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Equations satisfied by the extremal star-like functions

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Let G be a class of star-like functions in relation to the point $z=0$ having a development of the form

$$F(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots, \tag{1}$$

where $0 < |z| < 1$, $b_k = x_k + iy_k$. Let us define by H_m the class of functions belonging to the class G which map a circle $|z| < 1$ on the plane from which at most m segments of a straight line coming from the point $z=0$ have been eliminated. Each of those functions may be expressed by the formula

$$F(z) = \frac{1}{z} \prod_{k=1}^{m} (1-\sigma_k z)^{\beta_k} = \frac{1}{z} + b_0 + b_1 z + \ldots, \tag{2}$$

where $\sigma_k = e^{i\vartheta_k}$, ϑ_k is a real number, $\beta_k \geqslant 0$ and $\sum_{k=1}^{m} \beta_k = 2$.

Now we shall analyse the set of points V having their coordinates determined by n first coefficients of the function (1) — *i. e.* the set of points $(\ldots, x_k, \ldots; \ldots, y_k, \ldots)$ — in the Euclidean space of $2n$ dimensions.

This set includes the origin of the system of coordinates, for the function $1/z$ is of the class G. Moreover it is connected (each of the points $(\ldots, x_k, \ldots; \ldots, y_k, \ldots)$ may be joined by the continuous curve $(\ldots, x_k \varrho^{k+1}, \ldots; \ldots, y_k \varrho^{k+1}, \ldots)$, $0 \leqslant \varrho \leqslant 1$, situated inside the set V with the origin of the system of coordinates, because the function $\varrho F(\varrho z)$ belongs to the class G if $F(z)$ belongs to G), bounded (by the area theorem) and closed (the class G is normal).

Let E be a real function of the class C_1, dependent on $2n$ real variables, defined in an open set comprising the set V, for which at each point of the set V we have $\sum_{k=1}^{n} \left\{ \left(\frac{\partial E}{\partial x_k}\right)^2 + \left(\frac{\partial E}{\partial y_k}\right)^2 \right\} \neq 0$. Then we have the following

THEOREM 1. *If the functional E gets the extremal value for a function of the class H_m, this function is determined by the following system of equations:*

$$A_{n+1}\sigma_j^{2n+2}+A_n\sigma_j^{2n+1}+\ldots+A_1\sigma_j^{n+2}+\lambda\sigma_j^{n+1}+\bar{A}_1\sigma_j^{n}+\ldots+\bar{A}_{n+1}=0,$$

$$(2n+2)A_{n+1}\sigma_j^{2n+1}+(2n+1)A_n\sigma_j^{2n}+\ldots+(n+1)\lambda\sigma_j^{n}+n\bar{A}_1\sigma_j^{n-1}+\ldots+\bar{A}_n=0,$$

$$\sum_{l=1}^{m'}\beta_l=2,\quad \sigma_j=e^{i\vartheta_j},\quad A_k=\frac{1}{k}\sum_{l=-1}^{n-k}b_l\left\{\frac{\partial E}{\partial x_{l+k}}-i\frac{\partial E}{\partial y_{l+k}}\right\},$$

$$k=1,2,\ldots,n+1,\quad j=1,2,\ldots,m'\leqslant m.$$

Proof. The functional E determined for the functions of the class H_m is a real function of variables ϑ_k,β_k determined at the common points of the cylinder $-\infty<\vartheta_k<\infty$, $0\leqslant\beta_k\leqslant 2$ and the hyperplane $\sum\limits_{k=1}^{m}\beta_k=2$. The relation of ϑ_k is periodical. The extremum of the functional E will occur: 1° at the interior point, or 2° on the boundary of the region defined above.

Suppose the first case: In order to express the derivatives of the function $E(\ldots,\vartheta_k,\ldots;\ldots,\beta_k,\ldots)$ we have to find the coefficients $b_1,\ldots,b_n$ of the function (2) as the functions of ϑ_k and β_k and use the relations

$$x_p=\tfrac{1}{2}\{b_p+\bar{b}_p\},\quad y_p=\tfrac{1}{2}\{\bar{b}_p-b_p\}i.$$

From the formula (2) we have

$$\log F(z)=-\log z+\sum_{k=1}^{m}\beta_k\log(1-ze^{i\vartheta_k}).$$

Hence

$$\frac{d}{dz}\log F(z)=-\frac{1}{z}+\sum_{k=1}^{m}\beta_k\frac{-e^{i\vartheta_k}}{1-ze^{i\vartheta_k}}.$$

Because

$$\frac{e^{i\vartheta_k}}{1-ze^{i\vartheta_k}}=\sum_{p=0}^{\infty}e^{(p+1)i\vartheta_k}z^p,$$

introducing the notation

$$a_p=\sum_{k=1}^{m}\beta_k e^{pi\vartheta_k}$$

we get

$$z\frac{d\log F(z)}{dz}=-1-\sum_{p=1}^{\infty}a_pz^p.$$

Hence again

$$z\frac{d}{dz}F(z)=-\Big(1+\sum_{p=1}^{\infty}a_pz^p\Big)F(z),$$

and having compared the coefficients on both sides we get the relation

$$(p+1)b_p+b_{p-1}a_1+\ldots+b_0a_p+a_{p+1}=0,\quad p=0,1,\ldots$$

Thus we have obtained a system of equations from which we are able to define the coefficients of the function (2) by the variables ϑ_k and β_k

$$b_p=\frac{(-1)^{p+1}}{(p+1)!}\begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_p & a_{p+1}\\ p & a_1 & a_2 & \ldots & a_{p-1} & a_p\\ 0 & p-1 & a_1 & \ldots & a_{p-2} & a_{p-1}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & a_1 & a_2\\ 0 & 0 & 0 & \ldots & 1 & a_1\end{vmatrix}.$$

Hence we see that b_p takes the form of an algebraic polynomial of the variables β_k and a trigonometrical polynomial of the variables ϑ_k.

Further

$$\frac{\partial F}{\partial\beta_k}=\frac{1}{z}\prod_{j=1}^{m}(1-\sigma_jz)^{\beta_j}\log(1-\sigma_kz)=\frac{\partial b_0}{\partial\beta_k}+\frac{\partial b_1}{\partial\beta_k}z+\ldots+\frac{\partial b_p}{\partial\beta_k}z^p+\ldots$$

Developing the first part of the formula in a power series we get

$$\frac{\partial b_p}{\partial\beta_k}=-\left\{\frac{\sigma_k^{p+1}}{p+1}+b_0\frac{\sigma_k^p}{p}+\ldots+b_{p-2}\frac{\sigma_k^2}{2}+b_{p-1}\sigma_k\right\}. \tag{3}$$

Similarly we shall find

$$\frac{\partial b_p}{\partial\vartheta_k}=-i\beta_k\{\sigma_k^{p+1}+b_0\sigma_k^p+\ldots+b_{p-2}\sigma_k^2+b_{p-1}\sigma_k\}. \tag{4}$$

In order to find the extremum of the function $E(\ldots,\vartheta_k,\ldots;\ldots,\beta_k,\ldots)$ at the interior point we have to calculate the derivatives of the function $E^*=E+\lambda'(\sum\limits_{k=1}^{m}\beta_k-2)$, where λ' is a Lagrange multiplier connected with the condition $\sum_{k=1}^{m}\beta_k = 2$. We get

$$\frac{\partial E^*}{\partial\beta_k} = \sum_{j=1}^{n}\frac{\partial E}{\partial x_j}\cdot\frac{\partial x_j}{\partial\beta_k} + \sum_{j=1}^{n}\frac{\partial E}{\partial y_j}\cdot\frac{\partial y_j}{\partial\beta_k} + \lambda' = 0,$$

$$\frac{\partial E^*}{\partial\vartheta_k} = \sum_{j=1}^{n}\frac{\partial E}{\partial x_j}\cdot\frac{\partial x_j}{\partial\vartheta_k} + \sum_{j=1}^{n}\frac{\partial E}{\partial y_j}\cdot\frac{\partial y_j}{\partial\vartheta_k} = 0, \tag{5}$$

$$\sum_{j=1}^{m}\beta_j = 2, \qquad k = 1, 2, \ldots, m.$$

For $x_p = \operatorname{re} b_p$, $y_p = \operatorname{im} b_p$ we get from the formulae (3) and (4) and the equations (5) the following system of equations

$$\sum_{j=1}^{n+1}\frac{\sigma_k^j}{j}\sum_{p=-1}^{n-j} b_p\left\{\frac{\partial E}{\partial x_{p+j}} - i\frac{\partial E}{\partial y_{p+j}}\right\} + \sum_{j=1}^{n+1}\frac{\bar{\sigma}_k^j}{j}\sum_{p=-1}^{n-j}\bar{b}_p\left\{\frac{\partial E}{\partial x_{p+j}} + i\frac{\partial E}{\partial y_{p+j}}\right\} - 2\lambda' = 0,$$

$$\beta_k\left[\sum_{j=1}^{n+1}\sigma_k^j\sum_{p=-1}^{n-j} b_p\left\{\frac{\partial E}{\partial x_{p+j}} - i\frac{\partial E}{\partial y_{p+j}}\right\} - \sum_{j=1}^{n+1}\bar{\sigma}_k\sum_{p=-1}^{n-j}\bar{b}_p\left\{\frac{\partial E}{\partial x_{p+j}} + i\frac{\partial E}{\partial y_{p+j}}\right\}\right] = 0, \tag{6}$$

$$\sum_{j=1}^{m}\beta_j = 2, \qquad k = 1, 2, \ldots, m, \qquad b_{-1} = 1.$$

Ex definitione the extremum is inside the region; therefore we have $\beta_k \neq 0$, and thus $\bar{\sigma}_k = 1/\sigma_k$. For brevity let us substitute

$$A_j = \frac{1}{j}\sum_{p=-1}^{n-j} b_p\left\{\frac{\partial E}{\partial x_{p+j}} - i\frac{\partial E}{\partial y_{p+j}}\right\};$$

then we get the following system:

$$\sigma_k^{n+1}\sum_{j=1}^{n+1}\sigma_k^j A_j - 2\lambda'\sigma_k^{n+1} + \sum_{j=1}^{n+1}\sigma_k^{n+1-j}\bar{A}_j = 0,$$

$$\sigma_k^{n+1}\sum_{j=1}^{n+1}\sigma_k^j jA_j - \sum_{j=1}^{n+1}\sigma_k^{n+1-j} j\bar{A}_j = 0, \tag{7}$$

$$\sum_{j=1}^{m}\beta_j = 2, \qquad k = 1, 2, \ldots, m.$$

Multiplying the first equation of the system (7) by $n+1$, adding the result to the second equation and introducing $\lambda = -2\lambda'$, we obtain instead of the system (7) the equivalent system

$$A_{n+1}\sigma_k^{2n+2} + \ldots + A_1\sigma_k^{n+2} + \lambda\sigma_k^{n+1} + \bar{A}_1\sigma_k^{n} + \ldots + \bar{A}_{n+1} = 0,$$

$$(2n+2)A_{n+1}\sigma_k^{2n+1} + \ldots + (n+2)A_1\sigma_k^{n+1} + (n+1)\lambda\sigma_k^{n} + n\bar{A}_1\sigma_k^{n-1} + \ldots + \bar{A}_n = 0, \tag{8}$$

$$\sum_{j=1}^{m}\beta_j = 2, \qquad k = 1, 2, \ldots, m.$$

Thus — in the case of the extremum being inside — the theorem is proved.

Now suppose that the extremum appears on the boundary of the region of possible values for β_k and ϑ_k, *i. e.* for $\beta_k = 2$ or for $\beta_k = 0$. In the first case we find from the equality $\sum_{k=1}^{m}\beta_k = 2$ and from the condition $\beta_k \geqslant 0$ that

$$\beta_1 = \ldots = \beta_{k-1} = \beta_{k+1} = \ldots = \beta_m = 0;$$

thus the extremal function is expressed by the formula

$$F^* = \frac{1}{z}(1-\sigma z)^2 = \frac{1}{z} - 2\sigma + \sigma^2 z.$$

We can easily find whether the functional E takes its extremal value for this function. But if $\beta_k = 0$, then the function for which the extremum appears belongs to the class H_{m-1} and there we have to look for the extremum. All this leads to the solution of the system (8) under the condition that the number of the equations will be diminished by 2 (k varies from 1 to $m-1$ instead of m). In the case when some β_k equal 0 the extremum will be found relatively in the class $H_{m'}$ and k varies from 1 to $m' < m$. Thus the theorem is proved.

THEOREM 2. *The extremal value of the functional $E(\ldots, x_k, \ldots; \ldots, y_k, \ldots)$ is obtained in the class G for a function belonging to the class H_{n+1}.*

Proof. Theorem 1 implies that the extremal function of the functional E — and belonging to the class H_m (free m) — belongs to the class H_{n+1}; this follows from the fact that, although there may be many finite sequences of numbers $\{\sigma_1, \ldots, \sigma_m\}$ which solve the equations named in the theorems, but necessarily $m \leqslant n+1$, otherwise the polynomial of the $(2n+2)$-th degree and its derivative would have more than $n+1$ common roots.

Suppose that there is a function $\Phi(z)$ belonging to G and such that

$$E(\Phi(z)) > E(F(z)),$$

where $F(z)$ is a function of the class H_{n+1} given by the solution of the system of extremal equations and for which the functional E takes the largest value in the class H_{n+1}. Of course we are able to find the function $\Phi_1(z)$ belonging to the class H_m with sufficiently large m for which the first n coefficients would vary arbitrarily little from the first n coefficients of the function $\Phi(z)$, and hence also

$$E(\Phi_1(z)) > E(F(z)),$$

contrary to the assumption that for $F(z)$ the functional E takes the largest value in the class H_m (free m). The results obtained so far may be employed for estimating the coefficients of the star-like functions with a pole. Then we have to analyse the functional $E = \mathrm{re}\, b_n$. The equations which indicate the extremal function for this functional will be of the form

$$\sigma_k^{2(n+1)} + a_1\sigma_k^{2n+1} + \ldots + \lambda\sigma_k^{n+1} + \ldots + \bar{a}_1\sigma_k + 1 = 0,$$

$$(9)\qquad 2(n+1)\sigma_k^{2n+1} + (2n+1)a_1\sigma_k^{2n} + \ldots + (n+1)\lambda\sigma_k^{n} + \ldots + \bar{a}_1 = 0,$$

$$a_j = \frac{b_{j-1}}{n+1-j}(n+1), \qquad \sum_{l=1}^{m}\beta_l = 0, \qquad k = 1, 2, \ldots, m \leqslant n+1.$$

We can easily verify that these equations are satisfied by the sequences of numbers

$$(10)\qquad \beta_k = \frac{2}{n+1}, \quad \sigma_k = \exp\frac{2\pi k i}{n+1}, \qquad \beta_k = \frac{2}{n+1}, \quad \sigma_k = \exp\frac{\pi(2k+1)i}{n+1}$$

and even it will be sufficient to assume that $\beta_k = 2/(n+1)$ in order to obtain the sequences of numbers (10) as the only solution of the system. Hence we see that the hypothesis $|b_n| \leqslant 2/(n+1)$ may be held for the star-like functions.

A general solution of the system (9) seems to be difficult, but it may be done for $n=1$ and $n=2$.

Let us demonstrate the solution for $n=1$. The equations (9) are of the form for $n=1$

$$\sigma_k^4 + 2b_0\sigma_k^3 + \lambda\sigma_k^2 + 2\bar{b}_0\sigma_k + 1 = 0,$$

$$4\sigma_k^3 + 6b_0\sigma_k^2 + 2\lambda\sigma_k + 2b_0 = 0,$$

$$\sum_{l=1}^{m}\beta_l = 2, \qquad m \leqslant 2, \qquad k \leqslant 2.$$

Hence, either the extremum is obtained for the function

$$F(z) = \frac{1}{z}(1-\sigma z)^2 = \frac{1}{z} - 2\sigma + \sigma^2 z$$

and since $|\sigma| = 1$ we have $|b_1| \leqslant 1$ or σ_1 and σ_2 are the double roots of the equation

$$\sigma^4 + 2b_0\sigma^3 + \lambda\sigma^2 + 2\bar{b}_0\sigma + 1 = 0,$$

that is

$$2b_0 = -2(\sigma_1+\sigma_2), \qquad 1 = \sigma_1^2\sigma_2^2 \qquad \text{(Vieta's formulae)}.$$

Because $b_0 = -(\beta_1\sigma_1 + \beta_2\sigma_2)$ and $\beta_2 = 2-\beta_1$, we have $\beta_1\sigma_1 + (2-\beta_1)\sigma_2 = \sigma_1 + \sigma_2$ and hence, because $\sigma_1 \neq \sigma_2$, $\beta_1 = \beta_2 = 1$. Thus the form of the extremal function will be

$$F(z) = \frac{1}{z}(1-\sigma_1 z)(1-\sigma_2 z) = \frac{1}{z} - (\sigma_1+\sigma_2) + \sigma_1\sigma_2 z.$$

Because $\sigma_1^2\sigma_2^2 = 1$, we have $|b_1| \leqslant 1$.

Note 1. Theorem 2 may also be obtained by a special modification of the results comprised by Carathéodory's paper (see [1]).

Note 2. Analoguos results may be obtained for the functionals defined for the regular star-like functions.

Reference

[1] C. Carathéodory, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Functionen*, Rend. del Circolo Matematico di Palermo 32 (1911), p. 193-217.

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