Comme on a, d'après (36) et (31),

$$V(t_0) \leqslant (p^2Lc^2+2pL^2c+p^2L^2c^2)\frac{1}{M^2} = \frac{L_4}{M^2},$$

donc (cf. [6])

$$V(t) \leqslant \left(\frac{L_4}{M^2}+\frac{pK_1^2}{KM^2}\right)e^{K(t_1-t_0)}-\frac{pK_1^2}{KM^2} \leqslant \frac{L_5}{M^2}.$$

Mais on a, en vertu de (38)

$$V(t) \geqslant a\sum u_\nu^{\cdot 2}+bM^2\sum\left(|u_\nu|-\frac{pL^2}{bM^2}\right)^2-\frac{p^2L^4}{bM^2},$$

d'où

$$u_\nu^{\cdot 2} \leqslant \frac{L_6}{M^2}, \qquad \left(|u_\nu|-\frac{pL^2}{bM^2}\right)^2 \leqslant \frac{L_7}{M^4};$$

en posant $c_1 = \max(\sqrt{L_6}, \sqrt{L_7}+pL^2/b)$, nous obtenons donc

$$|u_\nu| \leqslant c_1/M^2, \quad |u_\nu^{\cdot}| \leqslant c_1/M \quad \text{dans} \quad \langle t_0, t'\rangle, \quad \text{c. q. f. d.}$$

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The first boundary value problem for a non-linear parabolic equation

by W. Mlak (Kraków)

We consider the first boundary problem for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}+f(x, t, u).$$

In the proof of the existence of a solution of that problem the topological method of Leray-Schauder is used. To obtain the so called a priori limitation of solutions, needed in this method, some qualitative conditions are formulated. These conditions make it possible to find in a simple way the topological degree of a suitable completely continuous vector field associated with our boundary problem. Conditions of a similar character have been discussed in [1], [2] and [6].

1. To begin with let us formulate the following condition:

(A) The function $\sigma(t, y)$ is continuous for $0 \leqslant t \leqslant b$ $(0 < b)$ and $y \geqslant 0$. For all $\eta \geqslant 0$ the right maximal integrals $\omega(t, \eta)$ of the differential equation $y' = \sigma(t, y)$ such that $\omega(0, \eta) = \eta$ exist in the interval $\langle 0, b\rangle$.

Theorem 3 of [5] implies the following lemma:

Lemma 1. *Assume that the function $\sigma(t, y)$ satisfies the condition* (A). *The function $f(x, t, u)$ is defined for $0 \leqslant x \leqslant a$ $(0 < a)$, $0 \leqslant t \leqslant b$ $(0 < b)$ and an arbitrary u. We assume that*

$$|f(x, t, u)| \leqslant \sigma(t, |u|).$$

Suppose that $v(x, t)$ is continuous in $R = \underset{(x,t)}{E}\{0 \leqslant x \leqslant a,\ 0 \leqslant t \leqslant b\}$ and possesses the continuous derivative $\partial^2 v/\partial x^2$ in the interior of R. Assume that $z = v(x, t)$ satisfies in the interior of R the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2+f(x, t, z)$$

and the boundary inequalities

$$|v(0, t)| \leqslant \eta, \quad |v(a, t)| \leqslant \eta, \quad 0 \leqslant t \leqslant b; \quad |v(x, 0)| \leqslant \eta, \quad 0 \leqslant x \leqslant a.$$

Under our assumptions we have the inequality

$$|v(x,t)| \leqslant \omega(t,\eta), \quad (x,t)\in R.$$

The second lemma is the following one (see [2] lemma III):

LEMMA 2. *Let the function $g(x,t)$ be continuous in $R = \underset{(x,t)}{E}\{0\leqslant x\leqslant a,\ 0\leqslant t\leqslant b\}$. Suppose that $|g(x,t)|\leqslant M$ for $(x,t)\in R$. Then there is such a constant K that the function*

$$p(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t\int_0^a \frac{\exp[-(x-\xi)^2/4(t-\tau)]}{\sqrt{t-\tau}}\, g(\xi,\tau)\,d\xi\,d\tau$$

satisfies the inequality

$$|p(x+h,t+k)-p(x,t)| \leqslant KM[|h|+|k|^{3/4}].$$

2. Suppose that the functions $u(x,t)$, $f(x,t,z)$ are continuous in R and $Q = \underset{(x,t,z)}{E}\{0\leqslant x\leqslant a,\ 0\leqslant t\leqslant b,\ -\infty<z<+\infty\}$ respectively. Take the function

$$(1)\qquad r(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t\int_0^a \frac{\exp[-(x-\xi)^2/4(t-\tau)]}{\sqrt{t-\tau}}\, f[\xi,\tau,u(\xi,\tau)]\,d\xi\,d\tau.$$

By means of the elementary theory of the heat equation (see [2], [3]) one finds the solution $q(x,t)$ of the equation

$$(2)\qquad \partial z/\partial t = \partial^2 z/\partial x^2$$

satisfying the conditions

$$q(0,t) = \varphi(t)-r(0,t), \quad q(a,t) = \psi(t)-r(a,t), \quad 0\leqslant t\leqslant b;$$
$$q(x,0)=0, \quad 0\leqslant x\leqslant a.$$

$\varphi(t)$ and $\psi(t)$ are fixed arbitrary continuous functions such that $\varphi(0) = \psi(0) = 0$, $r(x,t)$ is given by formula (1).

Now write $v(x,t) = q(x,t)+r(x,t)$. Hence to every $u(x,t)$ corresponds the uniquely determined function $v(x,t)$; the correspondence is given by the procedure mentioned above. Therefore we get the transformation law $u\to v$. Denote it by $T(u;f,\varphi,\psi)$ (f, φ and ψ play the role of parameters): $v = T(u;f,\varphi,\psi)$. It may easily be shown that $T(u;f,\varphi,\psi)$ is continuous with respect to u as considered in the space C of all functions continuous in R — the norm in C is defined as follows: $\|u\| = \max\limits_{(x,t)\in R}|u(x,t)|$. The operation T maps C in its part.

To complete our considerations we shall prove the following lemma:

LEMMA 3. *Suppose that $f(x,t,z)$ is continuous in Q. Let the functions $\varphi(t)$, $\psi(t)$ be continuous for $0\leqslant t\leqslant b$. Assume that $\varphi(0)=\psi(0)=0$. Then the operation $T(u;f,\varphi,\psi)$ is completely continuous with regard to u (by fixed f, φ and ψ).*

Proof. Suppose that $u_n\in C$ and $\|u_n\|\leqslant\varrho$. From lemma 2 we conclude that the functions

$$r_n(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t\int_0^a \frac{\exp[-(x-\xi)^2/4(t-\tau)]}{\sqrt{t-\tau}}\, f[\xi,\tau,u_n(\xi,\tau)]\,d\xi\,d\tau$$

are equibounded and equicontinuous in R. Therefore there exists a partial sequence $r_{a_n}(x,t)$ which converges uniformly to a certain continuous function $r(x,t)$: $r_{a_n}(x,t)\Rightarrow r(x,t)$ ($\Rightarrow$ denotes uniform convergence). Let q_{a_n} be a solution of the equation (2) such that

$$q_{a_n}(0,t) = \varphi(t)-r_{a_n}(0,t), \quad q_{a_n}(a,t) = \psi(t)-r_{a_n}(a,t), \quad 0\leqslant t\leqslant b;$$
$$q_{a_n}(x,0)=0, \quad 0\leqslant x\leqslant a.$$

The solutions of (2) depend in a continuous manner on boundary and initial conditions. Hence $q_{a_n}(x,t)\Rightarrow q(x,t)$ where $q(x,t)$ is a solution of (2) such that

$$q(0,t) = \varphi(t)-r(0,t), \quad q(a,t) = \psi(t)-r(a,t), \quad 0\leqslant t\leqslant b;$$
$$q(x,0)=0, \quad 0\leqslant x\leqslant a.$$

Obviously $v_{a_n} = T(u_{a_n};f,\varphi,\psi)\Rightarrow q+r$ q. e. d.

We say that the function $f(x,t,z)$ satisfies the condition (H) if:

(H) for every (x,t,z), $0<x<a$, $0\leqslant t\leqslant b$ and an arbitrary z there are such a neighbourhood $N(x,z)$ of (x,z) and such constants L, $0<\alpha<1$, $0<\beta<1$ depending in general on (x,t,z), that for $(\bar{x},\bar{z})$, $(\bar{\bar{x}},\bar{\bar{z}})\in N(x,z)$ the inequality

$$|f(\bar{x},t,\bar{z})-f(\bar{\bar{x}},t,\bar{\bar{z}})| \leqslant L[|\bar{x}-\bar{\bar{x}}|^\alpha+|\bar{z}-\bar{\bar{z}}|^\beta]$$

holds.

Suppose that $v = T(v;f,\varphi,\psi)$. Then $v(x,t) = q(x,t)+r(x,t)$ where

$$r(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t\int_0^a \frac{\exp[-(x-\xi)^2/4(t-\tau)]}{\sqrt{t-\tau}}\, f[\xi,\tau,v(\xi,\tau)]\,d\xi\,d\tau$$

and $q(x,t)$ is the solution of (2). From lemma 2 it follows that $r(x,t)$ satisfies the Hölder condition with respect to x. On the other hand $q(x,t)$, as the solution of (2), satisfies the Lipschitz condition in every compact lying in the interior of R. Therefore the function $f[x, t, v(x, t)]$, if $f(x, t, z)$ satisfies the (H) condition, possesses the following property: for every $x_0 \in (0, a)$, $t \in (0, b)$, there is a neighbourhood of x_0 in which the function $f[x, t, v(x, t)]$ satisfies the Hölder condition with respect to x. The Hölder constants depend on (x_0, t). From the classical results of Levy and Gevrey [3] it follows that $z = v(x, t)$ satisfies the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2 + f(x, t, z). \tag{3}$$

Moreover $v(x, t)$ satisfies the initial and boundary conditions

$$\begin{gathered} v(0, t) = \varphi(t), \quad v(a, t) = \psi(t), \quad 0 \leqslant t \leqslant b; \\ v(x, 0) = 0, \quad 0 \leqslant x \leqslant a \end{gathered} \tag{4}$$

where $\varphi(0) = \psi(0) = 0$.

In this way the boundary problem is reduced to the functional equation $z = T(z; f, \varphi, \psi)$.

THEOREM. *Suppose that the function $\sigma(t, y)$ satisfies the condition* (A). *Let the function $f(x, t, z)$ be continuous in Q and satisfy the condition* (H). *We assume that*

$$|f(x, t, u)| \leqslant \sigma(t, |u|).$$

The functions $\varphi(t), \psi(t)$ are continuous in $\langle 0, b\rangle$ and $\varphi(0) = \psi(0) = 0$. Under our assumptions there exists at least one solution of the equation (3) *satisfying conditions* (4).

Proof. Lemma 3 implies that the operations $T_\lambda = T(\cdot; \lambda f, \lambda\varphi, \lambda\psi)$ where $0 \leqslant \lambda \leqslant 1$ are completely continuous. If $v_\lambda = T_\lambda v_\lambda$ then $v_\lambda(x, t)$ satisfies the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2 + \lambda f(x, t, z)$$

and

$$v_\lambda(0, t) = \lambda\varphi(t), \quad v_\lambda(a, t) = \lambda\psi(t), \quad v_\lambda(x, 0) = 0.$$

But

$$|\lambda f(x, t, z)| \leqslant \lambda\sigma(t, |z|) \leqslant \sigma(t, |z|)$$

and

$$|v_\lambda(0, t)| \leqslant |\varphi(t)|, \quad |v_\lambda(a, t)| \leqslant |\psi(t)|, \quad |v_\lambda(x, 0)| = 0.$$

Applying lemma 1 we obtain

$$|v_\lambda(x, t)| \leqslant \omega(t, \eta), \quad (x, t) \in R, \quad 0 \leqslant \lambda \leqslant 1$$

where $\eta = \max\{\max_{\langle 0,b\rangle}|\varphi(t)|, \max_{\langle 0,b\rangle}|\psi(t)|\}$.

Now put $\varrho = \max_{\langle 0,b\rangle}\omega(t, \eta)$. Therefore $v \neq T_\lambda v$ if $\|v\| = \varrho+\delta$, $\delta > 0$.

Thus, considered on the sphere $\|v\| = \varrho+\delta$, the operation $T(v; f, \varphi, \psi)$ is homotopic with the operation identically equal to zero. Hence the topological degree of the vector field $Fv = v - T(v; f, \varphi, \psi)$ considered on the sphere $\|v\| = \varrho+\delta$ taken with respect to the zero vector of C is equal to $+1$. From the Leray-Schauder principle ([4]) we find that there exists such a v_0 that $v_0 = T(v_0; f, \varphi, \psi)$. This completes the proof.

EXAMPLE 1. Suppose that the function $f(x, t, z)$ is continuous in Q and satisfies the (H) condition. Assume that

$$|f(x, t, z)| \leqslant M|z| + K,$$

M and K being positive constants. The assumptions of our theorem are fulfilled if we put $\sigma = My+K$.

EXAMPLE 2. Let the function $f(x, t, z)$ satisfy in Q the Hölder condition of the following form:

$$|f(x, t, z) - f(\bar{x}, t, \bar{z})| \leqslant M[|x-\bar{x}|^\alpha + |z-\bar{z}|^\beta], \quad 0 < \alpha < 1, \ 0 < \beta < 1.$$

Write $K = \max_{\langle 0,b\rangle}|f(0, t, 0)|$. Therefore

$$|f(x, t, z)| \leqslant M|z|^\beta + (K + Ma^\alpha).$$

In that case we put $\sigma(t, y) = My^\beta + (K+Ma^\alpha)$.

3. Remark 1. Our investigations may be extended to systems of parabolic equations(¹):

$$\frac{\partial z_s}{\partial t} = \sum_{i,k=1}^{n} a_{ik}^s(x_1, \ldots, x_n; t)\frac{\partial^2 z_s}{\partial x_i \partial x_k} + f_s(x_1, \ldots, x_n, t, z_1, \ldots, z_m) \quad (s = 1, \ldots, m). \tag{$*$}$$

The functions a_{ik}^s and f_s are supposed to be sufficiently regular to permit the use of the potential theory. The condition $|f(x, t, u)| \leqslant \sigma(t, |u|)$ is replaced by the following one:

$$|f_s(x_1, \ldots, x_n, t, z_1, \ldots, z_m)| \leqslant \sigma_s(t, |z_1|, \ldots, |z_m|) \quad (s = 1, \ldots, m).$$

This enables us to formulate a lemma similar to lemma 1 (see [5], theorem 3). An analogous procedure as presented in this paper may be continued for the second and for the third boundary problem for parabolic equations of the form $(*)$. A priori limitation of solutions is then ensured by a suitable lemma (see [5], theorem 6).

(¹) The forms $\sum_{i,k=1}^{n} a_{ik}^s \lambda_i \lambda_k$ are non-negative.

Remark 2. It may easily be seen that if the assumptions of our existence theorem are satisfied for the functions f, φ and ψ they are satisfied also for $f+\varepsilon$, $\varphi+\varepsilon t$, $\psi+\varepsilon t$, $\sigma+\varepsilon$ and $\varepsilon>0$ small enough. Let the solutions of such a perturbed problem be denoted by v_ε. In a natural way, as in the theory of ordinary differential inequalities, one can prove the existence of the maximal solution. It is defined as the limit $\lim_{\varepsilon\to 0+} v_\varepsilon$. Similar constructions are possible for systems (*). The formal way is to be based on the Westphal-Prodi theorem concerning strong differential inequalities of parabolic type.

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Sur les courbes planes à courbure presque constante

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§ 1. Soit C une courbe plane de classe C^1 donnée par les équations

$$x = x(s), \quad y = y(s) \quad (s \geqslant 0)$$

où s est la longueur de l'arc de C compté à partir d'un point fixe. Désignons par $P(s)$ le point de la courbe C qui correspond à la valeur s du paramètre.

Définition. Nous dirons que *la courbe C s'enroule pour $s\to+\infty$ asymptotiquement autour d'une circonférence K*, si la distance entre le point $P(s)$ de la courbe et le plus proche des deux points de la circonférence K où les tangentes à la circonférence sont parallèles à la tangente à la courbe au point $P(s)$, tend vers zéro lorsque $s\to+\infty$.

Si donc la courbe C s'enroule asymptotiquement autour d'une circonférence, cela veut dire que non seulement le point mobile de la courbe $P(s)$ se rapproche, pour $s\to+\infty$, de cette circonférence, mais aussi qu'il tourne indéfiniment autour du centre de cette circonférence et que, s tendant vers l'infini, ces tours deviennent de plus en plus réguliers, des changements brusques de direction du mouvement étant impossibles.

Désignons par $\varphi(s)$ l'angle que fait le vecteur $\big(x'(s), y'(s)\big)$ tangent à la courbe C en point $P(s)$ avec l'axe des x. On voit facilement que si la courbe C s'enroule autour d'une circonférence, l'angle $\varphi(s)$ pour $s\to+\infty$ doit tendre ou bien vers $+\infty$, ou bien vers $-\infty$. Dans le premier cas nous dirons que le sens de l'enroulement est *positif* et dans le second cas qu'il est *négatif* (1). Cela posé, on peut démontrer le théorème suivant que nous nous bornons à énoncer, puisque sa démonstration est tout à élémentaire:

Théorème I. *Pour que la courbe C: $x = x(s)$, $y = y(s)$ de classe C^1 (s — paramètre intrinsèque) s'enroule asymptotiquement autour d'une circon-*

(1) Nous supposons que dans le système de coordonnées adopté le sens positif des rotations est celui de Ox vers Oy et que ce sens est contraire à celui des aiguilles d'une montre.