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On the summability almost everywhere of orthonormal series by the method of Euler-Knopp

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1. In the present paper we shall consider the Euler-Knopp summability-method of orthonormal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{1}$$

which are expansions of functions with an integrable square in $\langle 0, 1\rangle$, *i. e.* such, that

$$\sum_{n=0}^{\infty} a_n^2 < \infty. \tag{2}$$

We are especially interested in investigating the relation between the summability-method of Euler-Knopp and that of Cesàro.

I wish to express my deep gratitude to Professor W. Orlicz for his assistance and valuable suggestions in certain parts of this paper(*).

The *q-th Euler-Knopp means* of the sequence $\{s_n\}$ are defined as follows:

$$\tau_n^{(q)} = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \quad \text{for} \quad n = 0, 1, \ldots; \quad q > 0.$$

In place of $\tau_n^{(1)}$ we write τ_n. The series $\sum_0^\infty a_n$ with the n-th partial sum s_n is called *summable by the q-th Euler-Knopp method* (E, q) *to* s or, more concisely, (E, q)*-summable to* s, if $\lim_{n\to\infty} \tau_n^{(q)} = s$. It is known that the method of Euler-Knopp and that of Euler are equivalent. Therefore we shall write „the method of Euler-Knopp" for both methods.

(*) This paper originated in 1952 in connection with problems considered at the mathematical seminar directed by Professor W. Orlicz.

We shall denote by $\tau_n^{(q)}(x)$ the q-th Euler-Knopp mean of the series (1), writing $\tau_n^{(1)}(x) = \tau_n(x)$. Moreover, we shall write

$$s_n(x) = \sum_{k=0}^{n} a_k \varphi_k(x).$$

The methods (E, q) of Euler-Knopp and (C, r) of Cesàro are not equivalent (see [10], p. 340). There exist numerical series summable by one of those methods and not summable by the other. However, as we shall prove in this paper, every orthonormal series (1) summable almost everywhere by the method (E, q) of Euler-Knopp is also, for arbitrary $q > 0$ and $r > 0$, summable almost everywhere by the method (C, r) of Cesàro.

We have not determined whether there exists an orthonormal series summable by the method (C, r) in $\langle 0, 1\rangle$ and not summable by the method (E, q) for any set of positive measure. However, one may suppose that such series exist, because certain theorems holding for the method of Cesàro are not true for the method of Euler-Knopp.

To simplify further formulations we now introduce the following notation: If the sequence $|f_n(x)/g_n(x)|$ is bounded or convergent to zero for $n \to \infty$ almost everywhere in $\langle 0, 1\rangle$, then we shall write

$$f_n(x) \doteq O(g_n(x)) \quad \text{or} \quad f_n(x) \doteq o(g_n(x)),$$

respectively.

2. THEOREM 1. *The series*

$$\sum_{n=1}^{\infty} \int_0^1 n[\tau_n(x) - \tau_{n-1}(x)]^2 dx \tag{3}$$

is convergent if and only if $\sum_{n=1}^{\infty} a_n^2 \sqrt{n} < \infty$.

Proof. Applying the formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ we can write

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{2^n}\left\{s_n(x) + \sum_{k=0}^{n-1}\left[\binom{n}{k} - \binom{n-1}{k}\cdot 2\right] s_k(x)\right\}$$

$$= \frac{1}{2^n}\left\{s_n(x) + \sum_{k=0}^{n-1}\left[\binom{n-1}{k-1} - \binom{n-1}{k}\right] s_k(x)\right\}.$$

Writing $c_{ni} = \binom{n-1}{i-1} - \binom{n-1}{i}$, we have $\sum_{i=k}^{n} c_{ni} = \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$ and

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{2^n}\sum_{k=0}^{n} c_{nk} \sum_{i=0}^{k} a_i \varphi_i(x) = \frac{1}{2^n}\sum_{k=0}^{n} a_k \varphi_k(x) \sum_{i=k}^{n} c_{ni}.$$

Hence

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{n2^n}\sum_{k=0}^{n}\binom{n}{k} k a_k \varphi_k(x). \tag{4}$$

This and the orthonormality of the system $\{\varphi_n(x)\}$ imply

$$\int_0^1 n[\tau_n(x) - \tau_{n-1}(x)]^2 dx = \frac{1}{n4^n}\sum_{k=0}^{n}\binom{n}{k}^2 k^2 a_k^2. \tag{5}$$

Let $[x]$ be the integral part of the number x. We put $[n/2] = m$. Then (5) implies

$$\int_0^1 n[\tau_n(x) - \tau_{n-1}(x)]^2 dx < \frac{1}{4^n}\sum_{k=0}^{n}\binom{n}{k}^2 k a_k^2 < \frac{\sqrt{n}\binom{n}{m}}{2^n}\cdot\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sqrt{k}\, a_k^2.$$

If we apply Stirling's formula or, what is simpler, Knopp's inequality

$$1 < \frac{n!}{n^n e^{-n}\sqrt{n}} < 10 \quad \text{for} \quad n = 1, 2, \ldots \tag{6}$$

(see [6], p. 136) we obtain

$$\frac{\sqrt{n}\binom{n}{m}}{2^n} < 20e.$$

Thus we have

$$\int_0^1 n[\tau_n(x) - \tau_{n-1}(x)]^2 dx < 20e\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sqrt{k}\, a_k^2.$$

However, as is known for numerical series, the convergence of the series $\sum_0^{\infty} a_n$ implies the convergence of the series

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k} a_k$$

to the same sum (see [7], p. 7). Therefore, if we assume the convergence of the series $\sum_1^{\infty} a_k^2\sqrt{k}$, then

$$\sum_{n=1}^{\infty}\int_0^1 n[\tau_n(x) - \tau_{n-1}(x)]^2 dx < 40e\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\sqrt{k}\, a_k^2 = 40e\sum_{k=1}^{\infty} a_k^2\sqrt{k} < \infty.$$

Thus we have proved that the condition $\sum_1^{\infty} a_n^2\sqrt{n} < \infty$ in theorem 1 is sufficient.

Now we shall prove this condition to be necessary. Suppose that the series $\sum_1^\infty a_n^2\sqrt{n}$ diverges; then we shall prove that the series (3) also diverges.

According to (5)

$$\sum_{n=1}^{p}\int_0^1 n[\tau_n(x)-\tau_{n-1}(x)]^2dx=\sum_{n=1}^{p}\frac{1}{n4^n}\sum_{k=1}^{n}\binom{n}{k}^2k^2a_k^2=\sum_{k=1}^{p}k^2a_k^2\sum_{n=k}^{p}\binom{n}{k}^2\frac{1}{n4^n}$$

$$=\sum_{k=1}^{p}k^2a_k^2\sum_{r=0}^{p-k}\frac{\binom{k+r}{k}^2}{(k+r)4^{k+r}}\geqslant\sum_{k=4}^{[p/2]}k^2a_k^2\sum_{r=k-[\sqrt{k}]}^{k}\frac{\binom{k+r}{k}^2}{(k+r)4^{k+r}}.$$

Let $l=[\sqrt{k}]$. Applying inequalities (6) for $k\geqslant 4$ we shall now estimate the expression

$$\frac{1}{4^{2k-l}}\binom{2k-l}{k}^2=\frac{[(2k-l)!]^2}{(k!)^2[(k-l)!]^2 4^{2k-l}}.$$

Since $k\geqslant 4$ and $l^2\leqslant k$,

$$\frac{\binom{2k-l}{k}^2}{4^{2k-l}}=\frac{(2k-l)^{4k-2l}(2k-l)}{100^2 4^{2k-l}k^{2k}k(k-1)^{2k-2l}(k-l)}$$

$$=\frac{1}{100^2}\left(\frac{4k^2-4kl+l^2}{4k^2-4kl}\right)^{2k}\left(1-\frac{l}{2k-l}\right)^{2l}\frac{2k-l}{k-l}\cdot\frac{1}{k}$$

$$\geqslant\frac{1}{100^2}\left(1-\frac{l}{2k-l}\right)^{2l}\frac{2k-l}{k-l}\cdot\frac{1}{k}\geqslant\frac{1}{100^2}\left(1-\frac{1}{2l-1}\right)^{2l-1}\frac{2(l-1)}{2l-1}\cdot\frac{2k-l}{k-l}\cdot\frac{1}{k}.$$

However, for $k\geqslant 4$,

$$\frac{2(l-1)(2k-l)}{(2l-1)(k-l)}>1,$$

whence

$$\frac{\binom{2k-l}{k}^2}{4^{2k-l}}>\frac{1}{100^2}\left(1-\frac{1}{2l-1}\right)^{2l-1}\frac{1}{k}.$$

Since for $l\geqslant 2$ we have

$$\left(1-\frac{1}{2l-1}\right)^{-(2l-1)}=\left(1+\frac{1}{2l-2}\right)^{2l-2}\left(1+\frac{1}{2l-2}\right),$$

the following inequality

$$\left(1+\frac{1}{2l-2}\right)^{2l-2}<e$$

implies

$$\left(1-\frac{1}{2l-1}\right)^{2l-1}=\frac{1}{(1+1/(2l-2))^{2l-2}(1+1/(2l-2))}>\frac{1}{2e}.$$

Hence, for $4\leqslant l^2\leqslant k$,

$$\frac{\binom{2k-l}{k}^2}{4^{2k-l}}>\frac{1}{100^2\cdot 2e}\cdot\frac{1}{k}=\frac{C_1}{k},\quad\text{where}\quad C_1=\frac{1}{100^2\cdot 2e}.$$

Thus we have

$$\sum_{n=1}^{p}\int_0^1 n[\tau_n(x)-\tau_{n-1}(x)]^2dx>C_1\sum_{k=4}^{[p/2]}k^2a_k^2\frac{[\sqrt{k}]}{2k}\cdot\frac{1}{k}=C_1\cdot\frac{1}{2}\sum_{k=4}^{[p/2]}[\sqrt{k}]a_k^2$$

$$\geqslant\frac{C_1}{2}\sum_{k=4}^{[p/2]}(\sqrt{k}-1)a_k^2>\frac{C_1}{4}\sum_{k=4}^{[p/2]}a_k^2\sqrt{k},$$

implying the divergence of the series (3), as the series $\sum_{k=1}^{\infty}a_k^2\sqrt{k}$ is divergent.

3. Let us denote by $\sigma_n(x)$ the first Cesàro means of the sequence of the n-th partial sums of the series (1). We shall prove two lemmas.

Lemma 1. *If the orthonormal series* (1) *satisfies condition* (2), *then the series*

$$\sum_{n=1}^{\infty}\frac{1}{n}[\sigma_n(x)-\tau_n(x)]^2$$

converges almost everywhere.

Proof. The following equality holds:

$$\sigma_n(x)-\tau_n(x)=\sum_{k=0}^{n}a_k\varphi_k(x)\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}-\frac{k}{n+1}\right],$$

whence

$$\int_0^1[\sigma_n(x)-\tau_n(x)]^2dx=\sum_{k=1}^{n}a_k^2\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}-\frac{2k}{n+1}\right]+\frac{k^2}{(n+1)^2}\right\}$$

$$<\sum_{k=1}^{[n/3]+1}a_k^2\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2+\sum_{k=[n/3]+2}^{n}a_k^2\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}-\frac{2k}{n+1}\right]\right\}+$$

$$+\sum_{k=1}^{n}\frac{k^2a_k^2}{(n+1)^2}.$$

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We shall prove that the expression in square brackets in the second sum on the right-hand side of this inequality is negative for all k within the summation limits, *i. e.* writing

$$W_{nk} = \frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i} - \frac{2k}{n+1},$$

we have $W_{nk} < 0$ for $[n/3]+2 \leqslant k \leqslant n$. Let us consider two cases:

(a) $[n/3]+2 \leqslant k \leqslant [n/2]$,

(b) $[n/2]+1 \leqslant k \leqslant n$.

Since $[n/3] \geqslant (n-2)/3$, we may write in case (a),

$$W_{nk} < \frac{1}{2^n}\sum_{i=0}^{[n/2]-1}\binom{n}{i} - \frac{2\big((n-2)/3+2\big)}{n+1} \leqslant \frac{1}{2} - \frac{2n+8}{3(n+1)} < 0$$

and since $[n/2] \geqslant (n-1)/2$, in case (b) we have

$$W_{nk} < \frac{1}{2^n}\sum_{i=0}^{n-1}\binom{n}{i} - \frac{2\big((n-1)/2+1\big)}{n+1} < 1 - \frac{n-1+2}{n+1} = 0.$$

Thus, for $[n/3]+2 \leqslant k \leqslant n$ the inequality $W_{nk} < 0$ holds. Hence

$$\sum_{k=[n/3]+2}^{n} a_k^2\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i} - \frac{2k}{n+1}\right]\right\} < 0$$

and we have

$$\int_0^1 [\sigma_n(x)-\tau_n(x)]^2\,dx < \sum_{k=1}^{[n/3]+1} a_k^2\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 + \sum_{k=1}^{n}\frac{k^2 a_k^2}{(n+1)^2}. \tag{7}$$

Now we shall estimate the expression

$$\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 \quad \text{for} \quad 1 \leqslant k \leqslant \left[\frac{n}{3}\right]+1.$$

First, we shall prove that

$$\frac{1}{2^n}\binom{n}{[n/3]} < 10\sqrt{\frac{3}{2}}\left(\frac{27}{32}\right)^{[n/3]} \quad \text{for} \quad n = 1, 2, \ldots \tag{8}$$

Remark. If we assume $\frac{27}{28}$ in inequality (8) instead of $\frac{27}{32}$, then it can be proved that

$$\frac{1}{2^n}\binom{n}{[n/3]} < \left(\frac{27}{28}\right)^{[n/3]} \quad \text{for} \quad n = 1, 2, \ldots$$

When $k = [n/3]$, then the number k equals one of the three numbers $n/3$, $(n-1)/3$, $(n-2)/3$ and we have

$$\frac{\binom{3k}{k}}{2^{3k}} > \frac{\binom{3k+1}{k}}{2^{3k+1}} > \frac{\binom{3k+2}{k}}{2^{3k+2}}.$$

Therefore if we prove that

$$\frac{\binom{3k}{k}}{2^{3k}} < 10\sqrt{\frac{3}{2}}\left(\frac{27}{32}\right)^k \quad \text{for} \quad k = 1, 2, \ldots$$

then inequality (8) will be proved, too. However, applying inequalities (6), we get

$$\frac{1}{2^{3k}}\binom{3k}{k} < \frac{1}{2^{3k}}\cdot\frac{(3k)!}{(2k)!\,k!}\cdot\frac{10(3k)^{3k}e^{-3k}\sqrt{3k}}{2^{3k}(2k)^{2k}e^{-2k}\sqrt{2k}\,k^k e^{-k}\sqrt{k}}$$

$$= 10\sqrt{\frac{3}{2k}}\left(\frac{27}{32}\right)^k \leqslant 10\sqrt{\frac{3}{2}}\left(\frac{27}{32}\right)^k,$$

thus concluding the proof of inequality (8).

Returning to the estimation of the expression

$$\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 \quad \text{for} \quad 1 \leqslant k \leqslant \left[\frac{n}{3}\right]+1,$$

let us first note that the sequence $\left\{\frac{1}{2^n}\binom{n}{i}\right\}$ increases for $0 \leqslant i \leqslant [n/2]$. Thus for $1 \leqslant k \leqslant [n/3]+1$ we have

$$\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 \leqslant \frac{1}{4^n}\sum_{i=0}^{k-1}\binom{n}{i}^2\sum_{i=0}^{k-1}1 = \frac{k}{4^n}\sum_{i=0}^{k-1}\binom{n}{i}^2 < \frac{k^2}{4^n}\binom{n}{k-1}^2,$$

whence

$$\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 < \frac{k^2}{4^n}\binom{n}{[n/3]}^2.$$

Applying inequality (8), we obtain

$$\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 < 150k^2\left(\frac{27}{32}\right)^{2[n/3]} \leqslant C_2\frac{k^2}{n^2} \quad \text{for} \quad n = 1, 2, \ldots, \tag{9}$$

C_2 being a suitable constant.

Remark: Inequality (9) may also be obtained from an inequality referred to by G. G. Lorentz in [9](1), p. 180.

The inequalities (7) and (9) imply

$$(10)\qquad \int_0^1 [\sigma_n(x)-\tau_n(x)]^2\,dx < \frac{C_2+1}{n^2}\sum_{k=1}^{n} k^2 a_k^2.$$

Hence

$$\sum_{n=1}^{m}\int_0^1 \frac{1}{n}[\sigma_n(x)-\tau_n(x)]^2\,dx < (C_2+1)\sum_{n=1}^{m}\frac{1}{n^3}\sum_{k=1}^{n}k^2a_k^2 = (C_2+1)\sum_{k=1}^{m}k^2a_k^2\sum_{n=k}^{m}\frac{1}{n^3} = O\Big(\sum_{k=1}^{m}a_k^2\Big)$$

and consequently

$$\sum_{n=1}^{\infty}\int_0^1\frac{1}{n}[\sigma_n(x)-\tau_n(x)]^2\,dx<\infty.$$

Hence and from Levy's theorem follows lemma 1.

LEMMA 2. *If the orthonormal series* (1) *satisfies the assumption* (2) *and is summable almost everywhere by the* $(E,1)$*-method to* $s(x)$, *then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[s_k(x)-s(x)]^2 \doteq 0.$$

Proof. At first let us remark that the series

$$(11)\qquad \sum_{n=1}^{\infty}\frac{1}{n}[s_n(x)-\sigma_n(x)]^2$$

is convergent almost everywhere. We have

$$s_n(x)-\sigma_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}ka_k\varphi_k(x),$$

whence

$$\int_0^1[s_n(x)-\sigma_n(x)]^2\,dx=\frac{1}{(n+1)^2}\sum_{k=1}^{n}k^2a_k^2.$$

Hence

$$\sum_{n=1}^{m}\int_0^1\frac{1}{n}[s_n(x)-\sigma_n(x)]^2\,dx<\sum_{n=1}^{m}\frac{1}{n^3}\sum_{k=1}^{n}k^2a_k^2=\sum_{k=1}^{m}k^2a_k^2\sum_{n=k}^{m}\frac{1}{n^3}=O\Big(\sum_{k=1}^{m}a_k^2\Big),$$

which implies the convergence almost everywhere of the series (11).

Lemma 1 and Kronecker's theorem (see [8], p. 983) imply

$$(12)\qquad \frac{1}{n}\sum_{k=1}^{n}[\sigma_k(x)-\tau_k(x)]^2 \doteq 0 \quad\text{for}\quad n\to\infty.$$

Applying Kronecker's theorem to the series (11) we obtain

$$(13)\qquad \frac{1}{n}\sum_{k=1}^{n}[s_k(x)-\sigma_k(x)]^2 \doteq 0 \quad\text{for}\quad n\to\infty.$$

From Minkowski's inequality we have

$$(14)\qquad \Big\{\frac{1}{n}\sum_{k=1}^{n}[s_k(x)-s(x)]^2\Big\}^{1/2}\leqslant\Big\{\frac{1}{n}\sum_{k=1}^{n}[s_k(x)-\sigma_k(x)]^2\Big\}^{1/2}+\Big\{\frac{1}{n}\sum_{k=1}^{n}[\sigma_k(x)-\tau_k(x)]^2\Big\}^{1/2}+\Big\{\frac{1}{n}\sum_{k=1}^{n}[\tau_k(x)-s(x)]^2\Big\}^{1/2}.$$

According to the assumption, $\tau_n(x)\doteq s(x)$ and so the third expression on the right-hand side of the last inequality converges to zero almost everywhere. The first two expressions converge to zero almost everywhere according to (13) and (12). Thus, the right-hand side of inequality (14) converges to zero almost everywhere.

4. THEOREM 2. *If the orthonormal series* (1) *satisfies condition* (2) *and is summable almost everywhere by the* $(E,1)$*-method, then it is summable almost everywhere by the* $(C,1)$*-method to the same sum.*

Proof. Let $\tau_n(x)\doteq s(x)$. According to the inequality of Buniakowski-Schwarz we have

$$[\sigma_n(x)-s(x)]^2\leqslant\Big\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-s(x)|\Big\}^2\leqslant\frac{1}{n+1}\sum_{k=0}^{n}[s_k(x)-s(x)]^2.$$

Hence lemma 2 implies $\sigma_n(x)\doteq s(x)$.

(1) We put

$$p_\nu(t)=\binom{n}{\nu}t^\nu(1-t)^{n-\nu}\quad\text{for}\quad \nu=0,1,2,\dots,n.$$

Then the following inequality holds:

$$\sum_{0\leqslant\nu\leqslant n}p_\nu(t)<\frac{A}{n^2},\qquad\Big|\frac{\nu}{n}-t\Big|\geqslant n^{-1/3},$$

A being an absolute constant. This inequality yields for $t=\frac{1}{2}$ and $n\geqslant 6^3$,

$$p_\nu\Big(\frac{1}{2}\Big)=\frac{1}{2^n}\binom{n}{\nu}\quad\text{and}\quad\frac{1}{2^n}\sum_{\nu=0}^{[n/3]}\binom{n}{\nu}<\frac{A}{n^2},$$

since for $0\leqslant\nu\leqslant[n/3]$ we have

$$\Big|\frac{\nu}{n}-\frac{1}{2}\Big|=\frac{1}{2}-\frac{\nu}{n}\geqslant\frac{1}{6}\geqslant n^{-1/3}.$$

The last theorem may be generalized as follows.

THEOREM 3. *If the orthonormal series* (1) *satisfies condition* (2) *and is summable almost everywhere by the* (E, q)*-method for a certain* $q > 0$, *then it is summable almost everywhere by the* (C, r) *-method for every* $r > 0$.

In order to prove this theorem we apply two theorems given in the book by G. H. Hardy ([1], pp. 183, 210, theorems 128 and 147). These theorems concern Borel's method. We say that the series $\sum_{n=0}^{\infty} a_n$ with partial sums s_n is summable by Borel's method to s, if the following two conditions are satisfied:

(a) the power series $\sum_{n=0}^{\infty} s_n \frac{x^n}{n!}$ is convergent for every x,

(b) $\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!} = s$.

The above-mentioned two theorems are:

THEOREM I. *If the series* $\sum_{n=0}^{\infty} a_n$ *is* (E, q)*-summable for a certain* $q > 0$, *then it is summable by Borel's method.*

THEOREM II. *If* $a_n = o(n^{\varrho})$, $\varrho \geqslant -\frac{1}{2}$ *and the series* $\sum_{n=0}^{\infty} a_n$ *is summable by Borel's method then it is* $(C, 2\varrho+1)$*-summable.*

Proof of theorem 3. Since the series (1) is almost everywhere (E, q)-summable for a certain $q > 0$, according to theorem I it is almost everywhere summable by Borel's method. Since

$$\sum_{n=0}^{\infty} \int_0^1 a_n^2 \varphi_n^2(x)\, dx = \sum_{n=0}^{\infty} a_n^2 < \infty,$$

we have

$$a_n \varphi_n(x) \doteq o(1) \quad \text{for} \quad n \to \infty.$$

Therefore condition (2) implies that for the series (1) the assumptions of theorem II for $\varrho = 0$ are valid. Hence, according to theorem II the series (1) is almost everywhere $(C, 1)$-summable. Thus, the theorem of Kaczmarz-Zygmund (see [5], p. 105, theorem 12) allows us to conclude that the series (1) is almost everywhere (C, r)-summable for every $r > 0$.

Remark. Theorem 2 is a consequence of theorem 3; however, the proof of theorem 2 given here makes use of lemmas necessary in further considerations. On the other hand, in order to prove theorem 3 we use certain properties of Borel's method together with theorem II, the proof of which is based on facts belonging to the theory of analytic functions.

5. THEOREM 4. *If for the orthonormal series* (1) *the condition*

$$\sum_{n=2}^{\infty} a_n^2 (\log\log n)^2 < \infty$$

holds, then there exists almost everywhere $\lim_{n\to\infty} \tau_{2^n}(x)$.

Proof. Writing 2^n instead of n on both sides of inequality (10) we obtain

$$\int_0^1 [\sigma_{2^n}(x) - \tau_{2^n}(x)]^2 dx < \frac{C_2+1}{4^n} \sum_{k=1}^{2^n} k^2 a_k^2. \tag{15}$$

However,

$$\sum_{n=1}^{m} \frac{1}{4^n} \sum_{k=2}^{2^n} k^2 a_k^2 = \sum_{i=1}^{m} \sum_{p=i}^{m} \frac{1}{4^p} \sum_{\nu=2^{i-1}+1}^{2^i} k^2 a_k^2 < \sum_{i=1}^{m} \int_{i-1}^{\infty} \frac{dx}{4^x} \sum_{k=2^{i-1}+1}^{2^i} k^2 a_k^2$$
$$< 4 \sum_{i=1}^{m} \frac{1}{4^i} \sum_{k=2^{i-1}+1}^{2^i} k^2 a_k^2 < 4 \sum_{i=1}^{m} \sum_{k=2^{i-1}+1}^{2^i} a_k^2 < 4 \sum_{k=2}^{\infty} a_k^2 < \infty.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \sum_{k=1}^{2^n} k^2 a_k^2 < 4 \sum_{k=1}^{\infty} a_k^2$$

and according to (15), we have

$$\sum_{n=1}^{\infty} \int_0^1 [\sigma_{2^n}(x) - \tau_{2^n}(x)]^2 dx < 4(C_2+1) \sum_{k=1}^{\infty} a_k^2 < \infty,$$

whence

$$\sigma_{2^n}(x) - \tau_{2^n}(x) \doteq 0 \quad \text{for} \quad n \to \infty. \tag{16}$$

Applying our assumption and the theorem of Menchoff-Kaczmarz (see [5], p. 107, theorem 14) we obtain the $(C, 1)$-summability almost everywhere of the series (1). Hence there exists almost everywhere $\lim_{n\to\infty} \sigma_{2^n}(x)$, and (16) implies the existence of $\lim_{n\to\infty} \tau_{2^n}(x)$ almost everywhere.

6. We denote by $T[a_n]$ the n-th transform of the sequence $\{a_n\}$ by the method T. $T_1 T_2$ will indicate the method consisting in applying first the method T_2 to the given sequence $\{a_n\}$ and subsequently the method T_1 to the sequence of transforms thus obtained. We shall write C for the $(C, 1)$-method, E for the $(E, 1)$-method and $CC = H^{(2)}$ for the second iteration of Hölder's method. Further we denote by $T\text{-}\lim s_n$ the generalized limit of the sequence $\{s_n\}$ by the method T.

It is known in the theory of numerical series that if C-$\lim s_n$ exists, then CE-$\lim s_n$ also exists, and C-$\lim s_n = CE$-$\lim s_n$. But the existence of CE-$\lim s_n$ does not imply that of C-$\lim s_n$.

7. THEOREM 5. *If the orthonormal series* (1) *satisfies condition* (2) *and* CE-$\lim s_n(x)$ *exists almost everywhere, then* C-$\lim s_n(x)$ *exists almost everywhere, and* CE-$\lim s_n(x) \doteq C$-$\lim s_n(x)$.

Proof. Let

$$\lim_{n\to\infty} \frac{\tau_1(x)+\tau_2(x)+\ldots+\tau_n(x)}{n} \doteq s(x).$$

The formula (12) implies

$$\lim_{n\to\infty} \frac{\sigma_1(x)+\sigma_2(x)+\ldots+\sigma_n(x)}{n} \doteq s(x),$$

whence the series (1) is summable almost everywhere by the second iteration of Hölder's method. Thus, it is $(C, 2)$-summable almost everywhere. Hence the theorem of Kaczmarz-Zygmund ([5], p. 105, theorem 12) implies the $(C, 1)$-summability almost everywhere of the series (1) and we have C-$\lim s_n(x) \doteq s(x)$.

THEOREM 6. *The orthonormal series* (1) *satisfying the assumption* (2) *is* $(E, 1)$*-summable almost everywhere if and only if*

(a) *the series* (1) *is* $(C, 1)$*-summable almost everywhere,*

(b) EC-$\lim na_n\varphi_n(x) \doteq 0$.

Proof. Necessity. We suppose that the series (1) is $(E,1)$-summable almost everywhere. Then, by theorem 4, it is $(C, 1)$-summable almost everywhere, too. The condition (b) follows directly from a known theorem in the theory of summability of numerical series.

Sufficiency. We have

$$(17)\qquad s_n(x)-\sigma_n(x) = \sum_{k=0}^{n} a_k\varphi_k(x) - \frac{1}{n+1}\sum_{k=0}^{n}\sum_{\nu=0}^{k} a_\nu\varphi_\nu(x) = \frac{1}{n+1}\sum_{k=0}^{n} ka_k\varphi_k(x) = C[na_n\varphi_n(x)]$$

and

$$n[\sigma_n(x)-\sigma_{n-1}(x)] = \frac{1}{n+1}\sum_{k=0}^{n} ka_k\varphi_k(x).$$

Hence

$$(18)\qquad H^{(2)}[na_n\varphi_n(x)] = C\left[\frac{\sum_{k=0}^{n} ka_k\varphi_k(x)}{n+1}\right] = C\big[n\big(\sigma_n(x)-\sigma_{n-1}(x)\big)\big].$$

The lemma of Kaczmarz-Zygmund ([5], p. 104) and the assumption imply the convergence almost everywhere of the series

$$\sum_{k=1}^{\infty} k[\sigma_k(x)-\sigma_{k-1}(x)]^2.$$

Then, by Kronecker's theorem we have

$$\frac{1}{n+1}\sum_{k=1}^{n} k^2[\sigma_k(x)-\sigma_{k-1}(x)]^2 \doteq 0 \quad \text{for} \quad n\to\infty.$$

However, since

$$\left\{\frac{1}{n+1}\sum_{k=1}^{n} k|\sigma_k(x)-\sigma_{k-1}(x)|\right\}^2 < \frac{1}{n+1}\sum_{k=1}^{n} k^2[\sigma_k(x)-\sigma_{k-1}(x)]^2 \doteq 0,$$

we have

$$\frac{1}{n+1}\sum_{k=1}^{n} k[\sigma_k(x)-\sigma_{k-1}(x)] \doteq 0 \quad \text{for} \quad n\to\infty$$

and (18) yields

$$H^{(2)}\text{-}\lim na_n\varphi_n(x) \doteq 0 \quad \text{for} \quad n\to\infty.$$

Since

$$C\text{-}\lim s_n(x) \doteq CC\text{-}\lim s_n(x) = H^{(2)}\text{-}\lim s_n(x),$$

we have

$$(19)\qquad C[s_n(x)]\text{-}H^{(2)}[s_n(x)] \doteq o(1) \quad \text{for} \quad n\to\infty.$$

However, theorem 5 implies

$$C\text{-}\lim s_n(x) \doteq CE\text{-}\lim s_n(x) = EC\text{-}\lim s_n(x)$$

and, with (19) we obtain

$$(20)\qquad EC[s_n(x)]-H^{(2)}[s_n(x)] \doteq o(1) \quad \text{for} \quad n\to\infty.$$

Applying Euler's transformation to equality (17) we obtain

$$E[s_n(x)]-EC[s_n(x)] = EC[na_n\varphi_n(x)].$$

Now, adding (20) to the last equality and applying formula (19), we obtain

$$E[s_n(x)] \doteq EC[na_n\varphi_n(x)]+C[s_n(x)]+o(1).$$

Thus, conditions (a) and (b) in theorem 6 are sufficient.

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V. (1958)

A formula similar to Barnes' lemma

by F. M. Ragab (Princeton)

The formula to be established is

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(s)\Gamma(a-s)\Gamma(\beta-s)\Gamma(p-s)\Gamma(a-p+s)\Gamma(\beta-p+s)(-1)^s ds$$
$$=\frac{\Gamma(a)\Gamma(\beta)\Gamma(a-\frac{1}{2}p)\Gamma(\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}a+\frac{1}{2}\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}p)}{2^{2+p-a-\beta}\Gamma(\frac{1}{2})\Gamma(a+\beta-\frac{1}{2}p)\exp(-\frac{1}{2}i\pi p)},$$

where

(1) $\quad \mathrm{R}a>0,\quad \mathrm{R}\beta>0,\quad \mathrm{R}p>0,\quad \mathrm{R}(a-p)>0,\quad \mathrm{R}(\beta-p)>0.$

The path of integration is of Barnes' type and is curved, if necessary, to separate the increasing sequence of poles from the decreasing sequence. (1) is an extension of Barnes' lemma (see [1]) namely

(2) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)ds = \frac{\Gamma(a+\gamma)\Gamma(a+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(a+\beta+\gamma+\delta)}.$$

To prove (1), write I for the expression on the left. Let C be the semicircle of radius ϱ on the right of the imaginary axis with its centre at the origin and suppose that $\varrho\to\infty$ in such a way that the lower bound of the distance of C from the poles of $\Gamma(a-s)\Gamma(\beta-s)\Gamma(p-s)$ is definitely positive. Then the integrand is asymptotically equal to

$$O[s^{2a+2\beta-p-3}\exp(-3\pi|\mathrm{Im}\,s|)],$$

as $|s|\to\infty$ on the imaginary axis or on C. Thus the original integral converges and the integral round C tends to zero as $\varrho\to\infty$ when $\mathrm{R}(a+\beta-\frac{1}{2}p-1)<0$. The integral is therefore equal to minus $2\pi i$ times