On absolute convergence of multiple Fourier series

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1. Investigations concerning the absolute convergence of multiple Fourier series may be put under one of the following two headings: double Fourier series (papers [2], [7], [16]) or multiple Fourier series (papers [1], [4]). The former introduce the moduli of continuity defined as follows:

\[ \omega^{(1,2)}(h_1, h_2) = \sup_{(x_1, x_2) \in [0, h_1] \times [0, h_2]} |A^{1,2}(f; x_1, x_2)|, \]

\[ \omega^{(2)}(h_1, h_2) = \sup_{x_1 \in [0, h_1]} |A^{2}(f; x_1, h_2)|, \]

\[ \omega^{(1)}(h_1, h_2) = \sup_{x_2 \in [0, h_2]} |A^{1}(f; x_1, h_2)|, \]

where

\[ A^{1,2}(f; x_1, x_2; \delta_1, \delta_2) = f(x_1 + \delta_1, x_2 + \delta_2) - f(x_1, x_2) - f(x_1 + \delta_1, x_2) - f(x_1, x_2 + \delta_2), \]

\[ A^{2}(f; x_1, x_2; \delta_1, \delta_2) = f(x_1 + \delta_1, x_2 + \delta_2) - f(x_1, x_2) - f(x_1, x_2 + \delta_2) - f(x_1 + \delta_1, x_2), \]

\[ A^{1}(f; x_1, x_2; \delta_1, \delta_2) = f(x_1 + \delta_1, x_2 + \delta_2) - f(x_1, x_2 + \delta_2) - f(x_1 + \delta_1, x_2). \]

For functions of two variables those moduli make it possible to define certain generalized Lipschitz conditions. Čelidze [2] denotes by \( H^{\alpha, \beta}_{x, y} \) \((0 < \alpha, \beta, \alpha', \beta' < 1)\) the class of all continuous functions \( f(x_1, x_2) \) satisfying the inequalities

\[ \omega^{(1,2)}(h_1, h_2) \leq K_{0,2}(h_1 h_2^2), \]

\[ \omega^{(2)}(h_1, h_2) \leq K_{1,0}(h_1 h_2^2), \]

\[ \omega^{(1)}(h_1, h_2) \leq K_{1,0}(h_1 h_2^2), \]

the functions \( K_{0}(h_1) \) and \( K_{0}(h_2) \) being summable. The variations may be defined analogically (see [3], p. 345). With this notation sufficient conditions for the absolute convergence of double Fourier series may be formulated. E. g. if the function \( f(x_1, x_2) \) belongs to the class \( H^{\alpha, \beta}_{x, y} \) for some
with \( \beta_1, \beta_2 \geq 0 \). Sufficient convergence conditions for the series (1.5) are given by Reves and Szász in [7], and for the series (1.6) by Záková in [10].

In all these papers (2), (7), (10) the authors prove the convergence of the series with lower summation limits \( m_1 = m_2 = 1 \) using conditions of the form (1.1). They prove the convergence of the remaining parts of the series applying the known theorems on functions of one variable to the functions \( f(x, y) \text{d}x \) and \( \int f(x, y) \text{d}y \).

Bochner [1], and Minakshisundaram and Szász [4] introduce a quite different method in their considerations. Their proofs are based on the notion of spherical means introduced by Bochner. Bochner obtains a theorem (see [1], theorem X), with assumptions about derivatives of a suitably defined function \( f_0(x) \). Minakshisundaram and Szász assume Lipschitz conditions of the form \( |f(x) - f(y)| \leq M |x - y|^{1+\gamma} \), where \( x, y \) are two points of the \( n \)-dimensional Euclidean space and \( |x - y| \) the distance between \( x \) and \( y \). They obtain the convergence of a series of the form (1.5) for \( \gamma > 2m/(n+2a) \). For \( n > 1 \), however, this inequality does not embrace the exponent \( \gamma = 1 \).

In paper (6) lemma 3 and theorem 2 of the present paper are formulated without proof.

2. The object of the present paper (5) is to generalize the investigations made in [2], [7] and [10] to multiple Fourier series. The method of the proofs differs from that used in the papers mentioned above in that it is direct and does not reduce the \( n \)-dimensional case to the \( (n-1) \)-dimensional one. The notion of spherical means is not applied either. The author conceived the idea of considering this problem and especially of applying the \( r \)-th variations (see § 3) in its investigation at the mathematical seminar directed by Prof. Orlicz. I wish to thank Prof. Orlicz for his kind help.

We shall consider real- or complex-valued functions \( f(x_1, \ldots, x_n) \) of \( n \) real arguments \( x_1, \ldots, x_n \) defined in the whole \( n \)-dimensional Euclidean space, periodic with period 2 in each variable and integrable in the region \( 0 \leq x_i \leq 2; i = 1, 2, \ldots, n \) with the \( p \)-th power for a certain \( 1 < p \leq 2 \). For a given arbitrary subset of the set \( B = (1, 2, \ldots, n) \) and \( \Lambda \) the complement of \( A \) with respect to \( E \). The numbers

\[
\sum_{m_1, m_2, \ldots, m_n} (m_1 + 1)(m_2 + 1) \cdots (m_n + 1) \eta_1(m_1 + 1) \eta_2(m_2 + 1) \cdots \eta_n(m_n + 1)
\]

(1) This theorem is already given in paper (7) (corollary on p. 765) but its formulation is not adequate. The author assume only that the function \( f(x_1, x_2) \) is of finite variations and satisfies condition (1.1) for certain \( a, b > 0 \). This, however, is not sufficient for the absolute convergence of the Fourier series of \( f(x_1, x_2) \). As a counter-example we may take the function \( f(x, y) = g(x, y) \) being of finite variation and such that the Fourier series of \( g(x, y) \) is not absolutely convergent (for such an example, see the function given by the series (1) in [9], p. 136).

\[
\sum_{m_1, m_2, \ldots, m_n} (m_1 + 1) \eta_1(m_1 + 1) \sum_{m_2, m_2, \ldots, m_n} (m_2 + 1) \eta_2(m_2 + 1) \cdots (m_n + 1) \eta_n(m_n + 1)
\]

(2) The results of which were presented on June 7th, 1966 to the Polish Mathematical Society, Section of Poznań.

(3) We take the period 2 instead of 2\( \pi \) to simplify calculations.
will be called the Fourier coefficients of the function \( f(x_1, \ldots, x_n) \). Here \( A \subseteq \mathbb{R}^n \), \( m_1, \ldots, m_n = 0, 1, 2, \ldots \) and \( \prod_{i=1}^{n} a_i \) denotes the product of all \( a_i \) with indices \( i \in A \). The series

\[
\sum_{m_1, \ldots, m_n = 0}^{\infty} \lambda_{m_1 \ldots m_n} \sum_{f \in \mathcal{B}} \hat{m}_{m_1 \ldots m_n} (f) \prod_{i \in A} \cos m_i x_i + \prod_{j \in A^{c}} \sin m_j x_j,
\]

where \( \lambda_{m_1 \ldots m_n} = 2^{-n/2} \mu(B) \) and \( \mu(B) \) is the measure of all elements of the set \( B \), is the Fourier series of the function \( f(x_1, \ldots, x_n) \). The sign \( \sum \) indicates that the summation extends over all subsets \( A \) of the set \( B \), including the empty subset.

Using this notation we give some sufficient conditions for the convergence of the series

\[
\sum_{m_1, \ldots, m_n = 0}^{\infty} \beta_{m_1} \beta_{m_2} \ldots \beta_{m_n} \hat{m}_{m_1 \ldots m_n} (f),
\]

where \( \beta_1, \ldots, \beta_n \geq 0 \) and \( \gamma = 1 \) implies the absolute convergence of the series (2.2).

We shall need the following inequality of F. Riesz (see [8], p. 118):

If \( 1 < p \leq 2 \) and \( 1/p + 1/q = 1 \), then we have

\[
\sum_{m_1, \ldots, m_n = 0}^{\infty} \hat{m}_{m_1 \ldots m_n}^q (f)^q \leq M_n \int_{A} \left[ \sum_{i=1}^{n} \left( f(x_1, \ldots, x_n) \right)^p d\lambda_1 \ldots d\lambda_n \right]^{q/p},
\]

with the constant \( M_n \) not depending on the function \( f \).

The above notation is very useful. It will be seen in Lemma 3 that certain transformations of the functions \( f(x_1, \ldots, x_n) \) lead to functions whose Fourier coefficients can be obtained from the Fourier coefficients of the function \( f \) by certain operations on the set \( A \).

5. Now we shall introduce some symbols analogous to those given in \( \S \). Here \( H = (k_1, \ldots, k_n) \) will denote a non-empty subset of the set \( E \), where \( k_1, \ldots, k_n \) are the elements of \( H \).

For \( H = (k) \) we write

\[
D^H (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) = f(x_1, \ldots, x_{k-1}, x_k + \hat{h}_1, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n),
\]

and for \( H = (k_1, \ldots, k_n) \) \( (k_1 < \ldots < k_n) \)

\[
D^H (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) = \Delta_{g \in H} (\, D^{g \setminus H} (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) \},
\]

Further we put

\[
\omega^H (x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) = \sup_{|h| \leq C} |\Delta_{g \in \hat{h}} (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n)|,
\]

where \( (1, \ldots, h_{-m}) = H \), and

\[
\omega^H (h_1, \ldots, h_n) = \left( \sup_{|h| \leq C} |\Delta_{g \in \hat{h}} (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n)| \right)^{p/n},
\]

for \( p \geq 1 \). The functions \( \omega^H \) and \( \omega^g \) will be called the modulus of continuity and the \( p \)-th integral modulus of the function \( f(x_1, \ldots, x_n) \) with regard to the set \( H \), respectively. We obviously have

\[
\omega^g (h_1, \ldots, h_n) \leq \omega^g (x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) \leq 2^{2n/p - 1} \omega^g (h_1, \ldots, h_n) \text{ for } 1 \leq p \leq p'.
\]

Moreover, (3.1) implies

\[
\left( \sum_{i=1}^{n} \omega^H (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n)^{q/p} d\lambda_1 \ldots d\lambda_n \right)^{q/p} \leq \omega^g (2h_1, \ldots, 2h_n).
\]

We shall need the notion of \( r \)-th variation \( (r \geq 1) \) of the function \( f(x_1, \ldots, x_n) \) with respect to the set \( H \neq 0 \) in the region \( (a_1, \ldots, a_n) \) \( (1) \).

\[
\Delta^H (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) = \Delta_{g \in H} (\, D^{g \setminus H} (f; x_1, \ldots, x_n; \hat{h}_1, \ldots, \hat{h}_n) \}.
\]

(1) Instead of \( r \)-th variation one can consider \( \Phi \)-th variation, as introduced for one variable by L. C. Young. \( \Phi (u) \) is here a non-negative and non-decreasing function. In order to obtain the \( \Phi \)-th variation, \( \Phi (|D^g|) \) should be taken in (3.2) instead of \( |D^g| \) and the exponent \( 1/p \) omitted. A variation of this kind was used by the author in investigations of single Fourier series in \( \S \). We here give up this degree of generality to simplify the notation.
partition of the n-dimensional parallelepiped \((a_i \leq x_i \leq b_i; i = 1, 2, \ldots, n)\) into partial n-dimensional parallelepipeds:

\[ a_i = a_i^{(0)} < a_i^{(1)} < \ldots < a_i^{(N)} = b_i. \]

We introduce

\[
V_f^H(f) = \left( \sup_{x_i \leq a_i^{(0)}} \sup_{x_i \leq b_i} \sum_{|H_i| = 1}^{N} \left| \int_{x_i = a_i^{(0)}}^{a_i^{(1)}} \ldots \int_{x_i = b_i} \prod_{i=1} \right| f(x_1, \ldots, x_n) dH_1 \ldots dH_n \right)^N.
\]

4. We now proceed to give three lemmas not directly connected with the absolute convergence of Fourier series.

**Lemma 1.** If \(q > 2 > 0\) and \(a_1, \ldots, a_N > 0\) then

\[
2^{-N+1} \left( \frac{N}{a_1} \right)^q \leq \sum_{i=1}^{N} a_i^{-q} \leq 2^{-N} \left( \sum_{i=1}^{N} a_i^{-q} \right)^q.
\]

This known lemma can be proved by induction.

**Lemma 2.** Let the function \(f(x_1, \ldots, x_n)\) be defined in the whole n-dimensional Euclidean space and periodic in the variable \(x_i\) with the period \(a_i - b_i\) for each \(i\). We assume that \(H = (b_1, \ldots, b_n) \neq 0\) and that the numbers \(b_i > 0\) satisfy the following conditions: for each \(i\) there exists an integer \(p_i\) such that \(2b_i p_i - b_i = a_i - b_i\). Then

\[
\int_{x_1 = a_1^{(0)}}^{b_1} \ldots \int_{x_n = a_n^{(0)}}^{b_n} \prod_{i=1}^{N} |f(x_1, \ldots, x_n)| dx_1 \ldots dx_n \leq 2^N \left[ V_f^H(f) \right]^N \prod_{i=1}^{N} b_i.
\]

**Proof.** Putting \(x_i = a_i^{(0)} + 2t \quad (i = 1, 2, \ldots, N)\) and \(x_i = b_i - 2t \quad (i = 1, 2, \ldots, N)\), we have

\[
\int_{x_1 = a_1^{(0)}}^{b_1} \ldots \int_{x_n = a_n^{(0)}}^{b_n} \prod_{i=1}^{N} |f(x_1, \ldots, x_n)| dx_1 \ldots dx_n = \int_{x_1 = a_1^{(0)}}^{b_1} \ldots \int_{x_n = a_n^{(0)}}^{b_n} \prod_{i=1}^{N} |f(x_1, \ldots, x_n)| dx_1 \ldots dx_n.
\]

On the other hand,

\[
\int_{x_1 = a_1^{(0)}}^{b_1} \ldots \int_{x_n = a_n^{(0)}}^{b_n} \prod_{i=1}^{N} |f(x_1, \ldots, x_n)| dx_1 \ldots dx_n = \int_{x_1 = a_1^{(0)}}^{b_1} \ldots \int_{x_n = a_n^{(0)}}^{b_n} \prod_{i=1}^{N} |f(x_1, \ldots, x_n)| dx_1 \ldots dx_n.
\]

\[= \sum_{i=1}^{N} \sum_{|H_i| = 1}^{N} \left| \int_{x_i = a_i^{(0)}}^{a_i^{(1)}} \ldots \int_{x_i = b_i} \prod_{i=1}^{N} f(x_1, \ldots, x_n) dx_1 \ldots dx_n \right|^N.
\]

**Lemma 3.** For the function \(p_B^H\) introduced in § 3 we have, for arbitrary \(A \cap E\),

\[
\sum_{|H| > 0} \left( \sin m_1 \pi H_1 A_{m_1}^{H_1} \right) = \sum_{|H| > 0} \left( -1 \right)^{|H|} m_1 \pi H_1 A_{m_1}^{H_1} \sin m_1 \pi H_1 A_{m_1}^{H_1} \left( f \right),
\]

where \(\mu(B)\) denotes the number of all elements of the set \(B\) and \(A \cap B\) the symmetric difference of the sets \(A\) and \(B\). Moreover, given \(H\), the operation \(A \cap H\) defines a one-to-one correspondence from \(E\) to \(E\).

**Proof.** Let \(H = (b_1, \ldots, b_n)\). We prove formula (4.1) by induction with respect to \(\pi\). Suppose \(\pi = 1, i.e. H = (k)\). Let \(kA\). Then

\[
\int_{x_1 = a_1^{(0)}}^{b_1} \ldots \int_{x_n = a_n^{(0)}}^{b_n} \prod_{i=1}^{N} |f(x_1, \ldots, x_n)| dx_1 \ldots dx_n = \sum_{|H| > 0} \left( \cos m_1 \pi H_1 A_{m_1}^{H_1} \right) \sin m_1 \pi H_1 A_{m_1}^{H_1} x_1 \ldots \sin m_1 \pi H_1 A_{m_1}^{H_1} x_n.
\]

Putting \(-H_1\) instead of \(H_1\) in (4.2) and subtracting from (4.2) the equality thus obtained, we get

\[
a_{m_1, -m_1}^{H_1} \left( f \right) = 2 \sin m_1 \pi H_1 A_{m_1}^{H_1} \left( f \right).
\]

However, \(A \cap (H = (k)) = 0\), whence \(\left( -1 \right)^{|H|} = 1\) and \(A \cap H = A \cap (k)\); thus,

(4.3) implies (4.1). Now let \(k \neq \bar{A}\). Similarly to (4.3), the following equality is easily obtained:

\[
a_{m_1, -m_1}^{H_1} \left( f \right) = -2 \sin m_1 \pi H_1 A_{m_1}^{H_1} \left( f \right) \sin m_1 \pi H_1 A_{m_1}^{H_1} x_1 \ldots \sin m_1 \pi H_1 A_{m_1}^{H_1} x_n.
\]

Since \(A \cap (A = (k)) = 0\), \(\left( -1 \right)^{|H|} = -1\) and \(A \cap H = A \cup (k)\), we have (4.1).

Thus, formula (4.1) is proved for \(n \geq 1\).
Suppose formula (4.1) is true for $s - 1$ ($s \leq n$). We shall now prove it for $s$. Putting $H = (k_1, \ldots, k_n)$, we have

\begin{equation}
(4.4) \quad a_{p, m_i}^d \left[ P_{E_H}^n (f) \right] = a_{p, m_i}^d \left[ P_{E_H}^n (P_{E_H}^n (f)) \right] = (-1)^{ \frac{n}{2} } e^{-i \frac{\pi}{2} n \sum k_i} \sin \left( \frac{2 \pi}{n} \sum k_i \right) a_{p, m_i}^d (f) = (-1)^{ \frac{n}{2} } e^{-i \frac{\pi}{2} n \sum k_i} \sin \left( \frac{2 \pi}{n} \sum k_i \right) a_{p, m_i}^d (f)
\end{equation}

where

\begin{equation}
A_1 = \overline{A_2} \cap (k_1, \ldots, k_{s-1}), \quad A_2 = [A \cap (k_s)] \cap (k_1, \ldots, k_{s-1}).
\end{equation}

However,

\begin{equation}
A_1 \cap (k_1, \ldots, k_{s-1}) = \overline{A_2} \cap (k_1, \ldots, k_{s-1})
\end{equation}

whence

\begin{equation}
\mu (A \cap (k_s)) = \mu (A \cap H).
\end{equation}

Further

\begin{equation}
[A \cap (k_s)] \cap (k_1, \ldots, k_{s-1}) = A \cap H \cup \overline{A} \cap (k_s),
\end{equation}

therefore (4.5) implies

\begin{equation}
(4.7) \quad A_2 = A \cap H.
\end{equation}

(4.1) follows from (4.4), (4.6) and (4.7).

The one-to-one correspondence follows from the simple formula

\begin{equation}
A = (A \cap H) \cup \overline{A}.
\end{equation}

5. In order to formulate two auxiliary lemmas we put for $i = 1, 2, \ldots, n$:

\begin{equation}
A_{p, m_i} = \bigcup_{m_i = 1}^{n} \left\{ m_i \right\} \quad \text{for} \quad v_i \geq 1, \quad A_{p, m_i} = \bigcup_{m_i = 0}^{n} \left\{ m_i \right\} \quad \text{for} \quad v_i \geq 0.
\end{equation}

Here $\bigcup_{m_i = 0}^{n} \left\{ m_i \right\}$ denotes the set of all systems of $n$ non-negative integers $(m_1, \ldots, m_n)$ satisfying the condition contained in the brackets $\left\{ \right\}$. It is clear that for two different systems $(\nu_1, \ldots, \nu_n)$ (that is, for two systems having different integers in one place at least), the corresponding sets $A_{\nu_1, \ldots, \nu_n}$ are disjoint and the set

\begin{equation}
\bigcup_{\nu_1, \ldots, \nu_n \geq 0} A_{\nu_1, \ldots, \nu_n}
\end{equation}

is identical with the set of all systems $(m_1, \ldots, m_n)$ of non-negative integers. Further we put for a given system $(v_1, \ldots, v_n)$,

\begin{equation}
H = \bigcup_{v_i \geq 1} A_{p, m_i}
\end{equation}

and assume that $H \neq \emptyset$. It is now possible to formulate lemmas 4 and 5, denoting by $\sum$ summation over all systems $(m_1, \ldots, m_n) \in \mathcal{A}_{n-1}$.

**Lemma 4.** We take $1 \leq r \leq p \leq 2$, $1/p + 1/q = 1$ ($p \neq 1$), $H = (k_1, \ldots, k_n)$, $H = (l_1, \ldots, l_{n-1})$ and assume the function $f(x_1, \ldots, x_n)$ to be integrable in the region $(0 < x_i < 2; i = 1, 2, \ldots, n)$ with the $p$-th power.

Then

\begin{equation}
(5.1) \quad \sum_{m_1, \ldots, m_n \in \mathcal{A}_{n-1}} \sum_{v_1, \ldots, v_n \geq 1} \left| \hat{f} (m_1, \ldots, m_n) \right|^2 \lesssim M_{\infty}^{2 - \alpha} \sum_{k_1, \ldots, k_n} \left( \sin \left( \frac{2 \pi}{n} \sum k_i \right) \right)^{2 - \alpha} \times
\end{equation}

\begin{equation}
\times \left[ \int_{0}^{1} \ldots \int_{0}^{1} \left| P_{E_H}^n (f) (x_1, \ldots, x_n) \right|^{2 - \alpha} \prod_{i=1}^{n} dx_1 \ldots dx_n \right]^{\frac{1}{2 - \alpha}}
\end{equation}

**Proof.** According to lemma 3, the operation $A \cap H$ defines a one-to-one correspondence from $E$ to $H$. Hence

\begin{equation}
(5.2) \quad \sum_{m_1, \ldots, m_n \in \mathcal{A}_{n-1}} \sum_{v_1, \ldots, v_n \geq 1} \left| \hat{f} (m_1, \ldots, m_n) \right|^2 \lesssim M_{\infty}^{2 - \alpha} \sum_{k_1, \ldots, k_n} \left( \sin \left( \frac{2 \pi}{n} \sum k_i \right) \right)^{2 - \alpha} \times
\end{equation}

\begin{equation}
\times \left[ \int_{0}^{1} \ldots \int_{0}^{1} \left| P_{E_H}^n (f) (x_1, \ldots, x_n) \right|^{2 - \alpha} \prod_{i=1}^{n} dx_1 \ldots dx_n \right]^{\frac{1}{2 - \alpha}}
\end{equation}

Thus, formula (4.1) yields

\begin{equation}
(5.3) \quad \sum_{m_1, \ldots, m_n \in \mathcal{A}_{n-1}} \sum_{v_1, \ldots, v_n \geq 1} \left| \hat{f} (m_1, \ldots, m_n) \right|^2 \lesssim M_{\infty}^{2 - \alpha} \sum_{k_1, \ldots, k_n} \left( \sin \left( \frac{2 \pi}{n} \sum k_i \right) \right)^{2 - \alpha} \times
\end{equation}

\begin{equation}
\times \left[ \int_{0}^{1} \ldots \int_{0}^{1} \left| P_{E_H}^n (f) (x_1, \ldots, x_n) \right|^{2 - \alpha} \prod_{i=1}^{n} dx_1 \ldots dx_n \right]^{\frac{1}{2 - \alpha}}
\end{equation}

which implies (5.1).
Lemma 5. With the same notation and assumptions as in lemma 4, we have for \( \beta_1, \ldots, \beta_n \geq 0 \) and \( 0 < \gamma < 2 \):

\[
\begin{align*}
(5.4) \quad & \sum_{\text{(m}_1, \ldots, \text{m}_n)_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
& \leq M^2_{\gamma, \text{H}} \left\{ \left[ \sum_{i=1}^{n} \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} f_{\text{ak}}^{(0)}(x_1, \ldots, x_n) \right]^{\gamma} \right]^{\gamma/2} \right. \\
& \times \left. \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} \left[ \frac{2}{\delta_{x_1, \ldots, x_n}} \right]^{\gamma} \right]^{\gamma/2} \right\}^{1/\gamma}.
\end{align*}
\]

Proof. Applying the Hölder inequality \( \sum_{w} e^{\left( \sum_{w} x_{w} \right)^{\gamma/2}} \leq \gamma \left( \sum_{w} e^{x_{w}} \right)^{\gamma/2} \) with \( k = q/(q-\gamma) \), \( l = \gamma/\gamma \), \( u = (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \) and \( v = \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma/2} \) to the sum on the left-hand side of inequality (5.4), we obtain

\[
(5.5) \quad \sum_{\text{(m}_1, \ldots, \text{m}_n)_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
\leq \left\{ \left[ \sum_{i=1}^{n} \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} f_{\text{ak}}^{(0)}(x_1, \ldots, x_n) \right]^{\gamma} \right]^{1/\gamma} \right. \\
& \times \left. \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} \left[ \frac{2}{\delta_{x_1, \ldots, x_n}} \right]^{\gamma} \right]^{1/\gamma} \right\}^{1/\gamma}.
\]

However, it is easy to see that

\[
(5.6) \quad \sum_{\text{(m}_1, \ldots, \text{m}_n)_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma-n} \leq \frac{\delta}{\gamma(\gamma-\gamma)}.
\]

Applying inequality (5.6) and lemma 4 to the right-hand side of inequality (5.5) we obtain (5.4).

6. We shall prove three theorems on the absolute convergence of multiple Fourier series.

Theorem 1. Let \( E \) be a class of non-empty subsets of the set \( \{1, 2, \ldots, n\} \). Let us assume that the function \( f(x_1, \ldots, x_n) \) defined in the whole \( n \)-dimensional Euclidean space satisfies the following conditions:

1. \( \text{the function is periodic with period} \ 2 \ \text{in each variable} \)
2. \( \text{it is integrable in the region} \ \{0 \leq x_i \leq 2; \ i = 1, 2, \ldots, n\} \) with the \( p\)-th power for a certain \( 1 < p \leq 2 \),
3. \( \text{for every} \ \|f\|_{\text{H}} \text{there exists a number} \ r_{\text{H}} \text{such that} \ 1 \leq r_{\text{H}} < p \) and \( \|f\|_{\text{H}} < \infty \).

We further assume, for certain \( \beta_1, \ldots, \beta_n \geq 0 \) and \( 0 < \gamma < 2 \), the convergence of the series

\[
\begin{align*}
(a) \quad & \sum_{r_k=1}^{\infty} \sum_{m_{\text{ak}}=a_{r_k}} \sum_{m_{\text{ak}}=a_{r_k}} \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
& \times \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} \left[ \frac{2}{\delta_{x_1, \ldots, x_n}} \right]^{\gamma} \right]^{1/\gamma} \right\}^{1/\gamma} \right\}^{1/\gamma}.
\end{align*}
\]

(b) \( \sum_{r_k=1}^{\infty} \sum_{m_{\text{ak}}=a_{r_k}} \sum_{m_{\text{ak}}=a_{r_k}} \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
\times \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} \left[ \frac{2}{\delta_{x_1, \ldots, x_n}} \right]^{\gamma} \right]^{1/\gamma} \right\}^{1/\gamma} \right\}^{1/\gamma}.
\]

where \( H = (k_1, \ldots, k_s) \), \( \mathcal{H} = (l_1, \ldots, l_{n-s}) \). Then the series (2.3) is convergent.

Proof. According to lemma 1, the convergence of the series (2.3) is equivalent to that of the series

\[
(6.1) \quad \sum_{m_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
\]

with arbitrary \( q \geq 2 \), or to that of the series

\[
(6.2) \quad \sum_{m_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
\]

obtained from (6.1) by dropping the term with indices \( m_1 = \ldots = m_n = 0 \).

We denote by \( \sum_{\text{r_{\text{ak}}}=\text{a_{r_{\text{ak}}}}} \) summation extended over all non-empty subsets \( H \) of the set \( E \). Then, putting \( H = \{r_1 \neq 0\} = (k_1, \ldots, k_s) \), and applying lemma 5 (with \( r_{\text{H}} = p \) for \( H \notin E \)) we obtain

\[
\sum_{m_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
= \sum_{\text{r_{\text{ak}}}=\text{a_{r_{\text{ak}}}}} \sum_{r_{\text{ak}}=a_{r_{\text{ak}}}} \sum_{m_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
\leq M_{\gamma, \text{H}} \sum_{\text{r_{\text{ak}}}=\text{a_{r_{\text{ak}}}}} \sum_{r_{\text{ak}}=a_{r_{\text{ak}}}} \sum_{m_{\text{ak}}=a_{r_k}} (m_1+1)\beta_1 \ldots (m_n+1)\beta_n \left[ (\text{g}_{\text{ak}}^{(0)}, \text{m}_\text{ak})_{\text{f}} \right]^{\gamma} \\
\leq M_{\gamma, \text{H}} \left\{ \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} \left[ \frac{2}{\delta_{x_1, \ldots, x_n}} \right]^{\gamma} \right]^{1/\gamma} \right. \\
& \times \left. \left[ \int_{x_1, \ldots, x_n; 2^{-\gamma}, \ldots, 2^{-\gamma}} \left[ \frac{2}{\delta_{x_1, \ldots, x_n}} \right]^{\gamma} \right]^{1/\gamma} \right\}^{1/\gamma} \right\}^{1/\gamma} \right\}^{1/\gamma}.
\]

(*) The class \( \mathcal{H} \) may be empty.
This inequality proves that the series (6.2) and therefore also (2.3) are convergent if the series
\[
\sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{n^q} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} R \times
\left( \int_{1}^{\infty} \left[ \int_{1}^{\infty} \cdots \left( \int_{1}^{\infty} \cdots \left( \int_{1}^{\infty} \left[ \prod_{1}^{n} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} \right] \right) \right) \cdots \right) \right]^{\gamma p}
\]
are convergent for every \( H \neq 0, H \in E. \)

Suppose \( H \neq 0, H \neq 1. \) Then inequality (3.2) with \( r_H = p \) yields
\[
\left( \int_{1}^{\infty} \left[ \int_{1}^{\infty} \cdots \left( \int_{1}^{\infty} \left( \int_{1}^{\infty} \cdots \left( \int_{1}^{\infty} \left[ \prod_{1}^{n} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} \right] \right) \right) \cdots \right) \right) \right]^{\gamma p}
\leq \frac{c_H^{p}}{\gamma (2-n, 2-n)}
\]
and the convergence of the series (b) implies that of the series (6.3).

Suppose \( H \neq 0, H \neq 1. \) Then Lemma 2 yields
\[
\left( \int_{1}^{\infty} \left[ \int_{1}^{\infty} \cdots \left( \int_{1}^{\infty} \left( \int_{1}^{\infty} \cdots \left( \int_{1}^{\infty} \left[ \prod_{1}^{n} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} \right] \right) \right) \cdots \right) \right) \right]^{\gamma p}
\leq \frac{\left[ \prod_{1}^{n} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} \right]^{\gamma p}}{\gamma (2-n, 2-n)}.
\]
This, together with the convergence of the series (a) implies that of the series (6.3).

**Theorem 1.** Let the function \( f(x_1, \ldots, x_n) \) satisfy for a class \( \mathcal{K} \) of non-empty subsets of the set \( E = \{1, 2, \ldots, n\} \) the assumptions 1', 2' and 3' of Theorem 1. Further assume, for certain \( \beta_1, \ldots, \beta_n \geq 0 \) and \( 0 < \gamma < 2, \) the convergence of the series
\[
\sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \sum_{k=1}^{n} \left[ \prod_{1}^{n} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} \right]^{\gamma p}
\]
and the convergence of which is assured by inequality (6.4). Similarly (6.6) implies that the series (b) in Theorem 1 is majorized by the series
\[
\sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \left[ \prod_{1}^{n} \left[ \sum_{k=1}^{n} \left( a_k, a_{n-k} \right) \right]^{p-r} \right]^{\gamma p},
\]
which, according to (6.7), is convergent. Then the assumptions in Theorem 1 are satisfied, thus implying the convergence of the series (2.3).

7. It should be noted that the above theorems constitute generalizations of results concerning the absolute convergence of double Fourier series given in papers [2], [7] and [10]. Putting \( n = p = 2, \beta_1 = \beta_2 = 0, \gamma = 1, \mathcal{K} = 0 \) in Theorem 1, we obtain the condition
\[ a^2 > \frac{1}{4} \text{ (see [2]). Taking for } \gamma \text{ in theorem 2 the class of all non-empty subsets of the set } B = (1, 2), w = p = 2, r_{n^2} = y = 1, \text{ we obtain the condition of Zak, } a^2 > 2\delta_{n^2} \text{ (see [10]).}

The counter-examples given in these papers and concerning the strength of the results do not exhaust this problem. The author intends to take up this problem once more together with investigations concerning conditions of type (6.4) and (6.6).

Theorems 1', 2' generalize many known theorems holding for single Fourier series to multiple Fourier series, e.g. all theorems in §§ 6.3, 6.31-6.34 and 6.6.5-6 in [9](6).

The results obtained may be generalized to certain classes of almost periodic functions of \( n \) variables, by the method used in [5].

References


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ANNALES
POLONICI MATHEMATICI
V. (1958)

Sur un problème mixte pour l'équation du type hyperbolique

par G. Majcher (Kraków)

I. Cas de la dépendance linéaire entre la fonction \( u(x, y) \) et ses dérivées sur la courbe \( \Gamma \)

§ 1. Enoncé du problème. Considérons une équation du type hyperbolique à deux variables indépendantes, réduite à la forme canonique

\[ H[u] = u_x^2 + a(x, y)u_y^2 + b(x, y)u^2_x + c(x, y)u = f(x, y), \]

Soit \( \Omega \) un domaine limité par la demi-droite caractéristique \( y = 0, x \geq 0 \) de l'équation (1), par la droite \( x = a, y = 0 \) et par une courbe \( \Gamma \) issue de l'origine et représentée par l'équation:

\[ x = \theta(y), \quad y \geq 0 \quad \text{(ou } y = \tau(x), \quad x \geq 0), \]

où \( \theta'(y) > 0, \theta(0) = 0. \)

Nous nous proposons de trouver une intégrale \( u(x, y) \) de l'équation (1) qui soit de classe \( C^2 \), dans la fermeture \( \overline{\Omega} \) du domaine \( \Omega \), admette une dérivée partielle \( u^2_x \) continue dans cet ensemble et satisfasse aux conditions aux limites

\[ A(x, y)u^2_x(x, y) + B(y)u^2_x(x, y) + C(y)u(x, y) = g(y), \]

\( x = \theta(y), \quad y \geq 0 \) et

\[ u(x, 0) = h(x) \quad \text{pour } x \geq 0. \]

Dans la suite le problème posé sera appelé brièvement problème \((\mathcal{M})\)\(^{(1)}\).

Nous démontrerons dans cette partie du travail l'existence de la solution du problème \((\mathcal{M})\).

§ 2. Existence de la solution du problème \((\mathcal{M})\).

1. Supposons vérifiées les hypothèses suivantes:

\[ \text{(1) Ce problème a été posé par M. Kryžański.} \]