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# Limitation of solutions of parabolic equations

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We discuss some estimations concerning the solutions of parabolic equations. The epidermic theorems are used. In the second paragraph are given certain limitations for the increase of solutions with respect to the independent variables. As a rule we investigate non-linear equations.

**§ 1.** Suppose that $G$ is an open and bounded set of points $x = (x_1, \ldots, x_n)$ of the $n$-dimensional space $E^n$. Denote by $Z$ the Cartesian product of $G$ and of the interval $\Delta = (a, b\rangle$: $Z = G \times (a, b\rangle$; the boundary of $G$ is denoted by $\mathrm{F}G$. Suppose that we are given the functions $F_s(x_1, \ldots, x_n, t, u_1, \ldots, u_m, p_1, \ldots, q_{nn})$ shortly written as $F_s(x, t, u, p_i, q_{ik})$ $(s = 1, 2, \ldots, m)$ and the functions $\sigma_s(t, u_1, \ldots, u_m)$ $(s = 1, 2, \ldots, m)$. $F_s$ is defined for $x \in G$, $t \in \Delta$ and arbitrary $u_1, \ldots, u_m, p_i, q_{ik}$; $\sigma_s$ is defined for $t \in \langle a, b+\varepsilon)$ and arbitrary $u_1, \ldots, u_m$.

We say that the sequence of functions $u_1(x, t), \ldots, u_m(x, t)$ defined in $\bar{Z}$ is a *regular solution* of the system of equations

$$(1)\qquad \frac{\partial z_s}{\partial t} = F_s\left(x, t, z_1, \ldots, z_m, \frac{\partial z_s}{\partial x_i}, \frac{\partial^2 z_s}{\partial x_i \partial x_k}\right) \qquad (s = 1, \ldots, m)$$

if $u_s$ are continuous in $\bar{Z}$, possess continuous derivatives $\partial u_s/\partial x_i, \partial^2 u_s/\partial x_i \partial x_k$ in the interior of $Z$ and for $(x, b)$, $x \in G$, and satisfy the system (1) for $(x, t) \in \mathrm{int} Z$.

Let us introduce the following conditions:

(2) if $\sum_{i,k=1}^{n} q_{ik}\lambda_i\lambda_k \leqslant 0$ for arbitrary $\lambda_1, \ldots, \lambda_n$, then

$$F_s(x, t, u_1, \ldots, u_m, 0, q_{ik}) \leqslant \sigma_s(t, u_1, \ldots, u_m) \qquad (s = 1, \ldots, m);$$

(3) the functions $\sigma_s$ are continuous and satisfy the following condition:

(W) if $\bar{u}_i \leqslant \bar{\bar{u}}_i$ $(i \neq s)$, $\bar{u}_s = \bar{\bar{u}}_s$ then $\sigma_s(t, \bar{u}_1, \ldots, \bar{u}_m) \leqslant \sigma_s(t, \bar{\bar{u}}_1, \ldots, u_m)$; we assume that the right maximal integral $\omega_s(t; h_1, \ldots, h_m)$ of the system of ordinary differential equations

$$y_s' = \sigma_s(t, y_1, \ldots, y_m) \qquad (s = 1, \ldots, m)$$

such that $\omega_s(a; h_1, \ldots, h_m) = h_s$ is defined for $a \leqslant t < b+\bar{\varepsilon}$ $(0 < \bar{\varepsilon} \leqslant \varepsilon)$.

Remark that if (1) is a parabolic system (see [2]) and if

$$F_s(x, t, u_1, \ldots, u_m, 0, 0) \leqslant \sigma_s(t, u_1, \ldots, u_m)$$

then (2) is satisfied.

THEOREM 1. *Suppose that* (2) *and* (3) *are satisfied. Let* $u_1(x, t), \ldots$ $\ldots, u_m(x, t)$ *be a regular solution of the system* (1). *Assume that*

(*) $\quad u_s(x, a) \leqslant \omega_s(a; h_1, \ldots, h_m) = h_s, \quad x \in G, \quad s = 1, \ldots, m,$

(**) $\quad u_s(x, t) \leqslant \omega_s(t; h_1, \ldots, h_m), \quad x \in \mathrm{F}G, \quad a < t \leqslant b, \quad s = 1, \ldots, m.$

*Then*

$$u_s(x, t) \leqslant \omega_s(t; h_1, \ldots, h_m) \quad \text{for} \quad (x, t) \in \bar{Z}, \quad s = 1, \ldots, m.$$

Proof. Let us put

$$U_s(t) = \max_{x \in \bar{G}} u_s(x, t).$$

Suppose that for some $t$ we have $\omega_s(t; h_1, \ldots, h_m) < U_s(t)$. There is such an $\bar{x} \in \bar{G}$ that $U_s(t) = u_s(\bar{x}, t)$. According to the inequality (**) we find that $\bar{x} \in G$. The function $\varphi_s(x) = u_s(x, t)$ ($t$ being fixed) reaches in $\bar{x}$ a maximum. Therefore

$$\left(\frac{\partial \varphi_s}{\partial x_k}\right)_{\bar{x}} = \left(\frac{\partial u_s}{\partial x_k}\right)_{(\bar{x},t)} = 0 \quad (k = 1, \ldots, n) \quad \text{and} \quad \sum_{i,k=1}^{n} \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)_{(\bar{x},t)} \lambda_i \lambda_k \leqslant 0$$

for arbitrary $\lambda_1, \ldots, \lambda_n$.

By (2) we get

$$\left(\frac{\partial u_s}{\partial t}\right)_{(\bar{x},t)} = F_s\left(\bar{x}, t, u_1(\bar{x}, t), \ldots, u_m(\bar{x}, t), 0, \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)_{(\bar{x},t)}\right) \leqslant \sigma_s\big(t, u_1(\bar{x}, t), \ldots, u_m(\bar{x}, t)\big). \tag{3.1}$$

But

$$\bar{D}_- U_s(t) \leqslant (\partial u_s/\partial t)_{(\bar{x},t)}. \tag{3.2}$$

On the other hand $u_i(\bar{x}, t) \leqslant U_i(t)$ $(i = 1, \ldots, s-1, s+1, \ldots, m)$ and $u_s(\bar{x}, t) = U_s(t)$. This implies, by condition (W) for $\sigma_s$, the inequality

$$\sigma_s\big(t, u_1(\bar{x}, t), \ldots, u_m(\bar{x}\, t)\big) \leqslant \sigma_s\big(t, U_1(t), \ldots, U_m(t)\big). \tag{3.3}$$

From (3.1), (3.2) and (3.3) we obtain

$$\bar{D}_- U_s(t) \leqslant \sigma_s\big(t, U_1(t), \ldots, U_m(t)\big)$$



for such $t$ that $\omega_s(t; h_1, \ldots, h_m) < U_s(t)$. By (*) we have $U_i(a) \leqslant h_i$ $(i = 1, \ldots, m)$. Hence $U_1(t), \ldots, U_m(t)$ satisfy the assumptions of the epidermic theorem for right maximal integrals. Therefore (see [1] and [4]) $u_s(x, t) \leqslant U_s(t) \leqslant \omega_s(t; h_1, \ldots, h_m)$.

We formulate an analogous theorem concerning the limitation from below. We introduce the following conditions:

(4) if $\sum_{i,k=1}^{n} q_{ik} \lambda_i \lambda_k \geqslant 0$ for arbitrary $\lambda_1, \ldots, \lambda_n$, then

$$F_s(x, t, u_1, \ldots, u_m, 0, q_{ik}) \geqslant \sigma_s(t, u_1, \ldots, u_m);$$

(5) the functions $\sigma_s$ are continuous and satisfy condition (W); the right minimal integral $\omega_s(t; h_1, \ldots, h_m)$ $(s = 1, \ldots, m)$ of the system

$$y_s' = \sigma_s(t, y_1, \ldots, y_m) \quad (s = 1, \ldots, m)$$

such that $\omega_s(a; h_1, \ldots, h_m) = h_s$ exists in the interval $\langle a, b+\bar{\varepsilon})$ $(0 < \bar{\varepsilon} \leqslant \varepsilon)$.

THEOREM 2. *Suppose that* (4) *and* (5) *are satisfied. Let* $u_1(x, t), \ldots$ $\ldots, u_m(x, t)$ *be a regular solution of the system* (1). *Assume that*

$$h_s = \omega_s(a; h_1, \ldots, h_m) \leqslant u_s(x, a), \quad x \in G, \quad s = 1, \ldots, m,$$

$$\omega_s(t; h_1, \ldots, h_m) \leqslant u_s(x, t) \quad \textit{for} \quad x \in \mathrm{F}G, \quad a \leqslant t \leqslant b.$$

*Then*

$$\omega_s(t; h_1, \ldots, h_m) \leqslant u_s(x, t) \quad \textit{for} \quad (x, t) \in \bar{Z}.$$

Theorem 2 may have a similar proof to that of theorem 1; the lower epidermic effect is to be used.

Let us suppose now that $F_s$ have the form

$$F_s(x, t, u, p_\nu, q_{\nu\mu}) = \sum_{i,k=1}^{n} a_{ik}^s(x, t, u, p_\nu) q_{ik} + f_s(x, t, u, p_\nu).$$

The quadratic forms $\sum_{i,k=1}^{n} a_{ik}^s \lambda_i \lambda_k$ are assumed to be non-negative. If $F_s$ are of that form, the system (1) is a system of quasi-linear equations of parabolic type. It is a system as follows:

$$\frac{\partial z_s}{\partial t} = \sum_{i,k=1}^{n} a_{ik}^s\left(x, t, z_1, \ldots, z_m, \frac{\partial z_s}{\partial x_1}, \ldots, \frac{\partial z_s}{\partial x_n}\right) \frac{\partial^2 z_s}{\partial x_i \partial x_k} + f_s\left(x, t, z_1, \ldots, z_m, \frac{\partial z_s}{\partial x_1} \ldots \frac{\partial z_s}{\partial x_n}\right). \tag{6}$$

To receive the limitation theorem for solutions of (6) we formulate the following conditions:

(7) the non-negative functions $\eta_s(t, y_1, \ldots, y_m)$ $(s = 1, \ldots, m)$ are defined and continuous for $a \leqslant t \leqslant b+\varepsilon$ and $y_i \geqslant 0$; the right maximal integral $\omega_s(t; h_1, \ldots, h_m)$ $(s = 1, \ldots, m)$ of the system

$$y_s' = \eta_s(t, y_1, \ldots, y_m) \qquad (s = 1, \ldots, m)$$

such that $\omega_s(a; h_1, \ldots, h_m) = h_s$ exists in $\langle a, b+\bar{\varepsilon})$ $(0 < \bar{\varepsilon} \leqslant \varepsilon)$ ($h_i$ are arbitrary non-negative numbers);

(8) the functions $f_s$ and $\eta_s$ fulfil the inequalities

$$|f_s(x, t, y_1, \ldots, y_m, 0, \ldots, 0)| \leqslant \eta_s(t, |y_1|, \ldots, |y_m|) \qquad (s = 1, \ldots, m).$$

THEOREM 3. *Suppose that* (7) *and* (8) *are satisfied and let* $u_s(x, t)$ $(s = 1, \ldots, m)$ *be a regular solution of the system* (6) *satisfying the inequalities*

$$|u_s(x, a)| \leqslant h_s, \quad x \in \overline{G}, \quad s = 1, \ldots, m,$$

$$|u_s(x, t)| \leqslant h_s, \quad (x, t) \in \mathrm{F}G \times \langle a, b\rangle, \quad s = 1, \ldots, m.$$

*Under these assumptions we have*

$$|u_s(x, t)| \leqslant \omega_s(t; h_1, \ldots, h_m), \quad (x, t) \in \overline{Z}, \quad s = 1, \ldots, m.$$

Proof. Write

$$U_s(t) = \max_{x \in \overline{G}} |u_s(x, t)|$$

and suppose that for some $t$, $\omega_s(t; h_1, \ldots, h_m) < U_s(t)$. But $0 \leqslant \eta_s(t; y_1, \ldots, y_m)$, whence $0 \leqslant h_s \leqslant \omega_s(t; h_1, \ldots, h_m)$. There is such an $\bar{x}$ that $U_s(t) = |u_s(\bar{x}, t)|$. Of course $\bar{x} \notin \mathrm{F}G$. On the other hand $U_s(t) > 0$. Then $U_s(t) = -u_s(\bar{x}, t)$ or $U_s(t) = u_s(\bar{x}, t)$. If $U_s(t) = -u_s(\bar{x}, t)$ then, in view of

$$\overline{D}_- U_s(t) \leqslant \partial |u_s(\bar{x}, t)| / \partial t = \partial\big(-u_s(\bar{x}, t)\big)/\partial t,$$

we have

$$(9) \quad \overline{D}_- U_s(t) \leqslant \frac{\partial\big(-u_s(\bar{x}, t)\big)}{\partial t}$$

$$= \sum_{i,k=1}^{n} a_{ik}^s \frac{\partial^2(-u_s)}{\partial x_i \partial x_k} - f_s\Big(\bar{x}, t, u_1, \ldots, u_m, \frac{\partial u_s}{\partial x_1}, \ldots, \frac{\partial u_s}{\partial x_n}\Big).$$

In a sufficiently small neighbourhood of $\bar{x}$ we have

$$-u_s(x, t) = |u_s(x, t)| \leqslant |u_s(\bar{x}, t)| = -u_s(\bar{x}, t).$$



Hence

$$\big(\partial(-u_s)/\partial x_i\big)_{(\bar{x},t)} = (\partial u_s / \partial x_i)_{(\bar{x},t)} = 0 \qquad (i = 1, \ldots, n).$$

The form

$$\sum_{i,k=1}^{n} \big(\partial^2(-u_s)/\partial x_i \partial x_k\big)_{(\bar{x},t)} \lambda_i \lambda_k$$

is non-positive. From (9) we obtain $\overline{D}_- U_s(t) \leqslant -f_s\big(\bar{x}, t, u_1(\bar{x}, t), \ldots, u_m(\bar{x}, t), 0, \ldots, 0\big)$. According to (8) from the last inequality we obtain

$$\overline{D}_- U_s(t) \leqslant \eta_s\big(t, |u_1(\bar{x}, t)|, \ldots, |u_m(\bar{x}, t)|\big).$$

But $|u_i(\bar{x}, t)| \leqslant U_i(t)$ $(i \neq s)$ and $U_s(t) = |u_s(\bar{x}, t)|$. Therefore

$$(10) \qquad \overline{D}_- U_s(t) \leqslant \eta_s\big(t, U_1(t), \ldots, U_m(t)\big).$$

In a similar way one can show that if for some $t$ the inequality $\omega_s(t; h_1, \ldots, h_m) < U_s(t)$ holds and $U_s(t) = u_s(\bar{x}, t)$, then (10) is satisfied. By the epidermic theorem we obtain our theorem.

Let us assume now that for every $x \in \mathrm{F}G$ there is a normal $n_x$, inner with respect to $G$. Write

$$\frac{dz}{dn_x} = \lim_{\substack{x \to \bar{x} \\ x \in n_x}} \frac{z(x) - z(\bar{x})}{|x - \bar{x}|}.$$

THEOREM 4. *Suppose that* (2) *and* (3) *are satisfied. Let* $u_s(x, t)$ $(s = 1, \ldots, m)$ *be a regular solution of* (1). *The functions* $\varphi_s(x, t)$ $(s = 1, \ldots, m)$ *defined for* $(x, t) \in \mathrm{F}G \times (a, b\rangle$ *are negative:* $\varphi_s(x, t) < 0$ *for* $(x, t) \in \mathrm{F}G \times (a, b\rangle$. *We assume that* $u_s(x, a) \leqslant h_s$, $x \in \overline{G}$, $s = 1, \ldots, m$ *and*

$$du_s/dn_x + \varphi_s(x, t) u_s(x, t) \geqslant \varphi_s(x, t)\, \omega_s(t; h_1, \ldots, h_m), \quad (x, t) \in \mathrm{F}G \times (a, b\rangle.$$

*These assumptions imply that*

$$u_s(x, t) \leqslant \omega_s(t; h_1, \ldots, h_m), \quad (x, t) \in \overline{Z}, \quad s = 1, \ldots, m.$$

Proof. With the use of the same notation and arguments as in the proof of theorem 1 it is sufficient to show that if for some $t$ and $\bar{x} \in \overline{G}$ we have $\omega_s(t; h_1, \ldots, h_m) < U_s(t) = u_s(\bar{x}, t)$ then $\bar{x} \notin \mathrm{F}G$. If it were not so, then for $x \in n_{\bar{x}}$ and $x$ sufficiently near $\bar{x}$ we should have $u_s(x, t) \leqslant u_s(\bar{x}, t)$. Hence $du_s/dn_{\bar{x}} \leqslant 0$. However $\omega_s(t; h_1, \ldots, h_m) < u_s(\bar{x}, t)$ and $\varphi_s(\bar{x}, t) < 0$ thus

$$du_s/dn_{\bar{x}} + \varphi_s(\bar{x}, t) u_s(\bar{x}, t) < \varphi_s(\bar{x}, t)\, \omega_s(t; h_1, \ldots, h_m).$$

But this is impossible. Therefore $\bar{x} \notin \mathrm{F}G$.

In the same manner the following theorems may be proved:

THEOREM 5. *Assume that* (4) *and* (5) *are satisfied. Let* $u_s(x,t)$ $(s=1,\ldots,m)$ *form a regular solution of* (1). *Suppose that* $\varphi_s(x,t)<0$ *for* $(x,t)\in \mathrm{F}G\times(a,b\rangle$ $(s=1,\ldots,m)$. *We assume that* $h_s\leqslant u_s(x,a)$, $x\in\overline{G}$, $s=1,\ldots,m$ *and*

$$du_s/dn_{\bar{x}}+\varphi_s(x,t)u_s(x,t)\leqslant\varphi_s(x,t)\omega_s(t;h_1,\ldots,h_m),\ (x,t)\in \mathrm{F}G\times(a,b\rangle.$$

*Under these assumptions the following inequalities hold:*

$$\omega_s(t;h_1,\ldots,h_m)\leqslant u_s(x,t),\qquad (x,t)\in\overline{Z},\qquad s=1,\ldots,m.$$

THEOREM 6. *Suppose that* (7) *and* (8) *are satisfied. Let* $u_s(x,t)$ $(s=1,\ldots,m)$ *be a regular solution of the system* (6). *We assume that* $|u_s(x,a)|\leqslant h_s$, $x\in\overline{G}$, $s=1,\ldots,m$ *and*

$$|du_s/dn_x-\varphi_s(x,t)u_s(x,t)|\leqslant\varepsilon_s h_s,\qquad (x,t)\in \mathrm{F}G\times(a,b\rangle,\qquad s=1,\ldots,m$$

*where* $\varphi_s(x,t)\geqslant\varepsilon_s=\mathrm{const}>0$. *Then*

$$|u_s(x,t)|\leqslant\omega_s(t;h_1,\ldots,h_m)\quad in\quad \overline{Z},\qquad s=1,\ldots,m.$$

It may easily be observed that theorems 4, 5 and 6 remain true if instead of the normal derivative $du_s/dn_x$ one takes any derivative along an arbitrary direction, inner with regard to $G$.

EXAMPLE. Under the assumptions of theorem 3 the following assertion may be proved in a simple way: if $\omega_s(t)\equiv 0$ is a (L) stable solution (see [3]) of the system $y_s'=\eta_s(t,y_1,\ldots,y_m)$ then the solution $u_s\equiv 0$ of (6) is (L) stable in the wider sense.

**§ 2.** Consider a system of parabolic equations (see [4] and [3])

$$\frac{\partial z_s}{\partial t}=F_s\left(x,t,z_1,\ldots,z_m,\frac{\partial z_s}{\partial x_i},\frac{\partial^2 z_s}{\partial x_i\partial x_k}\right)\qquad(s=1,\ldots,m). \tag{11}$$

Suppose that for $0<h<\delta$ the systems of ordinary differential equations

$$y_s'=\sigma_s(t,y_1,\ldots,y_m;h) \tag{12}$$

where $\sigma_s(t,u_1,\ldots,u_m;h)$ satisfy condition (W) with respect to the variables $u_1,\ldots,u_m$ and admit right maximal integrals valid in $\langle a,b+\varepsilon\rangle$ for any non-negative initial conditions.

THEOREM 7. *Assume that*

$$|F_s(x,t+h,\bar{z}_1,\ldots,\bar{z}_m,p_i,q_{ik})-F_s(x,t,\bar{\bar{z}}_1,\ldots,\bar{\bar{z}}_m,p_i,q_{ik})|$$
$$\leqslant\sigma_s(t,|\bar{z}_1-\bar{\bar{z}}_1|,\ldots,|\bar{z}_m-\bar{\bar{z}}_m|;h)\quad for\quad 0<h<\delta,\qquad s=1,\ldots,m.$$



*Suppose that* $u_s(x,t)$ $(s=1,\ldots,m)$ *is a regular solution of* (11) *such that for* $s=1,2,\ldots,m$ *the following inequalities hold:*

$$|u_s(x,a+h)-u_s(x,a)|\leqslant\varepsilon_s(h),\qquad x\in\overline{G},\qquad 0<h<\delta,$$
$$|u_s(x,t+h)-u_s(x,t)|\leqslant\varepsilon_s(h),\qquad x\in \mathrm{F}G,\qquad a\leqslant t\leqslant b-\delta,\qquad 0<h<\delta.$$

*Under our assumptions we have*

$$|u_s(x,t+h)-u_s(x,t)|\leqslant\omega_s(t;h),\qquad s=1,\ldots,m$$

*for* $0<h<\delta$, $x\in\overline{G}$ *and* $a\leqslant t\leqslant b-\delta$ *where* $\omega_s(t;h)$ *is the right maximal integral of* (12) *such that* $\omega_s(a;h)=\varepsilon_s(h)$.

**Proof.** Write

$$v_s(x,t)=u_s(x,t+h),\qquad h \text{ is fixed}$$

and

$$G_s(x,t,z,p_i,q_{ik})=F_s(x,t+h,z,p_i,q_{ik}).$$

We have

$$\frac{\partial v_s}{\partial t}=G_s\left(x,t,v_1,\ldots,v_m,\frac{\partial v_s}{\partial x_i},\frac{\partial^2 v_s}{\partial x_i\partial x_k}\right)\qquad(s=1,\ldots,m).$$

This system is parabolic. On the other hand

$$|G_s(x,t,\bar{z}_1,\ldots,\bar{z}_m,p_i,q_{ik})-F_s(x,t,\bar{\bar{z}}_1,\ldots,\bar{\bar{z}}_m,p_i,q_{ik})|$$
$$\leqslant\sigma_s(t,|\bar{z}_1-\bar{\bar{z}}_1|,\ldots,|\bar{z}_m-\bar{\bar{z}}_m|;h).$$

According to the boundary and initial inequalities

$$|u_s(x,a)-v_s(x,a)|\leqslant\varepsilon_s(h),\qquad x\in\overline{G},$$
$$|u_s(x,t)-v_s(x,t)|\leqslant\varepsilon_s(h),\qquad (x,t)\in \mathrm{F}G\times(a,b-\delta\rangle$$

by means of the main theorem on the limitation of solutions of parabolic systems (see [4] theorem 2.1) one immediately obtains our theorem.

We will discuss the following example: Suppose we are given an equation

$$\frac{\partial z}{\partial t}=\sum_{i,k=1}^{n}a_{ik}(x)\frac{\partial^2 z}{\partial x_i\partial x_k}+f(x,t). \tag{13}$$

The quadratic form $\sum_{i,k=1}^{n}a_{ik}(x)\lambda_i\lambda_k$ is non-negative. Suppose that

$$|f(x,t+h)-f(x,t)|\leqslant Mh^{\alpha},\qquad M=\mathrm{const}.$$

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Let $u(x,t)$ be a regular solution of (13) in $G\times(0,b\rangle$ such that for $0<h<\delta$

$$u(x,0)=0,\quad |u(x,t+h)-u(x,t)|\leqslant Kh^{\beta},\quad (x,t)\in \mathrm{F}G\times(0,b-\delta\rangle$$

and $|u(x,h)|\leqslant Kh^{\beta}$ for $x\in\overline{G}$. From theorem 7, putting $\sigma_1=Mh^{\alpha}$, we obtain

$$|u(x,t+h)-u(x,t)|\leqslant Mh^{\alpha}t+Kh^{\beta}$$

for $0<h<\delta$ and $0\leqslant t\leqslant b-\delta$.

The second example is the following: Given a system of parabolic type

$$(14)\qquad \frac{\partial z_s}{\partial t}=\sum_{i,k=1}^{n}a_{ik}^{s}(x)\frac{\partial^2 z_s}{\partial x_i\partial x_k}+f_s(x,t,z_1,\dots,z_m)\qquad (s=1,\dots,m),$$

let us assume that

$$|f_s(x,t+h,\bar z_1,\dots,\bar z_m)-f_s(x,t,\bar{\bar z}_1,\dots,\bar{\bar z}_m)|\leqslant M\sum_{i=1}^{m}|\bar z_i-\bar{\bar z}_i|+Rh^{\gamma}\qquad (s=1,\dots,m).$$

If $u_s(x,t)$ $(s=1,\dots,m)$ forms a regular solution of (14) and

$$|u_s(x,t+h)-u_s(x,t)|\leqslant Kh^{\beta},\quad (x,t)\in\mathrm{F}G\times(0,b-\delta\rangle,$$

$$|u_s(x,0)-u_s(x,h)|\leqslant Kh^{\beta},\quad x\in\overline{G}$$

then

$$|u_s(x,t+h)-u_s(x,t)|\leqslant Ke^{Mmt}h^{\beta}+\frac{Rh^{\gamma}}{Mm}(e^{Mmt}-1)$$

for $x\in\overline{G}$, $0\leqslant t\leqslant b-\delta<b-h$.

Analogous theorems concerning the limitation of the differences $|u_s(x_1,\dots,x_r+h,\dots,x_n,t)-u_s(x_1,\dots,x_r,\dots,x_n,t)|$ may be proved. Similar investigations may be conducted in the case of the remaining boundary problems; this is possible because in that case theorems analogous to the main theorem are true (see [4] theorem 2.3).

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