Solvability of some singular and nonsingular nonlinear third order boundary value problems

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Abstract. Existence of positive solution to certain classes of singular and nonsingular third order nonlinear two point boundary value problems is examined using the idea of Topological Transversality.

1. In this paper we establish existence of positive solutions to some nonlinear singular and nonsingular third order boundary value problems of the form

\[
\begin{cases}
y'' + f(t, y, y') = 0, & 0 < t < 1, \\
y & \text{satisfies a boundary condition } B.
\end{cases}
\]

Here \( B \) will denote either

(i) \( y''(0) = c \leq 0, \quad y'(1) = b \geq 0, \quad y(0) = a \geq 0 \),

(ii) \( y''(0) = 0, \quad y'(0) = 0, \quad y(1) = a \geq 0 \), or

(iii) \( y''(0) = 0, \quad y(0) = a \geq 0, \quad y(1) = b \geq 0 \).

The above problems are singular because \( f \) is allowed to be singular at \( y = 0 \). Third order boundary value problems have become quite popular in the last ten years with most of the work concentrated on nonsingular problems; see [1], [2], [6], [7] and [8] for example. Also the author in [9] has discussed boundary value problems where \( f \) is allowed to be singular at \( t = 0 \) and \( t = 1 \). Results for singular initial value problems may be found in [3]. This paper was motivated from [11] where S. Taliaferro considered problems of the form \( y'' + a(t)y^\alpha = 0, \quad \alpha < 0 \) with \( y(0) = 0 \) and \( y(1) = 0 \). Finally here we summarize briefly the plan of the paper. We begin by showing that (1.1) has a \( C^3[0, 1] \) solution for all \( a > 0 \) with \( a \leq a_0 \) fixed. The key ideas used here are the Topological Transversality Theorem and the existence of a priori
bounds. We note that when \( a > 0 \), (1.1) does not involve singularities in \( y \).
To take care of the singular case (i.e. (1.1) with \( a = 0 \)) we pass to the limit in \( a \) (i.e. let \( a \to 0 \)). Here we deduce the existence of a \( C^1[0,1] \cap C^3(0,1) \) solution to (1.1) if \( B \) denotes either (i) or (iii) whereas we will obtain a \( C[0,1] \cap C^3(0,1) \) solution to (1.1) in the case where \( B \) denotes (ii). The main idea in this step is the Arzelà–Ascoli Theorem.

2. We begin by establishing existence of positive solutions on \((0,1]\) to

\[
\begin{cases}
y'''' + f(t,y) = 0, & 0 < t < 1, \\
y''(0) = 0, \\
y'(1) = b \geq 0, \\
y(0) = a \geq 0,
\end{cases}
\]

where \( f \) satisfies the following conditions:

\begin{enumerate}
\item[(2.2)]\( f \) is continuous on \([0,1] \times (0,\infty) \) with \( \lim_{y \to 0^+} f(t,y) = \infty \) uniformly on compact subsets of \((0,1)\),
\item[(2.3)]\( 0 < f(t,y) \leq g(y) \) on \([0,1] \times (0,\infty) \) where \( g \) is continuous and nonincreasing on \((0,\infty)\).
\end{enumerate}

In addition to the above we will have the following assumptions on \( g \):

\begin{enumerate}
\item[(2.4)] for any \( z \in [0,\infty) \), \( G(z) = \int_z^\infty g(u) \, du \leq \infty \),
\item[(2.5)] there exist constants \( A > 0, B > 0 \) and \( \alpha \) with \( G(z) \leq Az^\alpha + B \) for \( z \in [0,\infty) \) and \( 0 \leq \alpha < 2 \).
\end{enumerate}

We will discuss separately the cases (i) \( a > 0 \) and (ii) \( a = 0 \). It should be remarked here that the results for the case \( a > 0 \) will be used to discuss (2.1) with \( a = 0 \).

**Case 1:** \( a > 0 \). By a solution to (2.1) with \( a > 0 \) we mean a function \( y \in C^4[0,1] \) that satisfies the differential equation and the boundary conditions. Also if \( y \) is a solution to (2.1) then \( y > 0 \) on \((0,1)\) and so \( y'''' < 0 \) on \((0,1)\). Thus \( y'' < 0 \) on \((0,1)\), which in turn implies \( y' > 0 \) on \((0,1)\) so \( y \) is strictly increasing on \((0,1)\) and in particular \( y \geq a \) on \([0,1]\).

**Theorem 2.1** Suppose conditions (2.2)–(2.5) are satisfied. For \( \lambda \in [0,1] \) consider the family of problems

\[
\begin{cases}
y'''' + \lambda f(t,y) = 0, & 0 < t < 1, \\
y''''(0) = 0, & y'(1) = b, \quad y(0) = a.
\end{cases}
\]

Let \( 0 < a \leq a_0 \). Then there exist positive constants \( M_0, M_1, M_2, M_3 \) independent of \( \lambda \) with \( M_0, M_1 \) depending on \( a_0 \) such that for \( t \in [0,1] \)

\[
a \leq y(t) \leq M_0, \quad b \leq y'(t) \leq M_1, \quad -M_2 \leq y''(t) \leq 0, \quad -M_3 \leq y'''(t) \leq 0
\]

for each solution \( y \) to (2.6_\lambda) and \( a \leq a_0 \).
Remark. It should be noted here that for fixed positive \( a \), Theorem 2.1 holds if only conditions (2.2) and (2.3) are satisfied. To see this suppose \( y \) is a solution to (2.6). Then clearly \( y(t) \geq a \), \( y'(t) \geq b \), \( y''(t) \leq 0 \), and \( y'''(t) \leq 0 \) for \( t \in [0,1] \). The differential equation yields \(-y''' \leq \lambda g(y) \leq g(a) \equiv M_3 \) and integration gives \( M_2, M_1 \) and \( M_0 \) immediately. However, the goal of this paper is to examine singular problems and to achieve this we need to show that we can obtain \( M_0 \) and \( M_1 \) independent of \( a \) for a bounded \( a \). For these singular problems \( a = 1/n, n \in \mathbb{N}^+ = \{1,2, \ldots \} \). Thus we will prove Theorem 2.1 with this in mind.

Proof. Let \( y \) be a solution to (2.6). The case \( \lambda = 0 \) is trivial so assume \( 0 < \lambda \leq 1 \). Then \(-y''' \leq \lambda g(y) \leq g(a-y) \) since \( g \) is nonincreasing. Multiply by \( y' \) and integrate from 0 to \( t \) to obtain

\[-y'(t)y''(t) + \int_0^t [y''(s)]^2 \, ds \leq \frac{y(t)-a}{M} \int_0^t g(u) \, du = G(y(t)-a) .\]

Thus we have \(-y'(t)y''(t) \leq G(y(t)-a)\). Here \( G \) is an increasing map from \([0, \infty)\) into \([0, \infty)\) so we have \(-y'(t)y''(t) \leq G(y(t)) \leq G(y(1))\). Integrate the above inequality from \( t \) to 1 to obtain

\[-\frac{b^2}{2} + \frac{[y'(t)]^2}{2} \leq G(y(1)) \]

so \( y'(t) \leq \sqrt{2G(y(1)) + b^2} \). Hence \( y(1) \leq a + \sqrt{2G(y(1)) + b^2} \). Now assumption (2.5) implies there exists a constant \( M_0 \) (independent of \( \lambda \)) such that \( y(t) \leq y(1) \leq M_0 \) for \( t \in [0,1] \). In addition the above analysis yields

\[ y'(t) \leq \sqrt{2G(y(1)) + b^2} \leq \sqrt{2G(M_0) + b^2} = M_1 .\]

If \( a \leq a_0 \) then we can choose \( M_0 \) and \( M_1 \) independent of \( a \) since \( y(1) \leq a + \sqrt{2G(a) + b^2} \).

Also we have \( 0 \leq -y'''(t) \leq \lambda \sup_{[a,M_0]} g(u) \leq g(a) = M_2 \) and finally integration from 0 to \( t \) yields a bound for \( y'(t) \).

We will now apply the Topological Transversality Theorem [4, 5] to obtain our basic existence theorem. For notational purposes set

\[ C^3_B[0,1] = \{ u \in C^3[0,1]; u(0) = a, u'(1) = b, u''(0) = 0 \} , \]

\[ C^3_B[0,1] = \{ u \in C^3[0,1]; u(0) = 0, u'(1) = 0, u''(0) = 0 \} . \]

Theorem 2.2. Suppose conditions (2.2) and (2.3) are satisfied. Then a \( C^3[0,1] \) solution of (2.1) exists.

Proof. Consider the family of problems

(2.7,)

\[ \begin{cases} y''' + \lambda f(t,y) = 0, & 0 < \lambda < 1, \\ y'''(0) = 0, & y'(1) = b, \\ y'(0) = a. \end{cases} \]
where

$$\tilde{f}(t, y) = \begin{cases} f(t, y) & \text{for } y \geq a, \\ f(t, a) & \text{for } y \leq a. \end{cases}$$

Every solution $v$ of (2.7\lambda) satisfies $v \geq a$ and hence is a solution to (2.6\lambda). Also $f$ satisfies the hypothesis imposed on $f$ since for $0 < y \leq a$, $f(t, y) = f(t, u) \leq g(u) \leq g(y)$. Hence the conclusion of Theorem 2.1 remains valid for solutions to (2.7\lambda). Let $M_0 = \max\{M_0, b + a\}$ and

$$V = \{u \in C^2_B[0, 1] : a/2 < u(t) < M_0 + 1, |u'(t)| < M_1, |u''(t)| < M_2 + 1, |u'''(t)| < M_3 + 1\}.$$

$V$ is an open subset of $C^2_B[0, 1]$ which in turn is a convex subset of $C^3[0, 1]$.

Define mappings $F_\lambda : C[0, 1] \to C[0, 1], \quad j : C^3_B[0, 1] \to C[0, 1], \quad L : C^2_B[0, 1] \to C^2[0, 1]$ by $(F_\lambda u)(t) = -\lambda f(t, u(t)), \quad ju = u$ and $Lu(t) = u'''(t)$. $F_\lambda$ is continuous and $j$ is completely continuous by the Arzelà-Ascoli Theorem. Finally, we claim $L^{-1}$ exists and is continuous. To see this define $N : C^3_B[0, 1] \to C[0, 1]$ by $Nu = u'''$. $N^{-1}$ is a continuous linear operator by the Bounded Inverse Theorem; see [10, Theorem 5.10]. Thus $L^{-1}$ exists and is given by

$$(L^{-1}g)(x) = bx + a + (N^{-1}g)(x)$$

and so is continuous. Now define the map $H_\lambda : V \to C^2_B[0, 1]$ by $H_\lambda u = L^{-1}F_\lambda ju$. $H_\lambda$ is a compact homotopy and $H_\lambda u = u$ means $Lu = F_\lambda ju$, i.e. $u''' = -\lambda f(t, u(t))$ and $u$ satisfies the boundary conditions. Therefore $H_\lambda$ is fixed point free on $\partial V$ by construction of $V$ and Theorem 2.1. Finally, for any $u \in V$, $H_0(u) = u_0$, $u_0(x) = bx + a$, i.e. $H_0$ is a constant map and so is essential [4]. The topological transversality theorem [4, p. 86] implies that $H_1$ is essential, i.e. (2.7\lambda) has a solution and therefore (2.1) has a solution. 

It is also possible to consider in this section problems of the form

$$\begin{align*}
y''' + f(t, y, y') &= 0, & 0 < t < 1, \\
y'(0) &= c \leq 0, \\
y'(1) &= b > 0, \\
y(0) &= a > 0,
\end{align*}$$

(2.8)

where $f$ satisfies the following:

$$\begin{align*}
f \text{ is continuous on } [0, 1] \times (0, \infty) \times (-\infty, \infty) \text{ with } \lim_{y \to 0^+} f(t, y, p) &= \infty \text{ uniformly on compact subsets of } [0, 1] \times (-\infty, \infty) \setminus \{0\}, \\
0 < f(t, g, p) \leq g(y) p \text{ on } [0, 1] \times (0, \infty) \times [b, \infty) \text{ where } g \text{ is continuous and nonincreasing on } (0, \infty) \text{ with } f \geq 0 \text{ on } [0, 1] \times (0, \infty) \times (-\infty, \infty).
\end{align*}$$

(2.9) (2.10)
Theorem 2.3. Suppose that conditions (2.9), (2.10), (2.4) and (2.5) with 
0 \leq \alpha < 1 are satisfied. For \( \lambda \in [0, 1] \) consider
\begin{align*}
\frac{d^3y}{dt^3} + \lambda f(t, y, y') &= 0, \quad 0 < t < 1, \\
y''(0) &= c, \quad y'(1) = b, \quad y(0) = a.
\end{align*}
Then there exists constants \( M_0, M_1, M_2, M_3 \) independent of \( \lambda \) such that for 
\( t \in [0, 1] \)
\[ a \leq y(t) \leq M_0, \quad b \leq y(t) \leq M_1, \quad -M_2 \leq y''(t) \leq c, \quad -M_3 \leq y'''(t) \leq 0 \]
for each solution \( y \) to (2.11). 

Proof. Suppose \( y \) is a solution to (2.11). Assume \( 0 < \lambda \leq 1 \). Clearly (since (2.9) and (2.10) are satisfied) \( y(t) \geq a, \quad y'(t) \geq b, \quad y''(t) \leq c, \quad y'''(t) \leq 0 \)
for \( t \in [0, 1] \). We also have \( -y''' \geq g(y - a)y' \) since \( g \) is nonincreasing. 
Integrating from 0 to \( t \) yields \( -y'''(t) \leq G(y(t) - a) - c \leq G(y(t)) - c \) since \( G \) is an increasing map from \([0, \infty)\) into \([0, \infty)\). Now integrate from \( t \) to 1 to obtain
\[ y'(t) \leq \int_{t}^{1} G(y(u)) \, du + b - c \leq G(y(1)) + b - c \]
and so \( y(1) \leq G(y(1)) + b + a - c \). Assumption (2.5) with \( 0 \leq \alpha < 1 \) implies there exists a constant \( M_0 \) (independent of \( \lambda \)) such that \( y(1) \leq M_0 \). 
In addition the above analysis yields \( y'(t) \leq G(M_0) + b - c = M_1 \) and 
\[ -y'''(t) \leq G(y(t)) - c \leq G(M_0) - c = M_2 - c. \]

Remark. If \( a \leq a_0 \) then \( M_0, M_1, \) and \( M_2 \) can be chosen independent of \( a \).

Finally, we have \( 0 \leq -y'''(t) \leq \lambda \sup_{[a, M_0]} g(y) \sup_{[b, M_1]} \frac{y'}{y} = M_3. \]

Theorem 2.4. Suppose that conditions (2.9), (2.10), (2.4) and (2.5) with 
\( 0 \leq \alpha < 1 \) are satisfied. Then a \( C^2(0, 1) \) solution of (2.8) exists.

Proof. This follows the reasoning in Theorem 2.2. The only change is to define
\[ \tilde{f}(t, y, p) = \begin{cases} f(t, y, p) & \text{for } y \geq a, \\ f(t, a, p) & \text{for } y \leq a, \end{cases} \]
and here \( F_\lambda : C^4([0, 1]) \to C^0([0, 1]), \ j : C^3_0([0, 1]) \to C^4([0, 1]). \) Of course \( ju = u \) 
and \( (F_\lambda u)(t) = -\lambda \tilde{f}(t, u(t), u'(t)). \]

Case 2: \( a = 0 \). When \( a = 0 \), a solution to (2.1) will mean a function
\( y \in C^4([0, 1]) \cap C^3(0, 1) \) which satisfies the corresponding differential equation and boundary conditions.

Theorem 2.5. Suppose conditions (2.2)–(2.5) are satisfied. In addition suppose \( f \) satisfies:
For each constant $M > 0$ there exists $\psi(t)$ continuous on $[0, 1]$ and positive on $(0, 1)$ such that $f(t, y) \geq \psi(t)$ on $[0, 1] \times (0, M]$, and

(2.12) \[ \int_0^1 g(\theta(t)) \, dt < \infty \] where $\theta(t) = bt$, $b > 0$ and for $b = 0$,

\[
\theta(t) = \int_0^t (t - 5t^2 - 5s^2) \psi(s) \, ds + \int_1^t (1 - s) t \psi(s) \, ds.
\]

Then a $C^1[0, 1] \cap C^3(0, 1)$ solution of (2.1) exists.

Proof. We look at the family of problems (with $n \in \mathbb{N}^+$)

(2.13

\[
\begin{aligned}
y''' + f(t, y) &= 0, \quad 0 < t < 1, \\
y(0) &= 1/n, \\
y'(1) &= b, \\
y''(0) &= 0.
\end{aligned}
\]

Theorem 2.2 implies that (2.13) has a solution $y_n$ for each $n$. Moreover, there are constants $M_0$ and $M_1$ independent of $n$ (set $a_0 = 1$ in Theorem 2.1) such that $1/n \leq y_n(t) \leq M_0$, $|y''_n(t)| \leq M_1$ for $t \in [0, 1]$. In addition we claim that there is a constant $M_2$ independent of $n$ such that $\|y''_n\|_{L^2} \leq M_2$; to see this notice $-y'_n y''' \leq g(y_n)y'_n$ and integration from 0 to 1 yields

\[-by''_n(1) + \int_0^1 [y''_n(s)]^2 \, ds \leq \int_0^1 g(u) \, du.
\]

Since $y''_n(1) \leq 0$ our claim is established. The Arzelà–Ascoli Theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some function $y \in C^1[0, 1]$. Clearly $y \geq 0$ on $[0, 1]$ with $y(0) = a$ and $y'(1) = b$. In fact $y > 0$ on $(0, 1]$; $-y''_n(t) \geq \psi(t)$ so triple integration yields

\[y_n(t) \geq 1/n + bt + \int_0^t \psi(s) \left[ \int_s^t (1 - s) \, ds \right] \, du
\]

\[+ \int_0^t u(1 - u) \psi(u) \, du + \int_t^1 t(1 - u) \psi \, du.
\]

Now $y_{n'}$ satisfies the integral equation

\[y_{n'}(t) = y_{n'}(1) - b(t - 1) + \frac{y''_{n'}(1)}{2} (t - 1)^2
\]

\[- \int_0^1 (s - t)^2 f(s, y_{n'}(s)) \, ds.
\]

For $t \in (0, 1]$ we have $f(s, y_n(s)) \rightarrow f(s, y(s))$ uniformly in $s \in [t, 1]$ since $f$ is uniformly continuous on compact subsets of $[0, 1] \times (0, M_0]$. Thus letting
\(n' \to \infty\) yields (with \(q \in \mathbb{R}\))

\[
y(t) = y(1) - b(t - 1) + q \frac{(t - 1)^2}{2} - \frac{1}{t} \int (s - t)^2 f(s, y(s)) \, ds.
\]

From the integral equation we see that \(y''' = -f(t, y)\) and (2.12\(^*\)) together with the Lebesgue dominated convergence theorem implies

\[
0 = \lim_{n' \to \infty} y''_{n'}(0) = \lim_{n' \to \infty} \left[ \frac{1}{0} \int_0^1 f(s, y_{n'}(s)) \, ds - y''_{n'}(1) \right]
\]

\[
= \int_0^1 f(s, y(s)) \, ds - q = y''(0)
\]

since \(f(t, y_{n'}(t)) \leq g(y_{n'}(t)) \leq g(\theta(t)) \in L^1\); noting of course that \(-y''_{n'} \geq \psi(t)\) implies \(y_{n'}(t) \geq \theta(t)\) if \(b = 0\) whereas \(y_{n'}(t) \geq bt\) if \(b > 0\).

**Remark.** If \(b > 0\), assumption (2.12) can be removed in the statement of Theorem 2.5 and existence of a solution to (2.1) is guaranteed.

We can obtain a similar result for problems of the form (2.8).

**Theorem 2.6.** Suppose (2.9), (2.10), (2.4), (2.12\(^*\)) with \(b > 0\), and (2.5) with \(0 \leq \alpha < 1\) are satisfied. Then a \(C^1[0,1] \cap C^3(0,1)\) solution of (2.8) exists.

**Proof.** Examine the problems (with \(n \in \mathbb{N}^+\))

\[
\begin{aligned}
&y''' + f(t, y, y') = 0, \quad 0 < t < 1, \\
y'(0) = 1/n, \\
y'(1) = b, \\
y'(0) = c.
\end{aligned}
\]

Theorem 2.4 guarantees that (2.14\(n\)) has a solution \(y_n\) for each \(n\) and also there exist constants \(M_0, M_1\) and \(M_2\) independent of \(n\) such that \(1/n \leq y_n(t) \leq M_0, b \leq y'_n(t) \leq M_1, |y''_n(t)| \leq M_2\) for \(t \in [0,1]\). Consequently, we have a subsequence \(y_{n'}\) converging uniformly on \([0,1]\) to some function \(y \in C^4[0,1]\) and in addition \(y \geq 0, y' \geq b\) on \([0,1]\) with \(y(0) = 0\) and \(y'(1) = b\). Since \(b > 0\) we have \(y > 0\) on \((0,1]\). Moreover, \(y_{n'}\) satisfies

\[
y_{n'}(t) = y_{n'}(1) - b(t - 1) + \frac{y''_{n'}(1)}{2}(t - 1)^2 - \int_1^t (s - t)^2 f(s, y_{n'}(s), y'_{n'}(s)) \, ds
\]

so for \(t \in (0,1]\) and \(s \in [t,1]\) we have \(f(s, y_{n'}(s), y'_{n'}(s)) \to f(s, y(s), y'(s))\) uniformly since \(f\) is uniformly continuous on compact subsets of \([0,1] \times
(0, M_0] × [b, M_1]. The result follows by allowing n′ → ∞.

3. In this section we give a brief treatment to the problem of obtaining positive solutions on [0, 1) to

\[
\begin{aligned}
\begin{cases}
y'''' + f(t, y) = 0, & 0 < t < 1, \\
y''(0) = 0, \\
y'(0) = 0, \\
y(1) = a \geq 0.
\end{cases}
\end{aligned}
\]

(3.1)

**Case 1:** \( a > 0 \). Now if \( y \in C^3[0, 1] \) is a solution to (3.1) then \( y'''' < 0 \) on \( (0, 1) \) so \( y'' < 0 \) on \( (0, 1) \), which in turn implies \( y' < 0 \) on \( (0, 1) \) so \( y \) is strictly decreasing on \( (0, 1) \) and in particular \( y \geq a \) on \( [0, 1] \).

**Theorem 3.1.** Suppose \( f \) satisfies conditions (2.2) and (2.3). For \( \lambda \in [0, 1] \) consider

\[
\begin{aligned}
\begin{cases}
y'''' + \lambda f(t, y) = 0, & 0 < t < 1, \\
y''(0) = 0, \\
y'(0) = 0, \\
y(1) = a.
\end{cases}
\end{aligned}
\]

(3.2_\lambda)

Then there exist positive constants \( M_0, M_1, M_2 \) independent of \( \lambda \) such that for \( t \in [0, 1] \)

\[ a \leq y(t) \leq M_0, \quad -M_1 \leq y'(t) \leq 0, \quad -M_2 \leq y''(t) \leq 0, \quad -M_2 \leq y'''(t) \leq 0 \]

for each solution \( y \) to (3.2_\lambda).

**Proof.** Let \( 0 < \lambda \leq 1 \). For solutions \( y \) to (3.2_\lambda), \(-y''' \leq g(y)\) so integration from 0 to \( t \) yields

\[
-y'''(t) \leq \int_0^t g(y(u)) \, du \leq g(y(t)) \leq g(y(t) - a)
\]

since \( g \) is nonincreasing. Another integration from 0 to \( t \) will give \(-y'(t) \leq g(y(t) - a)\); finally, integrate the above inequality from \( t \) to 1 to obtain

\[
y(t) - a \geq \int_0^{y(t) - a} \frac{du}{g(u)} \leq 1.
\]

Since \( 1/g \) is nondecreasing and \( \int_0^a du/g(u) = \infty \), \( G \) is an increasing map from \( [0, \infty) \) onto \( [0, \infty) \) and therefore has an increasing inverse \( G^{-1} \). Thus we have \( y(t) \leq G^{-1}(1) + a = M_0 \) for \( t \in [0, 1] \). Returning to \(-y'''(t) \leq g(y(t))\) and multiplying by \( y' \) we obtain \(-y'y'' \geq g(y)y'\). Integrating from 0 to \( t \) yields

\[
-\frac{|y'(t)|^2}{2} \geq \int_0^{y(t)} g(u) \, du = -\int_{y(t)}^{y(0)} g(u) \, du.
\]
so
\[ [y'(t)]^2 \leq 2 \int_a^{M_0} g(u) \, du. \]

Thus we have \(-M_1 \leq y'(t) \leq 0\).

Remark. If \(a \leq a_0\) and \(\int_0^{M_0} g(u) \, du < \infty\) then it is possible to obtain \(M_0\) and \(M_1\) independent of \(a\).

In addition, \(0 \leq -y'''(t) \leq \sup_{[a,M_0]} g(y) \equiv M_2\) and integration yields the bound for \(y''(t)\).

Essentially the same reasoning as in Theorem 2.2 establishes

**Theorem 3.2.** Suppose \(f\) satisfies (2.2) and (2.3). Then a \(C^3[0,1]\) solution to (3.1) exists.

**Case 2:** \(a = 0\)

**Theorem 3.3.** Suppose \(f\) satisfies (2.2), (2.3) and (2.12). In addition, suppose \(g\) satisfies (2.4). Then a \(C[0,1] \cap C^3(0,1)\) solution to (3.1) exists.

**Proof.** This follows from a slight (and easy) modification of the proof of Theorem 2.5.

4. Finally, to conclude the paper we establish existence of positive solutions on \((0,1)\) to

\[
\begin{cases}
  y''' + f(t,y) = 0, & 0 < t < 1, \\
  y''(0) = 0, & \\
  y(0) = a \geq 0, & \\
  y(1) = b \geq 0, &
\end{cases}
\]  

(4.1)

where we assume without loss of generality that \(a \leq b\).

**Case 1:** \(a > 0\). Suppose \(y \in C^3[0,1]\) is any solution to (4.1). Then \(y > 0\) on \((0,1)\) and so \(y''' < 0\). Hence \(y'' < 0\) on \((0,1)\), which in turn implies \(y'\) is strictly decreasing on \((0,1)\) and of course \(y \geq a\) on \([0,1]\).

**Theorem 4.1.** Suppose \(f\) satisfies (2.2), (2.3) and \(g\) satisfies (2.4) and (2.5). In addition, suppose \(g\) satisfies the following:

\[(4.2) \quad yg(y) \text{ is nondecreasing on } (0,\infty).\]

For \(\lambda \in [0,1]\) consider

\[
\begin{cases}
  y''' + \lambda f(t,y) = 0, & 0 < t < 1, \\
  y''(0) = 0, & y(0) = a, \quad y(1) = b.
\end{cases}
\]  

(4.3)  

Then there exist constants \(M_0, M_1, M_2\) independent of \(\lambda\) such that

\[ a \leq y(t) \leq M_0, \quad |y'(t)| \leq M_1, \quad -M_2 \leq y''(t) \leq 0, \quad -M_2 \leq y'''(t) \leq 0 \]
for each solution $y$ to (4.3$\lambda$).

Remark. Again it should be noted that Theorem 4.1 is true if only conditions (2.2) and (2.3) are satisfied. However in hindsight we need to obtain $M_0$ and $M_1$ independent of $n$ if $a = 1/n$ and/or $b = 1/n$, $n \in \mathbb{N}^+$. 

Proof. Suppose $0 < \lambda \leq 1$. Let $y$ be a solution to (4.3$\lambda$) and $y_{\text{max}}$ the maximum of $y(t)$ on $[0,1]$. If the maximum occurs at the end points then $y_{\text{max}} \leq b$. On the other hand, suppose $y_{\text{max}}$ occurs at $t_0 \in (0,1)$, so $y'(t_0) = 0$. Now for $t \leq t_0$ we have $-y'y'' \leq g(y)y' \leq g(y-a)y'$ so integrating from 0 to $t$ yields

$$-y'(t)y''(t) + \int_0^t [y'(s)]^2 ds \leq \int_0^t g(u) du = G(y(t) - a).$$

Thus $-y'(t)y''(t) \leq G(y(t_0))$ for $t \leq t_0$ and integration from $t$ to $t_0$ will give

$$\frac{[y'(t)]^2}{2} \leq G(y(t_0)).$$

Hence $y'(t) \leq \sqrt{2G(y(t_0))}$, which in turn yields $y(t_0) \leq \sqrt{2G(y(t_0))} + a$. Consequently, there exists a constant $K_0$ (independent of $\lambda$) such that $y(t_0) \leq K_0$. This implies $a \leq y(t) \leq M_0 = \max\{K_0, b\}$.

Remark. If $a = 1/n$ and/or $b = 1/n$ where $n \in \mathbb{N}^+$ we can choose $M_0$ independent of $n$.

To find $M_1$ there are two cases to consider; either (A) there exists $t_0 \in (0,1)$ with $y'(t_0) = 0$ or (B) $y' > 0$ on $(0,1)$.

Case A. The above analysis yields for $t \leq t_0$

$$0 \leq y'(t) \leq \sqrt{2G(y(t_0))} \leq \sqrt{2 \int_0^{M_0} g(u) du} = K_1.$$ 

In addition we also have for $t \leq t_0$

$$-y(t)y'''(t) \leq y(t)g(y(t)) \leq y(t_0)g(y(t_0)) \leq M_0g(M_0) = K_2$$ 

since $g$ satisfies (4.2). Integration from 0 to $t_0$ now gives

$$-y(t_0)y'''(t_0) + \int_0^{t_0} y'(s)y''(s) ds \leq K_2$$

and so

$$-y(t_0)y''(t_0) \leq K_2 + \frac{[y'(0)]^2}{2} \leq K_2 + \frac{K_1^2}{2} = K_3.$$
Hence $-y(t_0)y''(t_0) \leq K_3$. For the case $t \geq t_0$ we have $-y(t)y'''(t) \leq y(t)g(y(t)) \leq K_2$ so integrating from $t_0$ to $1$ yields

$$y(t_0)y''(t_0) - y(1)y''(1) + \frac{|y'(1)|^2}{2} \leq K_2$$

from which we deduce

$$|y'(1)|^2 \leq 2K_2 + 2K_3, \quad y(1)y''(1) \leq 0.$$ 

Thus there exists a constant $K_4 > 0$ such that $0 \leq -y'(t) \leq K_4$ and consequently $0 \leq -y(t) \leq K_4$ for $t \geq t_0$.

Case B: $y' > 0$ on $(0,1)$. Since $y'' < 0$, $y'$ is decreasing on $(0,1)$. The mean value theorem implies there exists $\eta \in (0,1)$ with $y'(\eta) = b - a$. For $t \geq \eta$, $0 \leq y'(t) \leq b - a \leq b$. However, for $t < \eta$ we have $y'(t) \leq b - a$ and $-y(t)y'''(t) \leq g(y)y'$. Integrate from $0$ to $t < \eta$ to obtain $-y'(t)y'''(t) \leq \int_0^y g(u) du$ and then

$$\frac{|y'(t)|^2}{2} \leq \int_0^{y(\eta)} g(u) du + |y'(\eta)|^2 \leq \int_0^{M_0} g(u) du + \frac{(b-a)^2}{2}.$$ 

Combining both cases implies that we have shown $|y'(t)| \leq M_1$ for any solution $y$ to (4.3).

Remark. Once again if $a = 1/n$ and/or $b = 1/n$ then it is possible to choose $M_1$ independent of $n$.

To complete the proof note that $0 \leq -y'''(t) \leq \lambda \sup_{[a,M_0]} g(u) = M_2$ and integration yields the bound for $y'''$.

The reasoning in Theorem 2.2 immediately yields

**Theorem 4.2.** Suppose conditions (2.2) and (2.3) are satisfied. Then a $C^3[0,1]$ solution of (4.1) exists.

**Case 2:** $a = 0$

**Theorem 4.3.** Suppose conditions (2.2)–(2.5), (2.12) and (4.2) are satisfied. In addition, suppose (2.12*) holds with

$$\theta(t) = -\int_0^t (t-z) \int_0^z \psi(s) ds dz + t \int_0^1 (1-z) \int_0^z \psi(s) ds dz.$$ 

Then a $C^1[0,1] \cap C^3(0,1)$ solution of (4.1) exists.

**Proof.** This follows almost verbatim the ideas in Theorem 2.5. Here if $b > 0$ consider the family of problems

$$\begin{cases}
y''' + f(t,y) = 0, & 0 < t < 1, \\
y(0) = 1/n, & y(1) = b, & y''(0) = 0,
\end{cases}$$

where
whereas if $b = 0$ consider

$$
\begin{cases}
y'''(t) + f(t, y) = 0, & 0 < t < 1, \\
y(0) = 1/n, & y(1) = 1/n, & y''(0) = 0.
\end{cases}
$$

References


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