p}{k-1} b_{p+1}^{k-1} - \sigma \right). \]

If \(\sigma \leq \binom{n+p}{k-1} b_{p+1}^{k-1}\), that is, if \(\mu \leq \binom{n+p}{k-1} b_{p+1}^{k-1}\), and \(c_k = \binom{n+p+k-2}{k-1} a_{p+k-1}\), we have from (23) and (24)

\[ \left| c_k - \binom{n+p}{k-1} b_{p+1}^{k-1} \right| = |F(b_{p+1}, b_{p+2}, \ldots, b_{p+k-1})| \]

\[ \leq F(1, 1, \ldots, 1) = d_k - \binom{n+p}{k-1} b_{p+1}^{k-1}. \]

(19) follows from this, since \(b_{p+1} = a_{p+1}\). The coefficient bound in (19) is sharp for the function \(f(z) = z^p/(1 - z)\), which belongs to the class \(K_{n,p}(1/1 + z)\), for all \(n\). For \(p = 1\), this reduces to Ruscheweyh's result ([3], Theorem 4).

**Integral transform**

For a function \(f \in A(p)\) we consider the integral transform given by

\[ g(z) = \frac{p + c}{z^c} \int_0^z t^{c-1} f(t) \, dt \quad (n > -p, p \in \mathbb{N}). \]

We prove the following

**Theorem 1.6.** Let \(f \in A(p)\) be in the class \(K_{n+1,p}(h)\) for \(n > -p\) and \(p \in \mathbb{N}\). Then \(g(z) \in K_{n+1,p}(h)\), provided \(\text{Re}\{ (n+p+1)h - (n-c+1) \} > 0\).

**Proof.** By definition of \(g(z)\),

\[ zg'(z) + cg(z) = (p + c)f(z), \]

and therefore

\[ D^{n+p}(zg'(z)) + D^{n+p}(cg(z)) = D^{n+p}((p + c)f(z)). \]
By using $D^{n+p}(g'(z)) = z(D^{n+p}(g(z)))'$ and
\[(26) \quad z(D^{n+p}(g(z)))' = (n + p + 1)D^{n+p+1}g(z) - (n + 1)D^{n+p}g(z)\]
equation (25) reduces to
\[(n + p + 1)\frac{D^{n+p+1}g(z)}{D^{n+p}g(z)} - (n - c + 1) = (p + c)\frac{D^{n+p}f(z)}{D^{n+p}g(z)}.

Setting $D^{n+p+1}g(z)/D^{n+p}g(z) = R(z)$, this reduces to
\[R(z) = \frac{(n - c + 1)}{(n + p + 1)} = \frac{p + c}{n + p + 1} D^{n+p}f(z).

Taking logarithmic derivative and multiplying by $z$ we get
\[\frac{zR'(z)}{R(z) - (n - c + 1)/(n + p + 1)} = \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} - \frac{z(D^{n+p}g(z))'}{D^{n+p}g(z)}.

Using (26) and simplifying we get
\[\frac{zR'(z)}{(n + p + 1)R(z) - (n - c + 1)} + R(z) = \frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} < h(z),
\]
since $f \in K_{n+1}(h)$. Hence we conclude that $R(z) < h(z)$, that is,
\[D^{n+p+1}g(z)/D^{n+p}g(z) < h(z)\text{ if } \Re\{(n + p + 1)\ln h - (n - c + 1)\} > 0.
\]This completes the proof.

Remark. For $p = 1$, Theorem 1.6 reduces to Theorem 5 in [3].

2. The classes $A_{n,p}(a,h)$

Definition 2.1. Let $h$ be convex univalent in $E$ with $h(0) = 1$. The function $f(z) \in A(p)$ such that $D^{n+p-1}f(z) \neq 0$ and $D^{n+p}f(z) \neq 0$ for $0 < |z| < 1$ is said to be in $A_{n,p}(a,h)$ if
\[a \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (1 - a) \frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} < h(z) \quad (a \text{ real}).\]

Theorem 2.1. Let $n \in \mathbb{N}_0$, $p \in \mathbb{N}$, $0 \leq t \leq 1$. Then
\[A_{n,p}(a,h) \cap A_{n,p}(1,h) \subset A_{n,p}(a - 1)t + 1, h).
\]
Proof. If $f \in A_{n,p}(a,h)$ then
\[a \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (1 - a) \frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} < h(z).
\]
Again, $f \in A_{n,p}(1,h)$ implies $D^{n+p}f(z)/D^{n+p-1}f(z) < h(z)$. Let
\[a \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (1 - a) \frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} = h_1(z),
\]
\[\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = h_2(z).
\]
Then \( h_1 \prec h \) and \( h_2 \prec h \) so that \( th_1 + (1 - t)h_2 \prec h \). But

\[
[1 + t(a - 1)] \frac{D^{n+p}f}{D^{n+p-1}f} + (1 - a)t \frac{D^{n+p+1}f}{D^{n+p}f} = th_1 + (1 - t)h_2 \prec h.
\]

Thus \( f \in A_{n,p}((a - 1)t + 1, h) \).

References


