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# Solvability of some singular and nonsingular nonlinear third order boundary value problems

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**Abstract.** Existence of positive solution to certain classes of singular and nonsingular third order nonlinear two point boundary value problems is examined using the idea of Topological Transversality.

**1.** In this paper we establish existence of positive solutions to some nonlinear singular and nonsingular third order boundary value problems of the form

$$(1.1)\qquad \begin{cases} y''' + f(t, y, y') = 0, \quad 0 < t < 1, \\ y \text{ satisfies a boundary condition } B. \end{cases}$$

Here $B$ will denote either

(i) $y''(0) = c \le 0$, $y'(1) = b \ge 0$, $y(0) = a \ge 0$,
(ii) $y''(0) = 0$, $y'(0) = 0$, $y(1) = a \ge 0$, or
(iii) $y''(0) = 0$, $y(0) = a \ge 0$, $y(1) = b \ge 0$.

The above problems are singular because $f$ is allowed to be singular at $y = 0$. Third order boundary value problems have become quite popular in the last ten years with most of the work concentrated on nonsingular problems; see [1], [2], [6], [7] and [8] for example. Also the author in [9] has discussed boundary value problems where $f$ is allowed to be singular at $t = 0$ and $t = 1$. Results for singular initial value problems may be found in [3]. This paper was motivated from [11] where S. Taliaferro considered problems of the form $y'' + a(t)y^{\alpha} = 0$, $\alpha < 0$ with $y(0) = 0$ and $y(1) = 0$. Finally here we summarize briefly the plan of the paper. We begin by showing that (1.1) has a $C^3[0, 1]$ solution for all $a > 0$ with $a \le a_0$ fixed. The key ideas used here are the Topological Transversality Theorem and the existence of a priori

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bounds. We note that when $a > 0$, (1.1) does not involve singularities in $y$. To take care of the singular case (i.e. (1.1) with $a = 0$) we pass to the limit in $a$ (i.e. let $a \to 0$). Here we deduce the existence of a $C^1[0,1] \cap C^3(0,1)$ solution to (1.1) if $B$ denotes either (i) or (iii) whereas we will obtain a $C[0,1] \cap C^3(0,1)$ solution to (1.1) in the case where $B$ denotes (ii). The main idea in this step is the Arzelà–Ascoli Theorem.

**2.** We begin by establishing existence of positive solutions on $(0,1]$ to

$$\begin{cases} y''' + f(t,y) = 0, \quad 0 < t < 1, \\ y''(0) = 0, \\ y'(1) = b \geq 0, \\ y(0) = a \geq 0, \end{cases} \tag{2.1}$$

where $f$ satisfies the following conditions:

(2.2) $f$ is continuous on $[0,1] \times (0,\infty)$ with $\lim_{y \to 0^+} f(t,y) = \infty$ uniformly on compact subsets of $(0,1)$,

(2.3) $0 < f(t,y) \leq g(y)$ on $[0,1] \times (0,\infty)$ where $g$ is continuous and nonincreasing on $(0,\infty)$.

In addition to the above we will have the following assumptions on $g$:

(2.4) for any $z \in [0,\infty)$, $G(z) = \int_0^z g(u)\,du < \infty$,

(2.5) there exist constants $A > 0$, $B > 0$ and $\alpha$ with $G(z) \leq Az^\alpha + B$ for $z \in [0,\infty)$ and $0 \leq \alpha < 2$.

We will discuss separately the cases (i) $a > 0$ and (ii) $a = 0$. It should be remarked here that the results for the case $a > 0$ will be used to discuss (2.1) with $a = 0$.

Case 1: $a > 0$. By a solution to (2.1) with $a > 0$ we mean a function $y \in C^3[0,1]$ that satisfies the differential equation and the boundary conditions. Also if $y$ is a solution to (2.1) then $y > 0$ on $(0,1)$ and so $y''' < 0$ on $(0,1)$. Thus $y'' < 0$ on $(0,1)$, which in turn implies $y' > 0$ on $(0,1)$ so $y$ is strictly increasing on $(0,1)$ and in particular $y \geq a$ on $[0,1]$.

Theorem 2.1 *Suppose conditions* (2.2)–(2.5) *are satisfied. For* $\lambda \in [0,1]$ *consider the family of problems*

$$\begin{cases} y''' + \lambda f(t,y) = 0, \quad 0 < t < 1, \\ y''(0) = 0, \quad y'(1) = b, \quad y(0) = a. \end{cases} \tag{$2.6_\lambda$}$$

*Let* $0 < a \leq a_0$. *Then there exist positive constants* $M_0$, $M_1$, $M_2$, $M_3$ *independent of* $\lambda$ *with* $M_0$, $M_1$ *depending on* $a_0$ *such that for* $t \in [0,1]$

$$a \leq y(t) \leq M_0, \quad b \leq y'(t) \leq M_1, \quad -M_2 \leq y''(t) \leq 0, \quad -M_3 \leq y''(t) \leq 0$$

*for each solution* $y$ *to* $(2.6_\lambda)$ *and* $a \leq a_0$.

Remark. It should be noted here that for fixed positive $a$, Theorem 2.1 holds if only conditions (2.2) and (2.3) are satisfied. To see this suppose $y$ is a solution to $(2.6_\lambda)$. Then clearly $y(t) \geq a$, $y'(t) \geq b$, $y''(t) \leq 0$, $y'''(t) \leq 0$ for $t \in [0,1]$. The differential equation yields $-y''' \leq \lambda g(y) \leq g(a) \equiv M_3$ and integration gives $M_2$, $M_1$ and $M_0$ immediately. However, the goal of this paper is to examine singular problems and to achieve this we need to show that we can obtain $M_0$ and $M_1$ independent of $a$ for a bounded $a$. For these singular problems $a = 1/n$, $n \in \mathbb{N}^+ = \{1, 2, \ldots\}$. Thus we will prove Theorem 2.1 with this in mind.

Proof. Let $y$ be a solution to $(2.6_\lambda)$. The case $\lambda = 0$ is trivial so assume $0 < \lambda \leq 1$. Then $-y''' \leq \lambda g(y) \leq g(y-a)$ since $g$ is nonincreasing. Multiply by $y'$ and integrate from 0 to $t$ to obtain

$$-y'(t)y''(t) + \int_0^t [y''(s)]^2\, ds \leq \int_0^{y(t)-a} g(u)\, du = G(y(t)-a)\,.$$

Thus we have $-y'(t)y''(t) \leq G(y(t)-a)$. Here $G$ is an increasing map from $[0,\infty)$ into $[0,\infty)$ so we have $-y'(t)y''(t) \leq G(y(t)) \leq G(y(1))$. Integrate the above inequality from $t$ to 1 to obtain

$$-\frac{b^2}{2} + \frac{[y'(t)]^2}{2} \leq G(y(1))$$

so $y'(t) \leq \sqrt{2G(y(1)) + b^2}$. Hence $y(1) \leq a + \sqrt{2G(y(1)) + b^2}$. Now assumption (2.5) implies there exists a constant $M_0$ (independent of $\lambda$) such that $y(t) \leq y(1) \leq M_0$ for $t \in [0,1]$. In addition the above analysis yields

$$y'(t) \leq \sqrt{2G(y(1)) + b^2} \leq \sqrt{2G(M_0) + b^2} = M_1\,.$$

If $a \leq a_0$ then we can choose $M_0$ and $M_1$ independent of $a$ since $y(1) \leq a_0 + \sqrt{2G(y(1)) + b^2}$.

Also we have $0 \leq -y'''(t) \leq \lambda \sup_{[a,M_0]} g(u) \leq g(a) = M_2$ and finally integration from 0 to $t$ yields a bound for $y''(t)$. ■

We will now apply the Topological Transversality Theorem [4, 5] to obtain our basic existence theorem. For notational purposes set

$$C^3_B[0,1] = \{u \in C^3[0,1]; u(0) = a,\ u'(1) = b,\ u''(0) = 0\}\,,$$
$$C^3_{B_0}[0,1] = \{u \in C^3[0,1]; u(0) = 0,\ u'(1) = 0,\ u''(0) = 0\}\,.$$

THEOREM 2.2. *Suppose conditions* (2.2) *and* (2.3) *are satisfied. Then a* $C^3[0,1]$ *solution of* (2.1) *exists.*

Proof. Consider the family of problems

$$(2.7_\lambda) \qquad \begin{cases} y''' + \lambda \bar{f}(t,y) = 0, \quad 0 < \lambda < 1, \\ y''(0) = 0, \quad y'(1) = b, \quad y(0) = a, \end{cases}$$

where

$$\bar{f}(t,y) = \begin{cases} f(t,y) & \text{for } y \geq a, \\ f(t,a) & \text{for } y \leq a. \end{cases}$$

Every solution $v$ of $(2.7_\lambda)$ satisfies $v \geq a$ and hence is a solution to $(2.6_\lambda)$. Also $\bar{f}$ satisfies the hypothesis imposed on $f$ since for $0 < y \leq a$, $\bar{f}(t,y) = f(t,a) \leq g(a) \leq g(y)$. Hence the conclusion of Theorem 2.1 remains valid for solutions to $(2.7_\lambda)$. Let $\tilde{M}_0 = \max\{M_0, b+a\}$ and

$$V = \{u \in C^3_B[0,1] : a/2 < u(t) < \tilde{M}_0 + 1,\ |u'(t)| < M_1, \\ |u''(t)| < M_2 + 1,\ |u'''(t)| < M_3 + 1\}.$$

$V$ is an open subset of $C^3_B[0,1]$ which in turn is a convex subset of $C^3[0,1]$.

Define mappings $F_\lambda : C[0,1] \to C[0,1]$, $j : C^3_B[0,1] \to C[0,1]$, $L : C^3_B[0,1] \to C[0,1]$ by $(F_\lambda u)(t) = -\lambda \bar{f}(t,u(t))$, $ju = u$ and $Lu(t) = u'''(t)$. $F_\lambda$ is continuous and $j$ is completely continuous by the Arzelà–Ascoli Theorem. Finally, we claim $L^{-1}$ exists and is continuous. To see this define $N : C^3_{B_0}[0,1] \to C[0,1]$ by $Nu = u'''$. $N^{-1}$ is a continuous linear operator by the Bounded Inverse Theorem; see [10, Theorem 5.10]. Thus $L^{-1}$ exists and is given by

$$(L^{-1}g)(x) = bx + a + (N^{-1}g)(x)$$

and so is continuous. Now define the map $H_\lambda : \overline{V} \to C^3_B[0,1]$ by $H_\lambda u = L^{-1}F_\lambda ju$. $H_\lambda$ is a compact homotopy and $H_\lambda u = u$ means $Lu = F_\lambda ju$, i.e. $u''' = -\lambda \bar{f}(t,u(t))$ and $u$ satisfies the boundary conditions. Therefore $H_\lambda$ is fixed point free on $\partial V$ by construction of $V$ and Theorem 2.1. Finally, for any $u \in \overline{V}$, $H_0(u) = u_0$, $u_0(x) = bx + a$, i.e. $H_0$ is a constant map and so is essential [4]. The topological transversality theorem [4. p. 86] implies that $H_1$ is essential, i.e. $(2.7_1)$ has a solution and therefore (2.1) has a solution. ∎

It is also possible to consider in this section problems of the form

$$\begin{cases} y''' + f(t,y,y') = 0, \quad 0 < t < 1, \\ y''(0) = c \leq 0, \\ y'(1) = b > 0, \\ y(0) = a > 0, \end{cases} \tag{2.8}$$

where $f$ satisfies the following:

(2.9) $f$ is continuous on $[0,1] \times (0,\infty) \times (-\infty,\infty)$ with $\lim_{y\to 0^+} f(t,y,p) = \infty$ uniformly on compact subsets of $[0,1] \times (-\infty,\infty)\backslash\{0\}$,

(2.10) $0 < f(t,y,p) \leq g(y)p$ on $[0,1] \times (0,\infty) \times [b,\infty)$ where $g$ is continuous and nonincreasing on $(0,\infty)$ with $f \geq 0$ on $[0,1] \times (0,\infty) \times (-\infty,\infty)$.

THEOREM 2.3. *Suppose that conditions* (2.9), (2.10), (2.4) *and* (2.5) *with* $0 \le \alpha < 1$ *are satisfied. For* $\lambda \in [0,1]$ *consider*

$$(2.11_\lambda) \qquad \begin{cases} y''' + \lambda f(t,y,y') = 0, & 0 < t < 1, \\ y''(0) = c,\ y'(1) = b,\ y(0) = a. \end{cases}$$

*Then there exists constants* $M_0$, $M_1$, $M_2$, $M_3$ *independent of* $\lambda$ *such that for* $t \in [0,1]$

$$a \le y(t) \le M_0,\ b \le y(t) \le M_1,\ -M_2 \le y''(t) \le c,\ -M_3 \le y'''(t) \le 0$$

*for each solution* $y$ *to* $(2.11_\lambda)$.

Proof. Suppose $y$ is a solution to $(2.11_\lambda)$. Assume $0 < \lambda \le 1$. Clearly (since (2.9) and (2.10) are satisfied) $y(t) \ge a$, $y'(t) \ge b$, $y''(t) \le c$, $y'''(t) \le 0$ for $t \in [0,1]$. We also have $-y''' \ge g(y-a)y'$ since $g$ is nonincreasing. Integrating from 0 to $t$ yields $-y''(t) \le G(y(t)-a) - c \le G(y(t)) - c$ since $G$ is an increasing map from $[0,\infty)$ into $[0,\infty)$. Now integrate from $t$ to 1 to obtain

$$y'(t) \le \int_t^1 G(y(u))\,du + b - c \le G(y(1)) + b - c$$

and so $y(1) \le G(y(1)) + b + a - c$. Assumption (2.5) with $0 \le \alpha < 1$ implies there exists a constant $M_0$ (independent of $\lambda$) such that $y(1) \le M_0$. In addition the above analysis yields $y'(t) \le G(M_0) + b - c = M_1$ and $-y''(t) \le G(y(t)) - c \le G(M_0) - c = M_2 - c$.

Remark. If $a \le a_0$ then $M_0$, $M_1$, and $M_2$ can be chosen independent of $a$.

Finally, we have $0 \le -y'''(t) \le \lambda \sup_{[a,M_0]} g(y) \sup_{[b,M_1]} y' = M_3$. ∎

THEOREM 2.4. *Suppose that conditions* (2.9), (2.10), (2.4) *and* (2.5) *with* $0 \le \alpha < 1$ *are satisfied. Then a* $C^3[0,1]$ *solution of* (2.8) *exists.*

Proof. This follows the reasoning in Theorem 2.2. The only change is to define

$$\bar f(t,y,p) = \begin{cases} f(t,y,p) & \text{for } y \ge a, \\ f(t,a,p) & \text{for } y \le a, \end{cases}$$

and here $F_\lambda : C^1[0,1] \to C[0,1]$, $j : C^3_B[0,1] \to C^1[0,1]$. Of course $ju = u$ and $(F_\lambda u)(t) = -\lambda \bar f(t, u(t), u'(t))$. ∎

Case 2: $a = 0$. When $a = 0$, a solution to (2.1) will mean a function $y \in C^1[0,1] \cap C^3(0,1)$ which satisfies the corresponding differential equation and boundary conditions.

THEOREM 2.5. *Suppose conditions* (2.2)–(2.5) *are satisfied. In addition suppose* $f$ *satisfies*:

(2.12) *For each constant $M > 0$ there exists $\psi(t)$ continuous on $[0,1]$ and positive on $(0,1)$ such that $f(t,y) \ge \psi(t)$ on $[0,1] \times (0,M]$,*

$$(2.12^*) \qquad \begin{cases} \int_0^1 g(\theta(t))\,dt < \infty \text{ where } \theta(t) = bt,\ b > 0 \text{ and for } b = 0\,, \\ \theta(t) = \int_0^t (t - .5t^2 - .5s^2)\psi(s)\,ds + \int_t^1 (1-s)t\psi(s)\,ds\,. \end{cases}$$

*Then a $C^1[0,1] \cap C^3(0,1)$ solution of (2.1) exists.*

P r o o f. We look at the family of problems (with $n \in \mathbb{N}^+$)

$$(2.13_n) \qquad \begin{cases} y''' + f(t,y) = 0, \quad 0 < t < 1, \\ y(0) = 1/n, \\ y'(1) = b, \\ y''(0) = 0. \end{cases}$$

Theorem 2.2 implies that $(2.13_n)$ has a solution $y_n$ for each $n$. Moreover, there are constants $M_0$ and $M_1$ independent of $n$ (set $a_0 = 1$ in Theorem 2.1) such that $1/n \le y_n(t) \le M_0$, $|y_n'(t)| \le M_1$ for $t \in [0,1]$. In addition we *claim* that there is a constant $M_2$ independent of $n$ such that $\|y_n''\|_{L^2} \le M_2$; to see this notice $-y_n' y_n''' \le g(y_n) y_n'$ and integration from 0 to 1 yields

$$-by_n''(1) + \int_0^1 [y_n''(s)]^2\,ds \le \int_0^{M_0} g(u)\,du\,.$$

Since $y_n''(1) \le 0$ our claim is established. The Arzelà–Ascoli Theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0,1]$ to some function $y \in C^1[0,1]$. Clearly $y \ge 0$ on $[0,1]$ with $y(0) = a$ and $y'(1) = b$. In fact $y > 0$ on $(0,1]$; $-y_n'''(t) \ge \psi(t)$ so triple integration yields

$$y_n(t) \ge 1/n + bt + \int_0^t \psi(u)\Big[\int_u^t (1-s)\,ds\Big]\,du + \int_0^t u(1-u)\psi(u)\,du + \int_t^1 t(1-u)\psi\,du\,.$$

Now $y_n'$ satisfies the integral equation

$$y_{n'}(t) = y_{n'}(1) - b(t-1) + \frac{y_{n'}''(1)}{2}(t-1)^2 - \int_t^1 (s-t)^2 f(s, y_{n'}(s))\,ds\,.$$

For $t \in (0,1]$ we have $f(s, y_{n'}(s)) \to f(s, y(s))$ uniformly in $s \in [t,1]$ since $f$ is uniformly continuous on compact subsets of $[0,1] \times (0, M_0]$ . Thus letting

$n' \to \infty$ yields (with $q \in \mathbb{R}$)

$$y(t) = y(1) - b(t-1) + q\frac{(t-1)^2}{2} - \int_t^1 (s-t)^2 f(s, y(s))\, ds\,.$$

From the integral equation we see that $y''' = -f(t, y)$ and (2.12*) together with the Lebesgue dominated convergence theorem implies

$$\begin{aligned} 0 = \lim_{n'\to\infty} y''_{n'}(0) &= \lim_{n'\to\infty} \Big[ \int_0^1 f(s, y_{n'}(s))\, ds - y''_{n'}(1) \Big] \\ &= \int_0^1 f(s, y(s))\, ds - q = y''(0) \end{aligned}$$

since $f(t, y_{n'}(t)) \le g(y_{n'}(t)) \le g(\theta(t)) \in L^1$; noting of course that $-y'''_{n'} \ge \psi(t)$ implies $y_{n'}(t) \ge \theta(t)$ if $b = 0$ whereas $y_{n'}(t) \ge bt$ if $b > 0$. ∎

Remark. If $b > 0$, assumption (2.12) can be removed in the statement of Theorem 2.5 and existence of a solution to (2.1) is guaranteed.

We can obtain a similar result for problems of the form (2.8).

THEOREM 2.6. *Suppose* (2.9), (2.10), (2.4), (2.12*) *with* $b > 0$, *and* (2.5) *with* $0 \le \alpha < 1$ *are satisfied. Then a* $C^1[0,1] \cap C^3(0,1)$ *solution of* (2.8) *exists.*

Proof. Examine the problems (with $n \in \mathbb{N}^+$)

$$(2.14_n) \qquad \begin{cases} y''' + f(t, y, y') = 0, \quad 0 < t < 1, \\ y(0) = 1/n, \\ y'(1) = b, \\ y''(0) = c. \end{cases}$$

Theorem 2.4 guarantees that $(2.14_n)$ has a solution $y_n$ for each $n$ and also there exist constants $M_0$, $M_1$ and $M_2$ independent of $n$ such that $1/n \le y_n(t) \le M_0$, $b \le y'_n(t) \le M_1$, $|y''_n(t)| \le M_2$ for $t \in [0,1]$. Consequently, we have a subsequence $y_{n'}$ converging uniformly on $[0,1]$ to some function $y \in C^1[0,1]$ and in addition $y \ge 0$, $y' \ge b$ on $[0,1]$ with $y(0) = 0$ and $y'(1) = b$. Since $b > 0$ we have $y > 0$ on $(0,1]$. Moreover, $y_{n'}$ satisfies

$$\begin{aligned} y_{n'}(t) = y_{n'}(1) - b(t-1) &+ \frac{y''_{n'}(1)}{2}(t-1)^2 \\ &- \int_t^1 (s-t)^2 f(s, y_{n'}(s), y'_{n'}(s))\, ds \end{aligned}$$

so for $t \in (0,1]$ and $s \in [t,1]$ we have $f(s, y_{n'}(s), y'_{n'}(s)) \to f(s, y(s), y'(s))$ uniformly since $f$ is uniformly continuous on compact subsets of $[0,1] \times$

$(0, M_0] \times [b, M_1]$. The result follows by allowing $n' \to \infty$. ∎

**3.** In this section we give a brief treatment to the problem of obtaining positive solutions on $[0,1)$ to

$$\tag{3.1} \begin{cases} y''' + f(t,y) = 0, & 0 < t < 1, \\ y''(0) = 0, \\ y'(0) = 0, \\ y(1) = a \geq 0. \end{cases}$$

Case 1: $a > 0$. Now if $y \in C^3[0,1]$ is a solution to (3.1) then $y''' < 0$ on $(0,1)$ so $y'' < 0$ on $(0,1)$, which in turn implies $y' < 0$ on $(0,1)$ so $y$ is strictly decreasing on $(0,1)$ and in particular $y \geq a$ on $[0,1]$.

THEOREM 3.1. *Suppose $f$ satisfies conditions* (2.2) *and* (2.3). *For $\lambda \in [0,1]$ consider*

$$\tag{3.2$_\lambda$} \begin{cases} y''' + \lambda f(t,y) = 0, \quad 0 < t < 1, \\ y''(0) = 0, \quad y'(0) = 0, \quad y(1) = a. \end{cases}$$

*Then there exist positive constants $M_0$, $M_1$, $M_2$ independent of $\lambda$ such that for $t \in [0,1]$*

$$a \leq y(t) \leq M_0,\ -M_1 \leq y'(t) \leq 0,\ -M_2 \leq y''(t) \leq 0,\ -M_2 \leq y'''(t) \leq 0$$

*for each solution $y$ to* $(3.2_\lambda)$.

Proof. Let $0 < \lambda \leq 1$. For solutions $y$ to $(3.2_\lambda)$, $-y''' \leq g(y)$ so integration from 0 to $t$ yields

$$-y''(t) \leq \int_0^t g(y(u))\,du \leq g(y(t)) \leq g(y(t) - a)$$

since $g$ is nonincreasing. Another integration from 0 to $t$ will give $-y'(t) \leq g(y(t) - a)$; finally, integrate the above inequality from $t$ to 1 to obtain

$$\int_0^{y(t)-a} \frac{du}{g(u)} \leq 1\,.$$

Since $1/g$ is nondecreasing and $\int_0^z du/g(u) = \infty$, $G$ is an increasing map from $[0,\infty)$ onto $[0,\infty)$ and therefore has an increasing inverse $G^{-1}$. Thus we have $y(t) \leq G^{-1}(1) + a = M_0$ for $t \in [0,1]$. Returning to $-y''(t) \leq g(y(t))$ and multiplying by $y'$ we obtain $-y'y'' \geq g(y)y'$. Integrating from 0 to $t$ yields

$$-\frac{[y'(t)]^2}{2} \geq \int_{y(0)}^{y(t)} g(u)\,du = -\int_{y(t)}^{y(0)} g(u)\,du$$

so

$$[y'(t)]^2 \le 2 \int\limits_a^{M_0} g(u)\,du\,.$$

Thus we have $-M_1 \le y'(t) \le 0$.

Remark. If $a \le a_0$ and $\int_0^{M_0} g(u)\,du < \infty$ then it is possible to obtain $M_0$ and $M_1$ independent of $a$.

In addition, $0 \le -y'''(t) \le \sup_{[a,M_0]} g(y) \equiv M_2$ and integration yields the bound for $y''(t)$. ■

Essentially the same reasoning as in Theorem 2.2 establishes

Theorem 3.2. *Suppose $f$ satisfies* (2.2) *and* (2.3). *Then a $C^3[0,1]$ solution to* (3.1) *exists.*

Case 2: $a = 0$

Theorem 3.3. *Suppose $f$ satisfies* (2.2), (2.3) *and* (2.12). *In addition, suppose $g$ satisfies* (2.4). *Then a $C[0,1] \cap C^3(0,1)$ solution to* (3.1) *exists.*

Proof. This follows from a slight (and easy) modification of the proof of Theorem 2.5. ■

**4.** Finally, to conclude the paper we establish existence of positive solutions on $(0,1)$ to

$$\begin{cases} y''' + f(t,y) = 0, & 0 < t < 1, \\ y''(0) = 0, \\ y(0) = a \ge 0, \\ y(1) = b \ge 0, \end{cases} \tag{4.1}$$

where we assume without loss of generality that $a \le b$.

Case 1: $a > 0$. Suppose $y \in C^3[0,1]$ is any solution to (4.1). Then $y > 0$ on $(0,1)$ and so $y''' < 0$. Hence $y'' < 0$ on (0,1), which in turn implies $y'$ is strictly decreasing on (0,1) and of course $y \ge a$ on $[0,1]$.

Theorem 4.1. *Suppose $f$ satisfies* (2.2), (2.3) *and $g$ satisfies* (2.4) *and* (2.5). *In addition, suppose $g$ satisfies the following*:

(4.2) $\quad$ *$yg(y)$ is nondecreasing on $(0,\infty)$.*

*For $\lambda \in [0,1]$ consider*

$$\begin{cases} y''' + \lambda f(t,y) = 0, & 0 < t < 1, \\ y''(0) = 0, \quad y(0) = a, \quad y(1) = b. \end{cases} \tag{4.3$_\lambda$}$$

*Then there exist constants $M_0$, $M_1$, $M_2$ independent of $\lambda$ such that*

$$a \le y(t) \le M_0, \quad |y'(t)| \le M_1, \quad -M_2 \le y''(t) \le 0, \quad -M_2 \le y'''(t) \le 0$$

*for each solution $y$ to* $(4.3_\lambda)$.

Remark. Again it should be noted that Theorem 4.1 is true if only conditions (2.2) and (2.3) are satisfied. However in hindsight we need to obtain $M_0$ and $M_1$ independent of $n$ if $a = 1/n$ and/or $b = 1/n$, $n \in \mathbb{N}^+$.

Proof. Suppose $0 < \lambda \le 1$. Let $y$ be a solution to $(4.3_\lambda)$ and $y_{\max}$ the maximum of $y(t)$ on $[0,1]$. If the maximum occurs at the end points then $y_{\max} \le b$. On the other hand, suppose $y_{max}$ occurs at $t_0 \in (0,1)$, so $y'(t_0) = 0$. Now for $t \le t_0$ we have $-y'y''' \le g(y)y' \le g(y-a)y'$ so integrating from 0 to $t$ yields

$$-y'(t)y''(t) + \int_0^t [y''(s)]^2\,ds \le \int_0^{y(t)-a} g(u)\,du = G(y(t)-a)\,.$$

Thus $-y'(t)y''(t) \le G(y(t_0))$ for $t \le t_0$ and integration from $t$ to $t_0$ will give

$$\frac{[y'(t)]^2}{2} \le G(y(t_0))\,.$$

Hence $y'(t) \le \sqrt{2G(y(t_0))}$, which in turn yields $y(t_0) \le \sqrt{2G(y(t_0))} + a$. Consequently, there exists a constant $K_0$ (independent of $\lambda$) such that $y(t_0) \le K_0$. This implies $a \le y(t) \le M_0 = \max\{K_0, b\}$.

Remark. If $a = 1/n$ and/or $b = 1/n$ where $n \in \mathbb{N}^+$ we can choose $M_0$ independent of $n$.

To find $M_1$ there are two cases to consider; either (A) there exists $t_0 \in (0,1)$ with $y'(t_0) = 0$ or (B) $y' > 0$ on $(0,1)$.

Case A. The above analysis yields for $t \le t_0$

$$0 \le y'(t) \le \sqrt{2G(y(t_0))} \le \sqrt{2\int_0^{M_0} g(u)\,du} = K_1\,.$$

In addition we also have for $t \le t_0$

$$-y(t)y'''(t) \le y(t)g(y(t)) \le y(t_0)g(y(t_0)) \le M_0 g(M_0) = K_2$$

since $g$ satisfies (4.2). Integration from 0 to $t_0$ now gives

$$-y(t_0)y''(t_0) + \int_0^{t_0} y'(s)y''(s)\,ds \le K_2$$

and so

$$-y(t_0)y''(t_0) \le K_2 + \frac{[y'(0)]^2}{2} \le K_2 + \frac{K_1^2}{2} = K_3\,.$$

Hence $-y(t_0)y''(t_0) \le K_3$. For the case $t \ge t_0$ we have $-y(t)y'''(t) \le y(t)g(y(t)) \le K_2$ so integrating from $t_0$ to 1 yields

$$y(t_0)y''(t_0) - y(1)y''(1) + \frac{[y'(1)]^2}{2} \le K_2$$

from which we deduce

$$[y'(1)]^2 \le 2K_2 + 2K_3 \quad \text{since} \quad y(1)y''(1) \le 0\,.$$

Thus there exists a constant $K_4 > 0$ such that $0 \le -y'(1) \le K_4$ and consequently $0 \le -y'(t) \le K_4$ for $t \ge t_0$.

Case B: $y' > 0$ on $(0,1)$. Since $y'' < 0$, $y'$ is decreasing on (0,1). The mean value theorem implies there exists $\eta \in (0,1)$ with $y'(\eta) = b - a$. For $t \ge \eta$, $0 \le y'(t) \le b - a \le b$. However, for $t < \eta$ we have $y'(t) \ge b - a$ and $-y'(t)y'''(t) \le g(y)y'$. Integrate from 0 to $t < \eta$ to obtain $-y'(t)y'''(t) \le \int_0^{y(\eta)} g(u)\,du$ and then

$$\frac{[y'(t)]^2}{2} \le \int\limits_0^{y(\eta)} g(u)\,du + \frac{[y'(\eta)]^2}{2} \le \int\limits_0^{M_0} g(u)\,du + \frac{(b-a)^2}{2}\,.$$

Combining both cases implies that we have shown $|y'(t)| \le M_1$ for any solution $y$ to $(4.3_\lambda)$.

Remark. Once again if $a = 1/n$ and/or $b = 1/n$ then it is possible to choose $M_1$ independent of $n$.

To complete the proof note that $0 \le -y'''(t) \le \lambda \sup_{[a,M_0]} g(u) \equiv M_2$ and integration yields the bound for $y''$. ∎

The reasoning in Theorem 2.2 immediately yields

THEOREM 4.2. *Suppose conditions* (2.2) *and* (2.3) *are satisfied. Then a* $C^3[0,1]$ *solution of* (4.1) *exists.*

<u>Case</u> 2: $a = 0$

THEOREM 4.3. *Suppose conditions* (2.2)–(2.5), (2.12) *and* (4.2) *are satisfied. In addition, suppose* (2.12*) *holds with*

$$\theta(t) = -\int\limits_0^t (t-z)\int\limits_0^z \psi(s)\,ds\,dz + t\int\limits_0^1 (1-z)\int\limits_0^z \psi(s)\,ds\,dz\,.$$

*Then a* $C^1[0,1] \cap C^3(0,1)$ *solution of* (4.1) *exists.*

Proof. This follows almost verbatim the ideas in Theorem 2.5. Here if $b > 0$ consider the family of problems

$$\begin{cases} y''' + f(t,y) = 0, & 0 < t < 1, \\ y(0) = 1/n, \quad y(1) = b, & y''(0) = 0, \end{cases}$$

whereas if $b = 0$ consider

$$\begin{cases} y''' + f(t,y) = 0, & 0 < t < 1, \\ y(0) = 1/n, \quad y(1) = 1/n, \quad y''(0) = 0. \ \blacksquare \end{cases}$$

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