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A classification of certain submanifolds of an S-manifold

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Abstract. A classification theorem is obtained for submanifolds with parallel second fundamental form of an S-manifold whose invariant f-sectional curvature is constant.

0. Introduction. For manifolds with an f-structure, David E. Blair has introduced the analogue of the Kaehler structure in the almost complex case and the quasi-Sasakian structure in the almost contact case, defining the S-manifolds ([1]).

The purpose of this note is to present the following theorem about submanifolds with parallel second fundamental form of an S-manifold of constant invariant f-sectional curvature k:

Theorem 1. *Let M^{m+s} be a submanifold of an S-manifold $N^{2n+s}(k)$ $(k \neq s)$, tangent to the structure vector fields. If the second fundamental form σ of M^{m+s} is parallel, then M^{m+s} is one of the following submanifolds*:

(a) *an invariant submanifold of constant invariant f-sectional curvature k, immersed in $N^{2n+s}(k)$ as a totally geodesic submanifold*;

(b) *an anti-invariant submanifold immersed in $\overline{M}^{2m+s}(k)$, where $\overline{M}^{2m+s}(k)$ is an invariant and totally geodesic submanifold of $N^{2n+s}(k)$ of constant invariant f-sectional curvature $k \neq s$.*

1. Preliminaries. Let N^n be an n-dimensional Riemannian manifold and M^m an m-dimensional submanifold of N^n. Let g be the metric tensor field on N^n as well as the induced metric on M^m. We denote by $\widetilde{\nabla}$ the covariant differentiation in N^n and by ∇ the covariant differentiation in M^m determined by the induced metric. Let $T(N)$ (resp. $T(M)$) be the Lie

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algebra of vector fields on N^n (resp. on M^m) and $T(M)^\perp$ the set of all vector fields normal to M^m. The Gauss–Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X V = -A_V X + D_X V, \tag{1.1}$$

for any $X, Y \in T(M)$ and $V \in T(M)^\perp$, where D is the connection in the normal bundle, σ is the second fundamental form of M^m and A_V is the Weingarten endomorphism associated with V. A_V and σ are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V) .$$

We denote by $\widetilde{R}$ and R the curvature tensors associated with $\widetilde{\nabla}$ and ∇, respectively. The Gauss equation is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &\quad - g(\sigma(X,W),\sigma(Y,Z)) , \qquad X,Y,Z,W \in T(M) .\end{aligned} \tag{1.2}$$

Moreover, we have the following Codazzi equation:

$$(\widetilde{R}(X,Y)Z)^\perp = (\nabla'_X \sigma)(Y,Z) - (\nabla'_Y \sigma)(X,Z) , \tag{1.3}$$

$X, Y, Z \in T(M)$, where $\perp$ denotes the normal projection and the covariant derivative of the second fundamental form σ is defined as follows:

$$(\nabla'_X \sigma)(Y,Z) = D_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) , \tag{1.4}$$

$X, Y, Z \in T(M)$. The second fundamental form σ is said to be *parallel* if $\nabla'\sigma = 0$.

Finally, the submanifold M^m is said to be *totally geodesic* in N^n if $\sigma \equiv 0$.

2. Submanifolds of an S-manifold. Let (N^{2n+s}, g) be a $(2n+s)$-dimensional Riemannian manifold. N^{2n+s} is said to be an *S-manifold* if there exist on N^{2n+s} an f-structure f ([8]) of rank $2n$, and s global vector fields $\xi_1, \ldots, \xi_s$ (*structure vector fields*) such that ([1]):

(i) If $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then

$$f\xi_\alpha = 0 ; \quad \eta_\alpha \circ f = 0 ; \quad f^2 = -I + \sum_\alpha \xi_\alpha \otimes \eta_\alpha ; \tag{2.1}$$

$$g(X,Y) = g(fX, fY) + \Phi(X,Y) ,$$

for any $X, Y \in T(N)$, $\alpha = 1 \ldots, s$,, where $\Phi(X,Y) = \sum_\alpha \eta_\alpha(X)\eta_\alpha(Y)$.

(ii) The f-structure f is *normal*, that is,

$$[f,f] + 2\sum_\alpha \xi\alpha \otimes \mathrm{d}\eta_\alpha = 0 ,$$

where $[f,f]$ is the Nijenhuis torsion of f.

(iii) $\eta_1 \wedge \ldots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ and $d\eta_1 = \ldots = d\eta_s = F$, for any α, where F is the fundamental 2-form defined by $F(X,Y) = g(X, fY)$, $X, Y \in T(N)$.

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds are given in [1], [2], [3], [5].

For the Riemannian connection $\widetilde{\nabla}$ of g on an S-manifold N^{2n+s}, the following were also proved in [1]:

$$\widetilde{\nabla}_X \xi_\alpha = -fX\,, \quad X \in T(N)\,,\ \alpha = 1, \ldots, s\,, \tag{2.2}$$

$$(\widetilde{\nabla}_X f)Y = \sum_\alpha [g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X]\,, \quad X, Y \in T(Y)\,. \tag{2.3}$$

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complementary distribution. $\mathcal{M}$ is determined by $f^2 + I$ and spanned by $\xi_1, \ldots, \xi_s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any α, and if $X \in \mathcal{M}$, then $fX = 0$.

A plane section π is called an *invariant f-section* if it is determined by a vector $X \in \mathcal{L}(p)$, $p \in N^{2n+s}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature $K(X, fX)$, denoted by $H(X)$, is called an *invariant f-sectional curvature*. If N^{2n+s} is an S-manifold of constant invariant f-sectional curvature k, then its curvature tensor has the form ([6])

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) = \sum_{\alpha,\beta} \{& g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) \\ &- g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) \\ &- g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} + \tfrac{1}{4}(k + 3s)\{g(X, W)g(fY, fZ) \\ &- g(X, Z)g(fY, fW) + g(fY, fW)\Phi(X, Z) \\ &- g(fY, fZ)\Phi(X, W)\} + \tfrac{1}{4}(k - s)\{F(X, W)F(Y, Z) \\ &- F(X, Z)F(Y, W) - 2F(X, Y)F(Z, W)\}, \quad X, Y, Z, W \in T(N)\,. \end{aligned} \tag{2.4}$$

Then the S-manifold will be denoted by $N^{2n+s}(k)$.

Now, let M^m be an m-dimensional submanifold immersed in an S-manifold N^{2n+s}. For any $X \in T(M)$, we write

$$fX = TX + NX, \tag{2.5}$$

where TX is the tangential component of fX and NX is the normal component of fX. Then T is an endomorphism of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle.

The submanifold M^m is said to be *invariant* if all ξ_α ($\alpha = 1, \ldots, s$) are always tangent to M^m and N is identically zero, i.e., $fX \in T(M)$, for any $X \in T(M)$. It is easy to show that an invariant submanifold of an S-manifold is an S-manifold too and so $m = 2p + s$. On the other hand, M^m is said to be an *anti-invariant submanifold* if T is identically zero, i.e., $fX \in T(M)^\perp$, for any $X \in T(M)$.

From now on, we suppose that M^m is tangent to the structure vector fields (then $m \geq s$). From (2.2) and (2.5), we easily get

$$(2.6)\quad \nabla_X \xi_\alpha = -TX\,; \quad \sigma(X,\xi_\alpha) = -NX\,, \quad X \in T(M)\,, \quad \alpha = 1,\ldots,s\,.$$

LEMMA 2.1. *Let M^{2p+s} be an invariant submanifold of an S-manifold N^{2n+s}. Then, for any $X,Y \in T(M)$,*

$$(2.7)\qquad \sigma(X,fY) = f\sigma(X,Y) = \sigma(fX,Y)\,.$$

Proof. By using (2.3) and the Gauss–Weingarten formulas, we obtain

$$\begin{aligned}\sigma(X,fY) &= \widetilde{\nabla}_X fY - \nabla_X fY = (\widetilde{\nabla}_X f)Y + f\widetilde{\nabla}_X Y - \nabla_X fY \\ &= \sum_\alpha \{g(fX,fY)\xi_\alpha + \eta_\alpha(Y)f^2x\} + f\nabla_X Y + f\sigma(X,Y) - \nabla_X fY.\end{aligned}$$

Now, since M^{2p+s} is an invariant submanifold, comparing the normal parts yields (2.7).

PROPOSITION 2.2. *Let M^{2p+s} be an invariant submanifold of an S-manifold N^{2n+s}. If H denotes the invariant f-sectional curvature of M^{2p+s} and $\widetilde{H}$ denotes the invariant f-sectional curvature of N^{2n+s}, then $H \leq \widetilde{H}$ and equality holds if and only if M^{2p+s} is totally geodesic.*

Proof. By using the Gauss equation (1.2) and (2.7), we easily prove

$$(2.8)\qquad R(X,fX,fX,X) = \widetilde{R}(X,fX,fX,X) - 2\|\sigma(X,X)\|^2\,,$$

for any $X \in T(M)$. Then the first assertion is immediate from (2.8). Now, if M^{2p+s} is totally geodesic, then $\sigma(X,X) = 0$, for any $X \in T(M)$, and $H = \widetilde{H}$. Conversely, if $H = \widetilde{H}$, then $\sigma(X,X) = 0$, for any unit vector field $X \in T(M)$. Now, since σ is symmetric, the proof is complete.

PROPOSITION 2.3. *If the second fundamental form σ on an invariant submanifold M^{2p+s} of an S-manifold N^{2n+s} is parallel, then M^{2p+s} is totally geodesic.*

Proof. From (2.6), we have $\sigma(X,\xi_\alpha) = 0$, for any $X \in T(M)$ and any α, because M^{2p+s} is an invariant submanifold. Now, since M^{2p+s} is an S-manifold too, from (1.4) and (2.2) we get

$$0 = (\nabla'_X\sigma)(Y,\xi_\alpha) = f\sigma(X,Y)\,,$$

for any $X,Y \in T(M)$, so that $\sigma \equiv 0$ and M^{2p+s} is totally geodesic.

PROPOSITION 2.4. *Let M^{m+s} be a submanifold tangent to the structure vector fields of an S-manifold $N^{2n+s}(k)$ $(k \neq s)$. Then $(\widetilde{R}(X,Y)Z)^\perp = 0$, for any $X,Y,Z \in T(M)$, if and only if M^{m+s} is invariant or anti-invariant.*

Proof. If M^{m+s} is invariant or anti-invariant, from (2.4) we easily have $(\widetilde{R}(X,Y)Z)^\perp = 0\,,\ X,Y,Z \in T(M)$. Conversely, if $(\widetilde{R}(X,Y)Z)^\perp = 0$, from

(2.4) we get

$$0 = \widetilde{R}(X,Y,Z,V) = \tfrac{1}{4}(k-s)\{F(X,V)F(Y,Z) - F(X,Z)F(Y,V) - 2F(X,Y)F(Z,V)\}, \quad V \in T(M)^{\perp}.$$

Putting $X = Z$, we obtain $0 = g(Y, fX)g(X, fV)$, for any $X, Y \in T(M)$ and $V \in T(M)^{\perp}$. Then M^{m+s} is an invariant or anti-invariant submanifold.

3. Proof of Theorem 1. Let M^{m+s} be a submanifold of $N^{2n+s}(k)$ $(k \neq s)$, tangent to the structure vector fields and with parallel second fundamental form. Then the Codazzi equation (1.3) reduces to $(\widetilde{R}(X,Y)Z)^{\perp} = 0$, for any $X, Y, Z \in T(M)$. So, from Proposition 2.4, we find that M^{m+s} is invariant or anti-invariant. If M^{m+s} is invariant, Propositions 2.2 and 2.3 prove (a).

Now, assume that M^{m+s} is anti-invariant. Then the normal space $T_p(M)^{\perp}$, at any point $p \in M^{m+s}$, can be decomposed as

$$T_p(M)^{\perp} = fT_p(M) \oplus \nu_p(M),$$

where $\nu_p(M)$ is the orthogonal complement of $fT_p(M)$ in $T_p(M)^{\perp}$. Now, since σ is parallel, from (2.6) it is easy to prove that

$$D_X fY = f\nabla_X Y, \qquad X, Y \in T(M), \tag{3.1}$$

that is, $fT(M)$ is parallel with respect to the normal connection. Moreover, by using the Gauss–Weingarten formulas and (2.3), we get, for any $X, Y \in T(M)$,

$$A_{fY}X = -\widetilde{\nabla}_X fY + D_X fY = -\sum_{\alpha}\{g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X\} - f\nabla_X Y - f\sigma(X,Y) + D_X fY.$$

Therefore, we have

$$fA_{fY}X - \sum_{\alpha}\eta_\alpha(Y)fX - \sigma(X,Y) = 0.$$

So, for any $W \in \nu$, we obtain $g(\sigma(X,Y), W) = 0$, and consequently

$$A_W = 0. \tag{3.2}$$

Since $fT(M)$ is of constant dimension on M^{m+s} and taking account of (3.1) and (3.2), from the reduction theorem of Erbacher ([4]), there exists a totally geodesic invariant submanifold $\overline{M}^{2m+s}(k)$ in $N^{2n+s}(k)$, where M^{m+s} is immersed in $\overline{M}^{2m+s}(k)$ as an anti-invariant submanifold. This completes the proof.

4. Examples. Let E^{2n+s} be a euclidean space with cartesian coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s)$. Then an S-structure on E^{2n+s} is defined by (cf. [5])

$$\xi_\alpha = 2\partial/\partial z_\alpha \quad (\alpha = 1, \dots, s);$$

$$\eta_\alpha = \frac{1}{2}\Big(dz_\alpha - \sum_{i=1}^{n} y_i dx_i\Big) \quad (\alpha = 1, \dots, s);$$

$$fX = \sum_{i=1}^{n} Y^i \partial/\partial x_i - \sum_{i=1}^{n} X^i \partial/\partial y_i + \Big(\sum_{i=1}^{n} Y^i y_i\Big)\Big(\sum_\alpha \partial/\partial z_\alpha\Big);$$

$$g = \sum_\alpha \eta_\alpha \otimes \eta_\alpha + \frac{1}{4}\sum_{i=1}^{n}\Big(dx_i \otimes dx_i + dy_i \otimes dy_i\Big),$$

where $X = \sum_{i=i}^{n}\Big(X^i \partial/\partial x_i + Y^i \partial/\partial y_i\Big) + \sum_\alpha Z^\alpha \partial/\partial z_\alpha$.

With this structure, E^{2n+s} is an S-manifold of constant invariant f-sectional curvature $k = -3s$ ([5]).

(1) We consider the following natural imbedding of E^{n+s} into $E^{2n+s}(-3s)$:

$$(x_1, \dots, x_n, z_1, \dots, z_s) \mapsto (x_1, \dots, x_n, 0, \dots, 0, z_1, \dots, z_s).$$

A frame field for tangent vector fields in E^{n+s} is given by $\{X_1, \dots, X_n, \xi_1, \dots, \xi_s\}$, where $X_i = \partial/\partial x_i$ $(i = 1, \dots, n)$. Then it is easy to check that E^{n+s} is an anti-invariant submanifold of $E^{2n+s}(-3s)$. Moreover, we have $\sigma(X_i, X_j) = (s/2)(y_j fX_i + y_i fXj)$ and, from (2.6), $\sigma(X_i, \xi_\alpha) = -fX_i$, $\sigma(\xi_\alpha, \xi_\beta) = 0$, $(i, j = 1, \dots, n,\ \alpha, \beta = 1, \dots, s)$. Thus, the second fundamental form of E^{n+s} in $E^{2n+s}(-3s)$ is parallel.

On the other hand, $E^{2m+s}(-3s)$ is a totally geodesic and invariant submanifold of $E^{2n+s}(-3s)$ $(m < n)$.

(2) Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and put

$$M^{n+s} = S^1 \times E^{n-1} \times E^s.$$

Then consider an imbedding of M^{n+s} into $E^{2n+s}(-3s)$ given by

$$(\cos u, x_2, \dots, x_n, \sin u, 0, \dots, 0, z_1, \dots, z_s).$$

A frame field for tangent vector fields in M^{n+s} is given by $\{X_1, \dots, X_n, \xi_1, \dots, \xi_s\}$, where

$$X_1 = -\sin u\, \partial/\partial x_1 + \cos u\, \partial/\partial y_1;$$
$$X_i = \partial/\partial x_i \quad (i = 2, \dots, n).$$

Thus, M^{n+s} is an anti-invariant submanifold of $E^{2n+s}(-3s)$. Moreover, the second fundamental form of M^{n+s} in $E^{2n+s}(-3s)$ is given by

$$\sigma(X_1, X_1) = -(1 + s y_1^2) fX_1;$$
$$\sigma(X_1, X_i) = (s/2)(y_i fX_1 - y_1^2 fX_i) \quad (i = 2, \dots, n);$$

$$\sigma(X_i, X_j) = (s/2)(y_i f X_j + y_j f X_i) \qquad (i,j = 2,\dots,n)\,;$$
$$\sigma(X_i, \xi_\alpha) = -f X_i \qquad (i = 1,\dots,n\,,\ \alpha = 1,\dots,s)\,;$$
$$\sigma(\xi_\alpha, \xi_\beta) = 0 \qquad (\alpha,\beta = 1,\dots,s)\,.$$

Then the second fundamental form of M^{n+s} is parallel.

(3) Let S^{2n+1} be the $(2n+1)$-dimensional unit sphere with the standard Sasakian structure. Then S^{2n+1} is of constant invariant f-sectional curvature $k = 1$ (cf. [7]). If we consider the Clifford hypersurface $M_{p,q}$ defined by

$$M_{p,q} = S^p(\sqrt{(p/2n)}) \times S^q(\sqrt{(q/2n)})\,, \qquad p + q = 2n\,,$$

then $M_{p,q}$ is tangent to the structure vector field ξ, has parallel second fundamental form, but is neither an invariant nor an anti-invariant submanifold of S^{2n+1}.

Therefore, the assumption in Theorem 1 on the invariant f-sectional curvature $k \neq s$ of the ambient S-manifold is essential.

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