Limitation of solutions of parabolic equations

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We discuss some estimations concerning the solutions of parabolic equations. The epidemic theorems are used. In the second paragraph are given certain limitations for the increase of solutions with respect to the independent variables. As a rule we investigate non-linear equations.

§ 1. Suppose that $G$ is an open and bounded set of points $x = (x_1, \ldots, x_n)$ of the $n$-dimensional space $R^n$. Denote by $Z$ the Cartesian product of $G$ and of the interval $A = (a, b)$. $Z = G \times (a, b)$; the boundary of $G$ is denoted by $\partial G$. Suppose that we are given the functions $F_\nu(x, t, u_1, \ldots, u_m, p_1, \ldots, q_m)$, $\nu = 0, 1, 2, \ldots, m$. The functions $\sigma_\nu(t, u_1, \ldots, u_m)$, $\nu = 0, 1, 2, \ldots, m$, are defined for $(t, x) \in G$, $t \in A$ and arbitrary $u_1, \ldots, u_m$.

We say that the sequence of functions $u_t(x, t), u_{x_1}(x, t), \ldots, u_{x_n}(x, t)$ defined in $Z$ is a regular solution of the system of equations

\[ \frac{\partial u_t}{\partial t} + F_\nu(x, t, u_1, \ldots, u_m, \frac{\partial u}{\partial x}, \sum_{\nu=1}^{m} \frac{\partial^2 u_\nu}{\partial x^\nu} = 0 \quad (x \in \partial Z) \]

if $u_t$ are continuous in $Z$, possess continuous derivatives $\partial u_t/\partial x$, $\partial^2 u_t/\partial x^\nu\partial x^\mu$ in the interior of $Z$ and for $(x, t) \in \partial G$, and satisfy the system (1) for $(x, t) \in \partial Z$.

Let us introduce the following conditions:

(2) if $\sum_{\nu=1}^{m} t \nu \leq 0$ for arbitrary $t_1, \ldots, t_n$, then

\[ F_\nu(x, t, u_1, \ldots, u_m, 0, q_\nu) \leq \sigma_\nu(t, u_1, \ldots, u_m) \quad (\nu = 1, \ldots, m); \]

(3) the functions $\sigma_\nu$ are continuous and satisfy the following condition:

(W) if $u_\nu \leq \bar{u}_\nu$ (for $\nu \neq \bar{\nu}$), then $\sigma_\nu(t, u_1, \ldots, u_m) \leq \sigma_\nu(t, \bar{u}_1, \ldots, \bar{u}_m)$; we assume that the right maximal integral $\omega_0(t, x_1, \ldots, x_n)$ of the system of ordinary differential equations

\[ y'_\nu = \sigma_\nu(t, y_1, \ldots, y_m) \quad (\nu = 1, \ldots, m) \]
such that \( w_0(a; h_1, \ldots, h_m) = h_s \) is defined for \( a < t < b + \varepsilon \)
\((0 < \varepsilon \leq \delta)\).

Remark that if (1) is a parabolic system (see (2)) and if

\[ F_s(x, t, u_1, \ldots, u_m, 0, 0) \leq \sigma_s(t, u_1, \ldots, u_m) \]

then (2) is satisfied.

**Theorem 1.** Suppose that (2) and (3) are satisfied. Let \( u_s(x, t), \ldots, u_m(x, t) \) be a regular solution of the system (1). Assume that

1. \( u_s(x, a) \leq u_0(a; h_1, \ldots, h_m) = h_s, \quad x \in \Omega, \quad s = 1, \ldots, m, \)
2. \( u_s(x, t) \leq u_0(t; h_1, \ldots, h_m), \quad x \in \Gamma, \quad a < t < b, \quad s = 1, \ldots, m. \)

Then

\[ u_s(x, t) \leq u_0(t; h_1, \ldots, h_m) \quad (x, t) \in \bar{\Omega}, \quad s = 1, \ldots, m. \]

**Proof.** Let us put

\[ U_s(t) = \max_{x \in \Omega} u_s(x, t). \]

Suppose that for some \( t \) we have \( u_0(t; h_1, \ldots, h_m) < U_s(t) \). There is such an \( x \in \Omega \) that \( U_s(t) = u_s(x, t) \). According to the inequality (**) we find that \( \exists \xi \in \Omega \). The function \( u_s(x; x, t) \) (\( \xi \) being fixed) reaches in \( \bar{\Omega} \) a maximum. Therefore

\[ \frac{\partial u_s}{\partial \xi} \bigg|_{\bar{\Omega}} = 0 \quad (k = 1, \ldots, m) \quad \text{and} \quad \sum_{k=1}^{n} \frac{\partial^2 u_s}{\partial \xi_k \partial \xi_k} \bigg|_{\bar{\Omega}} \leq 0 \]

for arbitrary \( \lambda_1, \ldots, \lambda_m. \)

By (2) (3) we get

\[ \frac{\partial u_s}{\partial t} \bigg|_{\bar{\Omega}} = F_s \left( x, t, u_1, \ldots, u_m, 0, 0 ; \frac{\partial^2 u_s}{\partial \xi_k \partial \xi_k} \right) \leq \sigma_s(t, u_1, \ldots, u_m). \]

But

\[ D_s U_s(t) \leq \partial u_s/\partial t \bigg|_{\bar{\Omega}}. \]

On the other hand \( u_s(x, t) \leq U_s(t) \) (\( s = 1, \ldots, m, \) and \( u_0(x, t) = U_s(t) \). This implies, by condition (W) for \( \sigma_s \), the inequality

\[ \sigma_s(t, u_1, \ldots, u_m) \leq \sigma_s(t, U_s(t), \ldots, U_m(t)). \]

From (3.1), (3.2) and (3.3) we obtain

\[ D_s U_s(t) \leq \sigma_s(t, U_s(t), \ldots, U_m(t)) \]

for such \( t \) that \( u_0(t; h_1, \ldots, h_m) \leq U_s(t) \). By (s) we have \( U_s(t) \leq h_s (t = 1, \ldots, m) \). Hence \( U_1(t), \ldots, U_m(t) \) satisfy the assumptions of the epidermic theorem for right maximal integrals. Therefore (see (1) and (4))

\[ u_s(x, t) \leq u_0(t; h_1, \ldots, h_m). \]

We formulate an analogous theorem concerning the limitation from below. We introduce the following conditions:

4. \( \sum \limits_{\delta = 1}^{n} \eta \delta \lambda \delta \geq 0 \) for arbitrary \( \lambda_1, \ldots, \lambda_m, \) then

\[ F(x, t, u_1, \ldots, u_m, 0, 0 ; \sigma(t, u_1, \ldots, u_m)); \]

5. the functions \( \sigma_s \) are continuous and satisfy condition (W); the right minimal integral \( u_0(t; h_1, \ldots, h_m) \) \((s = 1, \ldots, m)\) of the system

\[ y_s = \sigma_s(t, y_1, \ldots, y_m) \quad (s = 1, \ldots, m) \]

such that \( u_0(a; h_1, \ldots, h_m) = h_s \) exists in the interval \((a, b + \varepsilon) (0 < \varepsilon \leq \delta)\).

**Theorem 2.** Suppose that (4) (5) are satisfied. Let \( u_s(x, t), \ldots, u_m(x, t) \) be a regular solution of the system (1). Assume that

\[ h_s = u_0(a; h_1, \ldots, h_m) \leq u_s(x, a) \quad x \in \Gamma, \quad a < t < b, \quad s = 1, \ldots, m, \]

\[ u_0(t; h_1, \ldots, h_m) \leq u_s(x, t) \quad x \in \Gamma, \quad a < t < b. \]

Then

\[ u_0(t; h_1, \ldots, h_m) \leq u_s(x, t) \quad (x, t) \in \bar{\Omega}. \]

Theorem 2 may have a similar proof to that of theorem 1; the lower epidermic effect is to be used.

Let us suppose now that \( F_s \) have the form

\[ F_s(x, t, u_1, \ldots, u_m, 0, 0) = \sum \limits_{i=1}^{n} \alpha_s(x, t, u_1, \ldots, u_m) u_i + f_s(x, t, u_1, \ldots, u_m). \]

The quadratic forms \( \sum \limits_{i=1}^{n} \alpha_s \lambda_i \lambda_i \) are assumed to be non-negative. If \( F_s \)
are of that form, the system (1) is a system of quasi-linear equations of parabolic type. It is a system as follows:

\[ \frac{\partial \sigma_i}{\partial t} = \sum_{i=1}^{n} \alpha_s \left( \begin{array}{c} \sigma_i(t, x_1, \ldots, x_m) \frac{\partial \sigma_i}{\partial x_1} + \cdots + \frac{\partial \sigma_i}{\partial x_m} \right) \]

\[ + f_s(x, t, x_1, \ldots, x_m, \sigma_1, \ldots, \sigma_m). \]
To receive the limitation theorem for solutions of (6) we formulate the following conditions:

(7) the non-negative functions \(\eta_i(t, y_1, \ldots, y_m)\) \((s = 1, \ldots, m)\) are defined and continuous for \(s < t < b + s\) and \(y_i \geq 0\); the right maximal integral \(a_\sigma(t; h_1, \ldots, h_m)\) \((s = 1, \ldots, m)\) of the system

\[ y'_e = \eta_i(t, y_1, \ldots, y_m) \quad (s = 1, \ldots, m) \]

such that \(a_\sigma(t; h_1, \ldots, h_m) = a_\sigma\) exists in \(\langle a, b + s \rangle \quad (0 < s < c)\)

\(a_\sigma\) are arbitrary non-negative numbers;

(8) the functions \(f_i\) and \(\eta_i\) fulfill the inequalities

\[ |f_i(x, t; y_1, \ldots, y_m, 0, \ldots, 0)| \leq \eta_i(t, y_i, \ldots, y_m) \quad (s = 1, \ldots, m). \]

Theorem 3. Suppose that (7) and (8) are satisfied and let \(u_s(x, t)\) \((s = 1, \ldots, m)\) be a regular solution of the system (6) satisfying the inequalities

\[ |u_s(x, t)| \leq a_\sigma, \quad x \in \mathbb{Z}, \quad s = 1, \ldots, m. \]

\[ |u_s(x, t)| \leq a_\sigma, \quad (x, t) \in \mathbb{P} \times \langle a, b \rangle, \quad s = 1, \ldots, m. \]

Under these assumptions we have

\[ |u_s(x, t)| \leq a_\sigma(t; h_1, \ldots, h_m), \quad (x, t) \in \mathbb{Z}, \quad s = 1, \ldots, m. \]

Proof. Write

\[ U_s(t) = \max_{x \in \mathbb{Z}} |u_s(x, t)| \]

and suppose that for some \(s\), \(a_\sigma(t; h_1, \ldots, h_m) < U_s(t)\). But \(0 < a_\sigma(t; h_1, \ldots, h_m) \leq \eta_i(t; y_1, \ldots, y_m)\), whence \(0 < h_s < a_\sigma(t; h_1, \ldots, h_m)\). There is such an \(x\) that \(U_s(t) = \eta_i(x, t)\). Of course \(x \in \mathbb{P}\). On the other hand \(U_s(t) > 0\).

Then \(U_s(t) = -u_s(x, t)\) or \(U_s(t) = u_s(x, t)\). If \(U_s(t) = -u_s(x, t)\) then, in view of

\[ \frac{\partial U_s(t)}{\partial t} \leq \frac{\partial |u_s(x, t)|}{\partial t} \leq \frac{\partial (-u_s(x, t))}{\partial t}, \]

we have

\[ \frac{\partial U_s(t)}{\partial t} \leq \sum_{i=1}^{n} \frac{\partial (-u_s)}{\partial x_i} = -u_s \left( x, t; y_1, \ldots, y_m, \frac{\partial u_s}{\partial y_1}, \ldots, \frac{\partial u_s}{\partial y_m} \right) \in \mathbb{Z}. \]

In a sufficiently small neighbourhood of \(x\) we have

\[ -u_s(x, t) = |u_s(x, t)| \leq |u_s(x, t)| = -u_s(x, t). \]

Hence

\[ \frac{\partial (-u_s)}{\partial x_i} = 0 \quad (i = 1, \ldots, m). \]

The form

\[ \sum_{i=1}^{n} \frac{\partial (-u_s)}{\partial x_i} \eta_i \lambda_i \]

is non-positive. From (9) we obtain \(\bar{D}_- U_s(t) \leq -f_\sigma(x, t; u_s(x, t), \ldots, u_m(x, t), 0, \ldots, 0)\). From (8) the last inequality we obtain

\[ \bar{D}_- U_s(t) \leq \eta_i(t; u_s(x, t), \ldots, u_m(x, t)) \]

But \(u_s(x, t) \leq U_s(t)\) \((i \neq s)\) and \(U_s(t) = |u_s(x, t)|\). Therefore

\[ \bar{D}_- U_s(t) \leq \eta_i(t; U_s(t), \ldots, u_m(x, t)). \]

In a similar way one can show that if for some \(t\) the inequality

\[ a_\sigma(t; h_1, \ldots, h_m) < U_s(t)\]

holds and \(U_s(t) = u_s(x, t)\), then (10) is satisfied. By the epidermic theorem we obtain our theorem.

Let us assume now that for every \(x \in \mathbb{P}\) there is a normal \(n_x\), inner with respect to \(\mathbb{P}\). Write

\[ \frac{dx}{dn_x} = \lim_{s \to \infty} x(x) - e(z) |z - x| \rightarrow 0. \]

Theorem 4. Suppose that (2) and (3) are satisfied. Let \(u_s(x, t)\) \((s = 1, \ldots, m)\) be a regular solution of (1). The functions \(f_i(x, t)\) \((s = 1, \ldots, m)\) defined for \((x, t) \in \mathbb{P} \times \langle a, b \rangle\) are negative: \(e_i(x, t) < 0\) for \((x, t) \in \mathbb{P} \times \langle a, b \rangle\). We assume that \(u_s(x, a) = \eta_i\), \(x \in \mathbb{Z}, s = 1, \ldots, m\) and

\[ d\eta_i/dn_x = e_i(x, t)u_s(x, t) \geq e_i(x, t) e_i(x, t) \eta_i(t; h_1, \ldots, h_m), \quad (x, t) \in \mathbb{P} \times \langle a, b \rangle. \]

These assumptions imply that

\[ u_s(x, t) \leq \eta_i(t; h_1, \ldots, h_m), \quad (x, t) \in \mathbb{Z}, \quad s = 1, \ldots, m. \]

Proof. With the use of the same notation and arguments as in the proof of Theorem 1 it is sufficient to show that if for some \(t\) and \(x \in \mathbb{Z}\) we have \(a_\sigma(t; h_1, \ldots, h_m) < U_s(t) = u_s(x, t)\) then \(x \in \mathbb{P}\). If it were not so, then for \(x \neq n_x\) and \(z\) sufficiently near \(x\) we should have \(u_s(x, t) \leq u_s(z, t)\). Hence \(d\eta_i/dn_x < 0\). However \(a_\sigma(t; h_1, \ldots, h_m) < u_s(x, t)\) and \(e_i(x, t) < 0\) thus

\[ d\eta_i/dn_x = e_i(x, t)u_s(x, t) \leq e_i(x, t) e_i(x, t) < 0. \]

But this is impossible. Therefore \(x \in \mathbb{P}\).
In the same manner the following theorems may be proved:

**Theorem 5.** Assume that (4) and (5) are satisfied. Let \( u_s(x, t) \) \((s = 1, \ldots, m)\) form a regular solution of (1). Suppose that \( q_{s}(x, t) < 0 \) for \((x, t) \in \mathcal{G} \times (a, b)\) \((s = 1, \ldots, m)\). We assume that \( h_s \leq u_s(x, a), \quad x \in \mathcal{G}, \ s = 1, \ldots, m \) and

\[
\frac{d u_s}{d t} + q_{s}(x, t) u_s(x, t) \leq q_{s}(x, t) \omega_s(t; h_1, \ldots, h_m), \quad (x, t) \in \mathcal{G} \times (a, b).
\]

Under these assumptions the following inequalities hold:

\[
\omega_s(t; h_1, \ldots, h_m) \leq u_s(x, t), \quad (x, t) \in \mathcal{G}, \ s = 1, \ldots, m.
\]

**Theorem 6.** Suppose that (7) and (8) are satisfied. Let \( u_s(x, t) \) \((s = 1, \ldots, m)\) be a regular solution of the system (6). We assume that \( |u_s(x, a)| \leq h_s, \quad x \in \mathcal{G}, \ s = 1, \ldots, m \) and

\[
|d u_s / d t - q_{s}(x, t) u_s(x, t)| \leq e_s h_s, \quad (x, t) \in \mathcal{G} \times (a, b), \quad s = 1, \ldots, m
\]

where \( q_{s}(x, t) \geq e_s = \text{const} > 0 \). Then

\[
|u_s(x, t)| \leq \omega_s(t; h_1, \ldots, h_m) \quad \text{in} \quad \mathcal{G}, \quad s = 1, \ldots, m.
\]

It may easily be observed that theorems 4, 5 and 6 remain true if instead of the normal derivative \( d u_s / d t \) one takes any derivative along an arbitrary direction, inner with regard to \( \mathcal{G} \).

**Example.** Under the assumptions of theorem 3 the following assertion may be proved in a simple way: If \( \omega_s(t) \equiv 0 \) is a (L) stable solution (see [3]) of the system \( y_s' = u_s(t; y_1, \ldots, y_m) \) then the solution \( u_s = 0 \) of (6) is (L) stable in the wider sense.

§ 2. Consider a system of parabolic equations (see [4] and [3])

\[
\begin{align*}
\frac{\partial y_s}{\partial t} &= \mathcal{F}_s(x, t, y_1, \ldots, y_m, \frac{\partial y_s}{\partial x_1}, \ldots, \frac{\partial y_s}{\partial x_n}) \quad (s = 1, \ldots, m) \tag{11}.
\end{align*}
\]

Suppose that for \( 0 < h < \delta \) the systems of ordinary differential equations

\[
\begin{align*}
y_s' &= \mathcal{G}_s(x, t, y_1, \ldots, y_m) \quad (s = 1, \ldots, m) \tag{12}
\end{align*}
\]

where \( \mathcal{G}_s(t, y_1, \ldots, y_m; h) \) satisfy condition (W) with respect to the variables \( y_1, \ldots, y_m \) and admit right maximal integrals valid in \((a, b + e)\) for any non-negative conditions.

**Theorem 7.** Assume that

\[
|\mathcal{F}_s(x, t + h, z_1, \ldots, z_m, p_1, q_1) - \mathcal{F}_s(x, t, z_1, \ldots, z_m, p_1, q_1)|
\leq \sigma_s(t, |\mathcal{Z}_1 - \mathcal{Z}|_h), \ldots, |\mathcal{Z}_m - \mathcal{Z}|; h \quad \text{for} \quad 0 < h < \delta, \ s = 1, \ldots, m.
\]

Suppose that \( u_s(x, t) \) \((s = 1, \ldots, m)\) is a regular solution of (11) such that for \( s = 1, 2, \ldots, m \) the following inequalities hold:

\[
|u_s(x, s + h) - u_s(x, s)| \leq \epsilon_s(h), \quad x \in \mathcal{G}, \quad 0 < h < \delta,
\]

\[
|u_s(x, t + h) - u_s(x, t)| \leq \epsilon_s(h), \quad x \in \mathcal{G}, \quad a \leq t \leq b - \delta, \quad 0 < h < \delta.
\]

Under our assumptions we have

\[
|u_s(x, t + h) - u_s(x, t)| \leq \epsilon_s(t; h), \quad s = 1, \ldots, m
\]

for \( 0 < h < \delta \), \( x \in \mathcal{G} \) and \( a \leq t \leq b - \delta \), where \( \omega_s(t; h) \) is the right maximal integral of (12) such that \( \omega_s(a; h) = \epsilon_s(h) \).

**Proof.** Write

\[
v_s(x, t) = u_s(x, t + h), \quad h \text{ is fixed}
\]

and

\[
G_s(x, t, z, p_1, q_1) = \mathcal{F}_s(x, t + h, z, p_1, q_1).
\]

We have

\[
\frac{\partial y_s}{\partial t} = G_s(x, t, y_1, \ldots, y_m, \frac{\partial y_s}{\partial z_1}, \ldots, \frac{\partial y_s}{\partial z_m}, \frac{\partial y_s}{\partial p_1}, \frac{\partial y_s}{\partial q_1}) \quad (s = 1, \ldots, m).
\]

This system is parabolic. On the other hand

\[
|G_s(x, t, z, p_1, q_1) - \mathcal{F}_s(x, t, z, p_1, q_1)|
\leq \sigma_s(t, |\mathcal{Z}_1 - \mathcal{Z}|_h), \ldots, |\mathcal{Z}_m - \mathcal{Z}|; h.
\]

According to the boundary and initial inequalities

\[
|u_s(x, a) - u_s(x, a)| \leq \epsilon_s(h), \quad x \in \mathcal{G},
\]

\[
|u_s(x, t) - u_s(x, t)| \leq \epsilon_s(h), \quad (x, t) \in \mathcal{G} \times (a, b - \delta)
\]

by means of the main theorem on the limitation of solutions of parabolic systems (see [4] theorem 2.1) one immediately obtains our theorem.

We will discuss the following example: Suppose we are given an equation

\[
\frac{\partial x}{\partial t} = \sum_{j=1}^{n} a_{n}(x) \frac{\partial^2 x}{\partial z_j \partial z_h} + f(x, t).
\]

The quadratic form \( \sum_{j=1}^{n} a_{n}(x) \lambda_j \lambda_h \) is non-negative. Suppose that

\[
|f(x, t + h) - f(x, t)| \leq M h^k, \quad M = \text{const}.
\]
Let $u(x, t)$ be a regular solution of (13) in $G \times (0, b)$ such that for $0 < \alpha < \beta$

$$u(x, 0) = 0, \quad |u(x, t+h) - u(x, t)| \leq KH'$$

and $|u(x, h)| \leq KH$ for $x \in \partial G$. From theorem 7, putting $\sigma = MN$, we obtain

$$|u(x, t+h) - u(x, t)| \leq KH't + KH$$

for $0 < h < \delta$ and $0 \leq t \leq b - \delta$.

The second example is the following: Given a system of parabolic type

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i} + f(x, t, x_1, \ldots, x_m) \quad (s = 1, \ldots, m),$$

let us assume that

$$|f_s(x, t+h, x_1, \ldots, x_m) - f_s(x, t, x_1, \ldots, x_m)| \leq M \sum_{i=1}^{m} |x_i - \tilde{x}_i| + KH'$$

$(s = 1, \ldots, m)$.

If $u_s(x, t)$ $(s = 1, \ldots, m)$ forms a regular solution of (14) and

$$|u_s(x, t+h) - u_s(x, t)| \leq KH'$$

$$|u_s(x, 0) - u_s(x, h)| \leq KH', \quad x \in \partial G$$

then

$$|u_s(x, t+h) - u_s(x, t)| \leq KE^{\frac{Mh}{Mm} + KH'}$$

for $x \in \partial G$, $0 \leq t \leq b - \delta < b - h$.

Analogous theorems concerning the limitation of the differences

$$|u_s(x_1, \ldots, x_r, \ldots, x_m, t) - u_s(x_1, \ldots, x_r, \ldots, x_m, 0)|$$

may be proved. Similar investigations may be conducted in the case of the remaining boundary problems; this is possible because in that case theorems analogous to the main theorem are true (see [4] theorem 2.3).

References
