A classification of certain submanifolds of an S-manifold

by José L. Cabrerizo, Luis M. Fernández
and Manuel Fernández (Sevilla)

Abstract. A classification theorem is obtained for submanifolds with parallel second fundamental form of an S-manifold whose invariant f-sectional curvature is constant.

0. Introduction. For manifolds with an f-structure, David E. Blair has introduced the analogue of the Kaehler structure in the almost complex case and the quasi-Sasakian structure in the almost contact case, defining the S-manifolds ([1]).

The purpose of this note is to present the following theorem about submanifolds with parallel second fundamental form of an S-manifold of constant invariant f-sectional curvature k:

Theorem 1. Let $M^{m+s}$ be a submanifold of an S-manifold $N^{2n+s}(k)$ ($k \neq s$), tangent to the structure vector fields. If the second fundamental form $\sigma$ of $M^{m+s}$ is parallel, then $M^{m+s}$ is one of the following submanifolds:

(a) an invariant submanifold of constant invariant f-sectional curvature $k$, immersed in $N^{2n+s}(k)$ as a totally geodesic submanifold;

(b) an anti-invariant submanifold immersed in $\overline{M}^{2m+s}(k)$, where $\overline{M}^{2m+s}(k)$ is an invariant and totally geodesic submanifold of $N^{2n+s}(k)$ of constant invariant f-sectional curvature $k \neq s$.

1. Preliminaries. Let $N^n$ be an $n$-dimensional Riemannian manifold and $M^m$ an $m$-dimensional submanifold of $N^n$. Let $g$ be the metric tensor field on $N^n$ as well as the induced metric on $M^m$. We denote by $\nabla$ the covariant differentiation in $N^n$ and by $\nabla$ the covariant differentiation in $M^m$ determined by the induced metric. Let $T(N)$ (resp. $T(M)$) be the Lie
algebra of vector fields on $N^n$ (resp. on $M^m$) and $T(M)\perp$ the set of all vector fields normal to $M^m$. The Gauss–Weingarten formulas are given by

\begin{align}
\tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y) \quad \text{and} \quad \tilde{\nabla}_X V = -A_V X + D_X V,
\end{align}

for any $X,Y \in T(M)$ and $V \in T(M)\perp$, where $D$ is the connection in the normal bundle, $\sigma$ is the second fundamental form of $M^m$ and $A_V$ is the Weingarten endomorphism associated with $V$. $A_V$ and $\sigma$ are related by

\[ g(A_V X, Y) = g(\sigma(X,Y), V). \]

We denote by $\tilde{R}$ and $R$ the curvature tensors associated with $\tilde{\nabla}$ and $\nabla$, respectively. The Gauss equation is given by

\begin{align}
\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)), \quad X,Y,Z,W \in T(M).
\end{align}

Moreover, we have the following Codazzi equation:

\begin{align}
(\tilde{R}(X,Y)Z)\perp &= (\nabla'_X \sigma)(Y,Z) - (\nabla'_Y \sigma)(X,Z),
\end{align}

for any $X,Y,Z \in T(M)$, where $\perp$ denotes the normal projection and the covariant derivative of the second fundamental form $\sigma$ is defined as follows:

\begin{align}
(\nabla'_X \sigma)(Y,Z) &= D_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),
\end{align}

for any $X,Y,Z \in T(M)$. The second fundamental form $\sigma$ is said to be parallel if $\nabla' \sigma = 0$.

Finally, the submanifold $M^m$ is said to be totally geodesic in $N^n$ if $\sigma \equiv 0$.

### 2. Submanifolds of an $S$-manifold.

Let $(N^{2n+s}, g)$ be a $(2n+s)$-dimensional Riemannian manifold. $N^{2n+s}$ is said to be an $S$-manifold if there exist on $N^{2n+s}$ an $f$-structure $f$ ([8]) of rank $2n$, and $s$ global vector fields $\xi_1, \ldots, \xi_s$ (structure vector fields) such that ([1]):

(i) If $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then

\begin{align}
\text{If } \eta_1, \ldots, \eta_s \text{ are the dual 1-forms of } \xi_1, \ldots, \xi_s, \text{ then}
\end{align}

\begin{align}
\eta_\alpha \circ f = 0; \quad \eta_\alpha \circ f = 0; \quad f^2 = -I + \sum_\alpha \xi_\alpha \otimes \eta_\alpha;
\end{align}

\[ g(X,Y) = g(fX,fY) + \Phi(X,Y), \]

for any $X,Y \in T(N)$, $\alpha = 1, \ldots, s$, where $\Phi(X,Y) = \sum_\alpha \eta_\alpha(X)\eta_\alpha(Y)$.

(ii) The $f$-structure $f$ is normal, that is,

\[ [f,f] + 2 \sum_\alpha \xi_\alpha \otimes d\eta_\alpha = 0, \]

where $[f,f]$ is the Nijenhuis torsion of $f$.

(iii) $\eta_1 \wedge \ldots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ and $d\eta_1 = \ldots = d\eta_s = F$, for any $\alpha$, where $F$ is the fundamental 2-form defined by $F(X,Y) = g(X,fY)$, $X,Y \in T(N)$. 

In the case $s = 1$, an $S$-manifold is a Sasakian manifold. For $s \geq 2$, examples of $S$-manifolds are given in [1], [2], [3], [5].

For the Riemannian connection $\nabla$ of $g$ on an $S$-manifold $N^{2n+s}$, the following were also proved in [1]:

\begin{align}
(2.2) & \quad \nabla_X \xi_\alpha = -f_X, \quad X \in T(N), \quad \alpha = 1, \ldots, s, \\
(2.3) & \quad \nabla_X fY = \sum_\alpha [g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X], \quad X, Y \in T(Y).
\end{align}

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complementary distribution. $\mathcal{M}$ is determined by $f^2 + I$ and spanned by $\xi_1, \ldots, \xi_s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha$, and if $X \in \mathcal{M}$, then $fX = 0$.

A plane section $\pi$ is called an invariant $f$-section if it is determined by a vector $X \in \mathcal{L}(p)$, $p \in N^{2n+s}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature $K(X, fX)$, denoted by $H(X)$, is called an invariant $f$-sectional curvature. If $N^{2n+s}$ is an $S$-manifold of constant invariant $f$-sectional curvature $k$, then its curvature tensor has the form ([6])

\begin{align}
(2.4) & \quad \tilde{R}(X, Y, Z, W) = \sum_{\alpha, \beta} \{g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) \\
& \quad - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) \\
& \quad - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z) + \frac{1}{2}(k + 3s)\{g(X, W)g(fY, fZ) \\
& \quad - g(X, Z)g(fY, fW) + g(fY, fW)\Phi(X, Z) \\
& \quad - g(fY, fZ)\Phi(X, W)\} + \frac{1}{2}(k - s)\{F(X, W)F(Y, Z) \\
& \quad - F(X, Z)F(Y, W) - 2F(X, Y)F(Z, W)\}, \quad X, Y, Z, W \in T(N).
\end{align}

Then the $S$-manifold will be denoted by $N^{2n+s}(k)$.

Now, let $M^m$ be an $m$-dimensional submanifold immersed in an $S$-manifold $N^{2n+s}$. For any $X \in T(M)$, we write

\begin{align}
(2.5) & \quad fX = TX + NX,
\end{align}

where $TX$ is the tangential component of $fX$ and $NX$ is the normal component of $fX$. Then $T$ is an endomorphism of the tangent bundle and $N$ is a normal-bundle valued 1-form on the tangent bundle.

The submanifold $M^m$ is said to be invariant if all $\xi_\alpha$ ($\alpha = 1, \ldots, s$) are always tangent to $M^m$ and $N$ is identically zero, i.e., $fX \in T(M)$, for any $X \in T(M)$. It is easy to show that an invariant submanifold of an $S$-manifold is an $S$-manifold too and so $m = 2p + s$. On the other hand, $M^m$ is said to be an anti-invariant submanifold if $T$ is identically zero, i.e., $fX \in T(M)^\perp$, for any $X \in T(M)$.

From now on, we suppose that $M^m$ is tangent to the structure vector.
fields (then \( m \geq s \)). From (2.2) and (2.5), we easily get
\[
(2.6) \quad \nabla_X \xi_\alpha = -T X; \quad \sigma(X, \xi_\alpha) = -N X, \quad X \in T(M), \quad \alpha = 1, \ldots, s.
\]

**Lemma 2.1.** Let \( M^{2p+s} \) be an invariant submanifold of an \( S \)-manifold \( N^{2n+s} \). Then, for any \( X, Y \in T(M) \),
\[
(2.7) \quad \sigma(X, fY) = f\sigma(X, Y) = \sigma(fX,Y).
\]

**Proof.** By using (2.3) and the Gauss–Weingarten formulas, we obtain
\[
\sigma(X, fY) = \tilde{\nabla}_X fY - \nabla_X fY = (\tilde{\nabla}_X f)Y + f\tilde{\nabla}_X Y - \nabla_X fY
\]
\[
= \sum_\alpha \{g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2x\} + f\nabla_X Y + f\sigma(X, Y) - \nabla_X fY.
\]

Now, since \( M^{2p+s} \) is an invariant submanifold, comparing the normal parts yields (2.7).

**Proposition 2.2.** Let \( M^{2p+s} \) be an invariant submanifold of an \( S \)-manifold \( N^{2n+s} \). If \( H \) denotes the invariant \( f \)-sectional curvature of \( M^{2p+s} \) and \( \tilde{H} \) denotes the invariant \( f \)-sectional curvature of \( N^{2n+s} \), then \( H \leq \tilde{H} \) and equality holds if and only if \( M^{2p+s} \) is totally geodesic.

**Proof.** By using the Gauss equation (1.2) and (2.7), we easily prove
\[
(2.8) \quad R(X, fX, fX, X) = \tilde{R}(X, fX, fX, X) - 2\|\sigma(X, X)\|^2,
\]
for any \( X \in T(M) \). Then the first assertion is immediate from (2.8). Now, if \( M^{2p+s} \) is totally geodesic, then \( \sigma(X, X) = 0 \), for any \( X \in T(M) \), and \( H = \tilde{H} \). Conversely, if \( H = \tilde{H} \), then \( \sigma(X, X) = 0 \), for any unit vector field \( X \in T(M) \). Now, since \( \sigma \) is symmetric, the proof is complete.

**Proposition 2.3.** If the second fundamental form \( \sigma \) on an invariant submanifold \( M^{2p+s} \) of an \( S \)-manifold \( N^{2n+s} \) is parallel, then \( M^{2p+s} \) is totally geodesic.

**Proof.** From (2.6), we have \( \sigma(X, \xi_\alpha) = 0 \), for any \( X \in T(M) \) and any \( \alpha \), because \( M^{2p+s} \) is an invariant submanifold. Now, since \( M^{2p+s} \) is an \( S \)-manifold too, from (1.4) and (2.2) we get
\[
0 = (\nabla_X \sigma)(Y, \xi_\alpha) = f\sigma(X, Y),
\]
for any \( X, Y \in T(M) \), so that \( \sigma \equiv 0 \) and \( M^{2p+s} \) is totally geodesic.

**Proposition 2.4.** Let \( M^{m+s} \) be a submanifold tangent to the structure vector fields of an \( S \)-manifold \( N^{2n+s} \) \( (k) \quad (k \neq s) \). Then \( (\tilde{R}(X, Y)Z)^\perp = 0 \), for any \( X, Y, Z \in T(M) \), if and only if \( M^{m+s} \) is invariant or anti-invariant.

**Proof.** If \( M^{m+s} \) is invariant or anti-invariant, from (2.4) we easily have \( (\tilde{R}(X, Y)Z)^\perp = 0 \), \( X, Y, Z \in T(M) \). Conversely, if \( (\tilde{R}(X, Y)Z)^\perp = 0 \), from
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(2.4) we get

\[ 0 = \tilde{R}(X, Y, Z, V) = \frac{1}{4}(k - s)\{F(X, V)F(Y, Z) - F(X, Z)F(Y, V) \]

\[ - 2F(X, Y)F(Z, V)\}, \quad V \in T(M)^\perp. \]

Putting $X = Z$, we obtain $0 = g(Y, fX)g(X, fV)$, for any $X, Y \in T(M)$ and $V \in T(M)^\perp$. Then $M^{m+s}$ is an invariant or anti-invariant submanifold.

3. Proof of Theorem 1. Let $M^{m+s}$ be a submanifold of $N^{2n+s}(k)$ ($k \neq s$), tangent to the structure vector fields and with parallel second fundamental form. Then the Codazzi equation (1.3) reduces to $(\tilde{R}(X, Y)Z)^\perp = 0$, for any $X, Y, Z \in T(M)$. So, from Proposition 2.4, we find that $M^{m+s}$ is invariant or anti-invariant. If $M^{m+s}$ is invariant, Propositions 2.2 and 2.3 prove (a).

Now, assume that $M^{m+s}$ is anti-invariant. Then the normal space $T_p(M)^\perp$, at any point $p \in M^{m+s}$, can be decomposed as

\[ T_p(M)^\perp = fT_p(M) \oplus \nu_p(M), \]

where $\nu_p(M)$ is the orthogonal complement of $fT_p(M)$ in $T_p(M)^\perp$. Now, since $\sigma$ is parallel, from (2.6) it is easy to prove that

\[ D_X fY = f\nabla_X Y, \quad X, Y \in T(M), \]

that is, $fT(M)$ is parallel with respect to the normal connection. Moreover, by using the Gauss–Weingarten formulas and (2.3), we get, for any $X, Y \in T(M)$,

\[ A_{fY}X = - \tilde{\nabla}_X fY + D_X fY = - \sum_{\alpha}\{g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X\} \]

\[ - f\nabla_X Y - f\sigma(X, Y) + D_X fY. \]

Therefore, we have

\[ fA_{fY}X - \sum_{\alpha}\eta_\alpha(Y)fX - \sigma(X, Y) = 0. \]

So, for any $W \in \nu$, we obtain $g(\sigma(X, Y), W) = 0$, and consequently

\[ A_W = 0. \]

Since $fT(M)$ is of constant dimension on $M^{m+s}$ and taking account of (3.1) and (3.2), from the reduction theorem of Erbacher ([4]), there exists a totally geodesic invariant submanifold $\overline{M}^{m+s}(k)$ in $N^{2n+s}(k)$, where $M^{m+s}$ is immersed in $\overline{M}^{m+s}(k)$ as an anti-invariant submanifold. This completes the proof.

4. Examples. Let $E^{2n+s}$ be a euclidean space with cartesian coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s)$. Then an $S$-structure on $E^{2n+s}$ is
defined by (cf. [5])
\[ \xi_\alpha = 2\partial/\partial z_\alpha \quad (\alpha = 1, \ldots, s); \]
\[ \eta_\alpha = \frac{1}{2} \left( dz_\alpha - \sum_{i=1}^{n} y_idx_i \right) \quad (\alpha = 1, \ldots, s); \]
\[ f_X = \sum_{i=1}^{n} Y^i\partial/\partial x_i - \sum_{i=1}^{n} X^i\partial/\partial y_i + \left( \sum_{\alpha}^{n} Y^\alpha y_\alpha \right) \left( \sum_{\alpha}^{n} \partial/\partial z_\alpha \right); \]
\[ g = \sum_{\alpha} \eta_\alpha \otimes \eta_\alpha + \frac{1}{4} \sum_{i=1}^{n} (dx_i \otimes dx_i + dy_i \otimes dy_i), \]
where \( X = \sum_{i=1}^{n} \left( X^i\partial/\partial x_i + Y^i\partial/\partial y_i \right) + \sum_{\alpha} Z^{\alpha}\partial/\partial z_\alpha. \)

With this structure, \( E^{n+s} \) is an \( S \)-manifold of constant invariant \( f \)-sectional curvature \( k = -3s \) ([5]).

(1) We consider the following natural imbedding of \( E^{n+s} \) into \( E^{2n+s}(-3s) \):
\[
(x_1, \ldots, x_n, z_1, \ldots, z_s) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0, z_1, \ldots, z_s).
\]

A frame field for tangent vector fields in \( E^{n+s} \) is given by \( \{X_1, \ldots, X_n, \xi_1, \ldots, \xi_s\} \), where \( X_i = \partial/\partial x_i \) (\( i = 1, \ldots, n \)). Then it is easy to check that \( E^{n+s} \) is an anti-invariant submanifold of \( E^{2n+s}(-3s) \). Moreover, we have \( \sigma(X_i, X_j) = (s/2)(y_j f X_1 + y_i f X_j) \) and, from (2.6), \( \sigma(X_i, \xi_\alpha) = -f X_i, \sigma(\xi_\alpha, \xi_\beta) = 0, (i, j = 1, \ldots, n, \alpha, \beta = 1, \ldots, s) \). Thus, the second fundamental form of \( E^{n+s} \) in \( E^{2n+s}(-3s) \) is parallel.

On the other hand, \( E^{2n+s}(-3s) \) is a totally geodesic and invariant submanifold of \( E^{2n+s}(-3s) \) \( (m < n) \).

(2) Let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \), and put
\[ M^{n+s} = S^1 \times E^{n-1} \times E^s. \]

Then consider an imbedding of \( M^{n+s} \) into \( E^{2n+s}(-3s) \) given by
\[ (\cos u, x_2, \ldots, x_n, \sin u, 0, \ldots, 0, z_1, \ldots, z_s). \]

A frame field for tangent vector fields in \( M^{n+s} \) is given by \( \{X_1, \ldots, X_n, \xi_1, \ldots, \xi_s\} \), where
\[ X_1 = -\sin u \partial/\partial x_1 + \cos u \partial/\partial y_1; \]
\[ X_i = \partial/\partial x_i \quad (i = 2, \ldots, n). \]

Thus, \( M^{n+s} \) is an anti-invariant submanifold of \( E^{2n+s}(-3s) \). Moreover, the second fundamental form of \( M^{n+s} \) in \( E^{2n+s}(-3s) \) is given by
\[ \sigma(X_1, X_1) = -(1 + sy_1^2)f X_1; \]
\[ \sigma(X_1, X_i) = (s/2)(y_i f X_1 - y_1^2 f X_i) \quad (i = 2, \ldots, n). \]
\[ \sigma(X_i, X_j) = \frac{s}{2}(y_i fX_j + y_j fX_i) \quad (i, j = 2, \ldots, n); \]
\[ \sigma(X_i, \xi_\alpha) = -fX_i \quad (i = 1, \ldots, n, \alpha = 1, \ldots, s); \]
\[ \sigma(\xi_\alpha, \xi_\beta) = 0 \quad (\alpha, \beta = 1, \ldots, s). \]

Then the second fundamental form of \( M^{n+s} \) is parallel.

(3) Let \( S^{2n+1} \) be the \((2n+1)\)-dimensional unit sphere with the standard Sasakian structure. Then \( S^{2n+1} \) is of constant invariant \( f \)-sectional curvature \( k = 1 \) (cf. [7]). If we consider the Clifford hypersurface \( M_{p,q} \) defined by

\[
M_{p,q} = S^p(\sqrt{p/2n}) \times S^q(\sqrt{q/2n}), \quad p + q = 2n,
\]

then \( M_{p,q} \) is tangent to the structure vector field \( \xi \), has parallel second fundamental form, but is neither an invariant nor an anti-invariant submanifold of \( S^{2n+1} \).

Therefore, the assumption in Theorem 1 on the invariant \( f \)-sectional curvature \( k \neq s \) of the ambient \( S \)-manifold is essential.

References

[8] K. Yano, On a structure defined by a tensor field \( f \) of type \((1,1)\) satisfying \( f^3 + f = 0 \), Tensor 14 (1963), 99–109.