Les ANNALES POLONICI MATHEMATICI constituent une continuation des ANNALES DE LA SOCIÉTÉ POLONAISE DE MATHÉMATIQUE (vol. I-XXV) fondées en 1921 par Stanisław Zaremba.

Les ANNALES POLONICI MATHEMATICI publient, en langues des congrès internationaux, des travaux consacrés à l'Analyse Mathématique, la Géométrie et la Théorie des Nombres. Chaque volume paraît en 3 fascicules.

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

ANNALES
POLONICI MATHEMATICI
V. (1958)

On absolute convergence of multiple Fourier series

by J. Musielak (Poznań)

1. Investigations concerning the absolute convergence of multiple Fourier series may be put under one of the following two headings: double Fourier series (papers [2], [7], [10]) or multiple Fourier series (papers [1], [4]). The former introduce the *moduli of continuity* defined as follows:

$$\omega^{(1,2)}(h_1, h_2) = \sup_{\substack{a\leqslant x_1,x_2\leqslant b \\ |\delta_1|\leqslant h_1;\,|\delta_2|\leqslant h_2}} |\Delta^{(1,2)}(f; x_1, x_2; \delta_1, \delta_2)|,$$

$$\omega^{(2)}(x_1; h_1, h_2) = \sup_{\substack{a\leqslant x_2\leqslant b \\ |\delta_2|\leqslant h_2}} |\Delta^{(2)}(f; x_1, x_2; \delta_1, \delta_2)|,$$

$$\omega^{(1)}(x_2; h_1, h_2) = \sup_{\substack{a\leqslant x_1\leqslant b \\ |\delta_1|\leqslant h_1}} |\Delta^{(1)}(f; x_1, x_2; \delta_1, \delta_2)|,$$

where

$$\Delta^{(1,2)}(f; x_1, x_2; \delta_1, \delta_2) = f(x_1+\delta_1, x_2+\delta_2)-f(x_1+\delta_1, x_2)- \\ -f(x_1, x_2+\delta_2)+f(x_1, x_2),$$

$$\Delta^{(2)}(f; x_1, x_2; \delta_1, \delta_2) = f(x_1, x_2+\delta_2)-f(x_1, x_2),$$

$$\Delta^{(1)}(f; x_1, x_2; \delta_1, \delta_2) = f(x_1+\delta_1, x_2)-f(x_1, x_2).$$

For functions of two variables those moduli make it possible to define certain *generalized Lipschitz conditions*. Čelidze [2] denotes by $H_{\alpha\beta}^{\alpha'\beta'}$ $(0<\alpha, \beta, \alpha', \beta'<1)$ the class of all continuous functions $f(x_1, x_2)$ satisfying the inequalities

$$\omega^{(1,2)}(h_1, h_2) \leqslant K_{(1,2)} h_1^{\alpha} h_2^{\beta}, \tag{1.1}$$

$$\omega^{(2)}(x_1; h_1, h_2) \leqslant K_{(2)}(x_1) h_2^{\beta'}, \tag{1.2}$$

$$\omega^{(1)}(x_2; h_1, h_2) \leqslant K_{(1)}(x_2) h_1^{\alpha'}, \tag{1.3}$$

the functions $K_{(2)}(x_1)$ and $K_{(1)}(x_2)$ being summable. The *variations* may be defined analogically (see [3], p. 345). With this notation sufficient conditions for the absolute convergence of double Fourier series may be formulated. *E. g.* if the function $f(x_1, x_2)$ belongs to the class $H_{\alpha\beta}^{\alpha'\beta'}$ for some

$\alpha, \beta, \alpha', \beta' > \frac{1}{2}$, then its Fourier series is absolutely convergent. This follows immediately from theorem III in the paper by Reves and Szász ([7], p. 697). Moreover, the proof of this fact is the object of a subsequent paper by Čelidze [2]. Further it is known that if a function $f(x_1, x_2)$ of bounded variations belongs to the class $H_{\alpha\ \beta}^{\alpha'\beta'}$ for some $\alpha, \beta, \alpha', \beta' > 0$, then its Fourier series is absolutely convergent. This is proved by Žak [10](1).

The above theorems may be generalized in various ways. Instead of assuming the Lipschitz conditions, we may use the functions $\omega^{(1,2)}(h_1, h_2)$, $\omega_\cdot^{(2)}(x_1; h_1, h_2)$ and $\omega^{(1)}(x_2; h_1, h_2)$ to formulate more general conditions. The moduli ω can be replaced by integral moduli.

It is known that the absolute convergence (in a set of positive measure) of the double Fourier series

$$f(x_1, x_2) \sim \sum_{m_1, m_2=0}^{\infty} \lambda_{m_1 m_2}(a_{m_1 m_2}\cos m_1 x_1 \cos m_2 x_2 + b_{m_1 m_2}\sin m_1 x_1 \cos m_2 x_2 + c_{m_1 m_2}\cos m_1 x_1 \sin m_2 x_2 + d_{m_1 m_2}\sin m_1 x_1 \sin m_2 x_2),$$

where

$$\lambda_{m_1 m_2} = \begin{cases} \frac{1}{4} & \text{for} \quad m_1 = m_2 = 0, \\ \frac{1}{2} & \text{for} \quad m_1 = 0,\ m_2 > 0 \quad \text{or} \quad m_1 > 0,\ m_2 = 0, \\ 1 & \text{for} \quad m_1 > 0,\ m_2 > 0 \end{cases}$$

and the coefficients $a_{m_1 m_2}, b_{m_1 m_2}, c_{m_1 m_2}, d_{m_1 m_2}$ are defined by the known Euler-Fourier formulas, is equivalent to the convergence of the series

$$(1.4) \qquad \sum_{m_1, m_2=0}^{\infty} (|a_{m_1 m_2}| + |b_{m_1 m_2}| + |c_{m_1 m_2}| + |d_{m_1 m_2}|)$$

(see [7], theorem II). Instead of the convergence of the series (1.4) one can more generally investigate that of the series

$$(1.5) \qquad \sum_{m_1, m_2=0}^{\infty} (|a_{m_1 m_2}|^\gamma + |b_{m_1 m_2}|^\gamma + |c_{m_1 m_2}|^\gamma + |d_{m_1 m_2}|^\gamma),$$

where $0 < \gamma < 2$, or of the series

$$(1.6) \qquad \sum_{m_1, m_2=0}^{\infty} (m_1+1)^{\beta_1}(m_2+1)^{\beta_2}(|a_{m_1 m_2}| + |b_{m_1 m_2}| + |c_{m_1 m_2}| + |d_{m_1 m_2}|)$$

(1) This theorem is already given in paper [7] (corollary on p. 705) but its formulation is not adequate. The authors assume only that the function $f(x_1, x_2)$ is of finite variations and satisfies condition (1.1) for certain $\alpha, \beta > 0$. This, however, is not sufficient for the absolute convergence of the Fourier series of $f(x_1, x_2)$. As a counter-example we may take the function $f(x_1, x_2) = g(x_1)$, $g(x_1)$ being of finite variation and such that the Fourier series of $g(x_1)$ is not absolutely convergent (for such an example, see the function given by the series (1) in [9], p. 136).

with $\beta_1, \beta_2 \geqslant 0$. Sufficient convergence conditions for the series (1.5) are given by Reves and Szász in [7], and for the series (1.6) by Žak in [10]

In all these papers ([2], [7], [10]) the authors prove the convergence of the series with lower summation limits $m_1 = m_2 = 1$ using conditions of the form (1.1). They prove the convergence of the remaining parts of the series applying the known theorems on functions of one variable to the functions $\int_0^{2\pi} f(x, y)\,dx$ and $\int_0^{2\pi} f(x, y)\,dy$.

Bochner [1], and Minakshisundaram and Szász [4] introduce a quite different method in their considerations. Their proofs are based on the notion of *spherical means* introduced by Bochner. Bochner obtains a theorem (see [1], theorem X), with assumptions about derivatives of a suitably defined function $f_x(t)$. Minakshisundaram and Szász assume Lipschitz conditions of the form $|f(x) - f(y)| \leqslant K|x-y|^\alpha$, where x and y are two points of the n-dimensional Euclidean space and $|x-y|$ the distance between x and y. They obtain the convergence of a series of the form (1.5) for $\gamma > 2n/(n+2\alpha)$. For $n > 1$, however, this inequality does not embrace the exponent $\gamma = 1$.

In paper [6] lemma 3 and theorem 2 of the present paper are formulated without proof.

2. The object of the present paper(2) is to generalize the investigations made in [2], [7] and [10] to multiple Fourier series. The method of the proofs differs from that used in the papers mentioned above in that it is direct and does not reduce the n-dimensional case to the $(n-1)$-dimensional one. The notion of spherical means is not applied either. The author conceived the idea of considering this problem and especially of applying the r-th variations (see § 3) in its investigation at the mathematical seminar directed by Prof. Orlicz. I wish to thank Prof. Orlicz for his kind help.

We shall consider real- or complex-valued functions $f(x_1, \ldots, x_n)$ of n real arguments $x_1, \ldots, x_n$, defined in the whole n-dimensional Euclidean space, periodic with period 2 (3) in each variable and integrable in the region $(0 \leqslant x_i \leqslant 2;\ i = 1, 2, \ldots, n)$ with the p-th power for a certain $1 < p \leqslant 2$. Let A be an arbitrary subset of the set $E = (1, 2, \ldots, n)$ and $\bar{A}$ the complement of A with regard to E. The numbers

$$(2.1) \qquad a_{m_1 \ldots m_n}^{A}(f) = \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_n) \prod_{i \in A} \cos m_i \pi x_i \prod_{j \in \bar{A}} \sin m_j \pi x_j \, dx_1 \ldots dx_n$$

(2) The results of which were presented on June 7th, 1956 to the Polish Mathematical Society, Section of Poznań.

(3) We take the period 2 instead of 2π to simplify calculations.

will be called the *Fourier coefficients* of the function $f(x_1, \ldots, x_n)$. Here $A \subset E$, $m_1, \ldots, m_n = 0, 1, 2, \ldots$ and $\prod_{i \in A} a_i$ denotes the product of all a_i with indices $i \in A$. The series

$$(2.2)\qquad f(x_1, \ldots, x_n) \sim \sum_{m_1,\ldots,m_n=0}^{\infty} \lambda_{m_1\ldots m_n} \sum_{A\subset E} a^A_{m_1\ldots m_n}(f) \prod_{i\in A} \cos m_i \pi x_i \prod_{j\in \bar{A}} \sin m_j \pi x_j,$$

where $\lambda_{m_1\ldots m_n} = 2^{-\mu(\bar{H})}$, $H = \underset{\nu}{E}\{m_\nu \neq 0\}$ and $\mu(\bar{H})$ is the number of all elements of the set $\bar{H}$, is the *Fourier series* of the function $f(x_1, \ldots, x_n)$. The sign $\sum_{A\subset E}$ indicates that the summation extends over all subsets A of the set E, including the empty subset.

Using this notation we give some sufficient conditions for the convergence of the series

$$(2.3)\qquad \sum_{m_1,\ldots,m_n=0}^{\infty} (m_1+1)^{\beta_1} \ldots (m_n+1)^{\beta_n} \varrho^{(\gamma)}_{m_1\ldots m_n}(f),$$

where $\beta_1, \ldots, \beta_n \geqslant 0$, $0 < \gamma < 2$ and

$$(2.4)\qquad \varrho^{(\gamma)}_{m_1\ldots m_n}(f) = \sum_{A\subset E} |a^A_{m_1\ldots m_n}(f)|^\gamma.$$

It is clear that the convergence of the series (2.3) with $\beta_1 = \ldots = \beta_n = 0$ and $\gamma = 1$ implies the absolute convergence of the series (2.2).

We shall need the following inequality of F. Riesz (see [8], p. 118):

If $1 < p \leqslant 2$ and $1/p + 1/q = 1$, then we have

$$(2.5)\quad \Big[\sum_{m_1,\ldots,m_n=0}^{\infty} \varrho^{(q)}_{m_1\ldots m_n}(f)\Big]^{1/q} \leqslant M_n \Big[\int_0^2 \ldots \int_0^2 |f(x_1, \ldots, x_n)|^p \, dx_1 \ldots dx_n\Big]^{1/p}$$

with the constant M_n not depending on the function f.

The above notation is very useful. It will be seen in lemma 3 that certain transformations of the function $f(x_1, \ldots, x_n)$ lead to functions whose Fourier coefficients can be obtained from the Fourier coefficients of the function f by certain operations on the set A.

3. Now we shall introduce some symbols analogical to those given in § 1. Here $H = (k_1, \ldots, k_s)$ will denote a non-empty subset of the set E, where $k_1, \ldots, k_s$ are the elements of H.

For $H = (k)$ we write

$$\Delta^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n) = f(x_1, \ldots, x_{k-1}, x_k+h_k, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n),$$

$$F^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n) = f(x_1, \ldots, x_{k-1}, x_k+h_k, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x_k-h_k, x_{k+1}, \ldots, x_n)$$

and for $H = (k_1, \ldots, k_s)$ $(s > 1, k_1 < \ldots < k_s)$

$$\Delta^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n) = \Delta^{(k_s)}[\Delta^{H-(k_s)}(f; x_1, \ldots, x_n; h_1, \ldots, h_n)],$$

$$F^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n) = F^{(k_s)}[F^{H-(k_s)}(f; x_1, \ldots, x_n; h_1, \ldots, h_n)].$$

These definitions imply, for $H = (k_1, \ldots, k_s)$,

$$(3.1)\qquad F^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n) = \Delta^H(f; x_1, \ldots, x_{k_1-1}, x_{k_1}-h_{k_1}, x_{k_1+1}, \ldots, x_{k_s-1}, x_{k_s}-h_{k_s}, x_{k_s+1}, \ldots, x_n; h_1, \ldots, h_{k_1-1}, 2h_{k_1}, h_{k_1+1}, \ldots, h_{k_s-1}, 2h_{k_s}, h_{k_s+1}, \ldots, h_n).$$

Further we put

$$\omega^H(x_{l_1}, \ldots, x_{l_{n-s}}; h_1, \ldots, h_n) = \sup_{\substack{0\leqslant x_i\leqslant 2\\ |\delta_i|\leqslant h_i; i\in H}} |\Delta^H(f; x_1, \ldots, x_n; \delta_1, \ldots, \delta_n)|,$$

where $(l_1, \ldots, l_{n-s}) = \bar{H}$, and

$$\omega^H_p(h_1, \ldots, h_n) = \Big[\sup_{|\delta_i|\leqslant h_i; i=1,\ldots,n} \int_0^2 \ldots \int_0^2 |\Delta^H(f; x_1, \ldots, x_n; \delta_1, \ldots, \delta_n)|^p dx_1 \ldots dx_n\Big]^{1/p}$$

for $p \geqslant 1$. The functions ω^H and ω^H_p will be called the *modulus of continuity* and the *p-th integral modulus* of the function $f(x_1, \ldots, x_n)$ with regard to the set H, respectively. We obviously have

$$\omega^H_{p'}(h_1, \ldots, h_n) \leqslant 2^{n(1/p'-1/p'')} \omega^H_{p''}(h_1, \ldots, h_n) \quad \text{for} \quad 1 \leqslant p' \leqslant p''.$$

Moreover, (3.1) implies

$$(3.2)\quad \Big[\int_0^2 \ldots \int_0^2 |F^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n)|^p dx_1 \ldots dx_n\Big]^{1/p} \leqslant \omega^H_p(2h_1, \ldots, 2h_n).$$

We shall need the notion of *r-th variation* $(r \geqslant 1)$ of the function $f(x_1, \ldots, x_n)$ with respect to the set $H \neq 0$ in the region $(a_i \leqslant x_i \leqslant b_i;\ i = 1, 2, \ldots n)$[4]. Let $H = (k_1, \ldots, k_s) \neq 0$ and let Π be the following

(4) Instead of *r-th variations* one can consider *Φ-th variations*, as introduced for one variable by L. C. Young. $\Phi(u)$ is here a non-negative and non-decreasing function. In order to obtain the *Φ-th wariation*, $\Phi(|\Delta^H|)$ should be taken in (3.3) instead of $|\Delta^H|^r$ and the exponent $1/r$ omitted. A variation of this kind was used by the author in investigations of single Fourier series in [5]. We here give up this degree of generality to simplify the notation.

partition of the n-dimensional parallelepiped ($a_i \leqslant x_i \leqslant b_i$; $i = 1, 2, \ldots, n$) into partial n-dimensional parallelepipeds:

$$a_i = x_i^{(0)} < x_i^{(1)} < \ldots < x_i^{(N_i)} = b_i.$$

We introduce

$$(3.3)\quad V_r^H(f) = \Big[\sup_{\substack{a_i \leqslant x_i \leqslant b_i \\ i \in H}} \sup_{\Pi} \sum_{i_{k_1}=1}^{N_{k_1}} \ldots \sum_{i_{k_s}=1}^{N_{k_s}} |\Delta^H(f; x_1, \ldots, x_{k_1-1}, x_{k_1}^{(i_{k_1}-1)}, x_{k_1+1}, \ldots, x_{k_s-1}, x_{k_s}^{(i_{k_s}-1)}, x_{k_s+1}, \ldots, x_n; x_1^{(i_1)} - x_1^{(i_1-1)}, \ldots, x_n^{(i_n)} - x_n^{(i_n-1)})|^r\Big]^{1/r}.$$

4. We now proceed to give three lemmas not directly connected with the absolute convergence of Fourier series.

LEMMA 1. *If $q > \gamma > 0$ and $a_1, \ldots, a_N > 0$ then*

$$2^{-N+1}\Big(\sum_{i=1}^{N} a_i^q\Big)^{\gamma/q} \leqslant \sum_{i=1}^{N} a_i^{\gamma} \leqslant 2^{N-1}\Big(\sum_{i=1}^{N} a_i^q\Big)^{\gamma/q}.$$

This known lemma can be proved by induction.

LEMMA 2. *Let the function $f(x_1, \ldots, x_n)$ be defined in the whole n-dimensional Euclidean space and periodic in the variable x_i with the period $b_i - a_i$ for each $i \in H$. We assume that $H = (k_1, \ldots, k_s) \neq 0$ and that the numbers $h_i > 0$ satisfy the following conditions: for each $i \in H$ there exists an integer p_i such that $2h_i p_i = b_i - a_i$*[5]. *Then*

$$\int_{a_{k_1}}^{b_{k_1}} \ldots \int_{a_{k_s}}^{b_{k_s}} |F^H(f; x_1, \ldots, x_n; h_1, \ldots, h_n)|^r dx_{k_1} \ldots dx_{k_s} \leqslant 2^s [V_r^H(f)]^r \prod_{i \in H} h_i.$$

Proof. Putting $x_j^{(i_j)} - x_j^{(i_j-1)} = 2h_j$ $(i_j = 1, 2, \ldots, N_j;\ j \in H)$ and $x_{k_j} = x_{k_j}^{(i_j-1)} + t_j$ for $k_j \in H$ we have

$$\int_{a_{k_1}}^{b_{k_1}} \ldots \int_{a_{k_s}}^{b_{k_s}} |F^H(f; x_1, \ldots, x_n; h_1 \ldots, h_n)|^r dx_{k_1} \ldots dx_{k_s}$$

$$= \int_{a_{k_1}}^{b_{k_1}} \ldots \int_{a_{k_s}}^{b_{k_s}} |\Delta^H(f; x_1, \ldots, x_n; 2h_1, \ldots, 2h_n)|^r dx_{k_1} \ldots dx_{k_s}$$

$$= \sum_{i_{k_1}=1}^{N_{k_1}} \ldots \sum_{i_{k_s}=1}^{N_{k_s}} \int_{x_{k_1}^{(i_{k_1}-1)}}^{x_{k_1}^{(i_{k_1})}} \ldots \int_{x_{k_s}^{(i_{k_s}-1)}}^{x_{k_s}^{(i_{k_s})}} |\Delta^H(f; x_1, \ldots, x_n; 2h_1, \ldots, 2h_n)|^r dx_{k_1} \ldots dx_{k_s}$$

[5] The author admits the possibility of certain of these assumptions being superfluous; however, they make the proof simpler.

$$= \sum_{i_{k_1}=1}^{N_{k_1}} \ldots \sum_{i_{k_s}=1}^{N_{k_s}} \int_{x_{k_1}^{(i_{k_1}-1)}}^{x_{k_1}^{(i_{k_1})}} \ldots \int_{x_{k_s}^{(i_{k_s}-1)}}^{x_{k_s}^{(i_{k_s})}} |\Delta^H(f; x_1, \ldots, x_n; x_1^{(i_1)} - x_1^{(i_1-1)}, \ldots, x_n^{(i_n)} - x_n^{(i_n-1)})|^r dx_{k_1} \ldots dx_{k_s}$$

$$= \sum_{i_{k_1}=1}^{N_{k_1}} \ldots \sum_{i_{k_s}=1}^{N_{k_s}} \int_0^{2h_{k_1}} \ldots \int_0^{2h_{k_s}} |\Delta^H(f; x_1, \ldots, x_{k_1-1}, x_{k_1}^{(i_{k_1}-1)} + t_1, x_{k_1+1}, \ldots, x_{k_s-1}, x_{k_s}^{(i_{k_s}-1)} + t_s, x_{k_s+1}, \ldots, x_n; x_1^{(i_1)} - x_1^{(i_1-1)}, \ldots, x_n^{(i_n)} - x_n^{(i_n-1)})|^r dt_1 \ldots dt_s$$

$$\leqslant 2^s [V_r^H(f)]^r \prod_{i \in H} h_i.$$

LEMMA 3. *For the function F^H introduced in § 3 we have, for arbitrary $A \subset E$,*

$$(4.1)\qquad a_{m_1 \ldots m_n}^A(F^H) = (-1)^{\mu(\bar{A} \cap H)} 2^{\mu(H)} \prod_{i \in H} \sin m_i \pi h_i \, a_{m_1 \ldots m_n}^{A \triangle H}(f),$$

where $\mu(B)$ denotes the number of all elements of the set B and $A \triangle H$ the symmetric difference of the sets A and H. Moreover, given H, the operation $A \triangle H$ defines a one-to-one correspondence from E to E.

Proof. Let $H = (k_1, \ldots, k_s)$. We prove formula (4.1) by induction with respect to s. Suppose $s = 1$, *i. e.* $H = (k)$. Let $k \in A$. Then

$$(4.2)\qquad \int_0^2 \ldots \int_0^2 f(x_1, \ldots, x_{k-1}, x_k + h_k, x_{k+1}, \ldots, x_n) \prod_{i \in A} \cos m_i \pi h_i \times$$

$$\times \prod_{j \in \bar{A}} \sin m_j \pi h_j dx_1 \ldots dx_n = a_{m_1 \ldots m_n}^A(f) \cos m_k \pi h_k + a_{m_1 \ldots m_n}^{A-(k)}(f) \sin m_k \pi h_k.$$

Putting $-h_k$ instead of h_k in (4.2) and subtracting from (4.2) the equality thus obtained, we get

$$(4.3)\qquad a_{m_1 \ldots m_n}^A(F^H) = 2 a_{m_1 \ldots m_n}^{A-(k)}(f) \sin m_k \pi h_k.$$

However, $\bar{A} \cap H = 0$, whence $(-1)^{\mu(\bar{A} \cap H)} = 1$ and $A \triangle H = A - (k)$; thus, (4.3) implies (4.1). Now let $k \in \bar{A}$. Similarly to (4.3), the following equality is easily obtained:

$$a_{m_1 \ldots m_n}^A(F^H) = -2 a_{m_1 \ldots m_n}^{A \cup (k)}(f) \sin m_k \pi h_k.$$

Since $\bar{A} \cap H = (k)$, $(-1)^{\mu(\bar{A} \cap H)} = -1$ and $A \triangle H = A \cup (k)$, we have (4.1). Thus, formula (4.1) is proved for $s = 1$.

Suppose formula (4.1) is true for $s-1$ ($s \leqslant n$). We shall now prove it for s. Putting $H = (k_1, \ldots, k_s)$, we have

$$a^{A}_{m_1\ldots m_n}[F^{H}(f)] = a^{A}_{m_1\ldots m_n}[F^{(k_s)}(F^{H-(k_s)})] \tag{4.4}$$

$$= (-1)^{\mu[\bar{A}\cap(k_s)]} 2\sin m_{k_s}\pi h_{k_s} a^{A\triangle(k_s)}_{m_1\ldots m_n}[F^{H-(k_s)}(f)]$$

$$= (-1)^{\mu[\bar{A}\cap(k_s)]} 2\sin m_{k_s}\pi h_{k_s}(-1)^{\mu(A_1)} 2^{s-1}\prod_{i=1}^{s-1}\sin m_{k_i}\pi h_{k_i} a^{A_2}_{m_1\ldots m_n}(f)$$

$$= (-1)^{\mu[\bar{A}\cap(k_s)]+\mu(A_1)} 2^{\mu(H)}\prod_{i\in H}\sin m_i\pi h_i a^{A_2}_{m_1\ldots m_n}(f),$$

where

$$A_1 = \overline{A\triangle(k_s)}\cap(k_1, \ldots, k_{s-1}), \qquad A_2 = [A\triangle(k_s)]\triangle(k_1, \ldots, k_{s-1}).$$

However,

$$\overline{A\triangle(k_s)}\cap(k_1, \ldots, k_{s-1}) = \bar{A}\cap(k_1, \ldots, k_{s-1}) \tag{4.5}$$

whence

$$\mu[\bar{A}\cap(k_s)]+\mu(A_1) = \mu(\bar{A}\cap H). \tag{4.6}$$

Further

$$[A\triangle(k_s)]\cap\overline{(k_1, \ldots, k_{s-1})} = A\cap\bar{H}\cup\bar{A}\cap(k_s),$$

therefore (4.5) implies

$$A_2 = A\triangle H. \tag{4.7}$$

(4.1) follows from (4.4), (4.6) and (4.7).

The one-to-one correspondence follows from the simple formula $A = (A\triangle H)\triangle H$.

5. In order to formulate two auxiliary lemmas we put for $i = 1, 2, \ldots, n$:

$$A_{i;\nu} = \underset{(m_1,\ldots,m_n)}{E}\{2^{\nu-1} \leqslant m_i < 2^{\nu}\} \quad \text{for} \quad \nu \geqslant 1, \quad A_{i;0} = \underset{(m_1,\ldots,m_n)}{E}\{m_i = 0\},$$

$$A_{\nu_1\ldots\nu_n} = \bigcap_{i=1}^{n} A_{i;\nu_i} \quad \text{for} \quad \nu_i \geqslant 0.$$

Here $\underset{(m_1,\ldots,m_n)}{E}\{\ \}$ denotes the set of all systems of n non-negative integers $(m_1, \ldots, m_n)$ satisfying the condition contained in the brackets $\{\ \}$. It is clear that for two different systems $(\nu_1, \ldots, \nu_n)$ (that is, for two systems having different integers in one place at least), the corresponding sets $A_{\nu_1\ldots\nu_n}$ are disjoint and the set

$$\bigcup_{\nu_1,\ldots,\nu_n=0}^{\infty} A_{\nu_1\ldots\nu_n}$$

is identical with the set of all systems $(m_1, \ldots, m_n)$ of non-negative integers. Further we put for a given system $(\nu_1, \ldots, \nu_n)$,

$$H = \underset{i}{E}\{\nu_i \neq 0\}$$

and assume that $H \neq 0$. It is now possible to formulate lemmas 4 and 5, denoting by $\sum_{(m_1,\ldots,m_n)\in A_{\nu_1\ldots\nu_n}}$ summation over all systems $(m_1, \ldots, m_n) \in A_{\nu_1\ldots\nu_n}$.

Lemma 4. *We take* $1 \leqslant r \leqslant p \leqslant 2$, $1/p+1/q = 1$ ($p \neq 1$), $H = (k_1, \ldots, k_s)$, $\bar{H} = (l_1, \ldots, l_{n-s})$ *and assume the function* $f(x_1, \ldots, x_n)$ *to be integrable in the region* ($0 \leqslant x_i \leqslant 2$; $i = 1, 2, \ldots, n$) *with the p-th power Then*

$$\sum_{(m_1,\ldots,m_n)\in A_{\nu_1\ldots\nu_n}} \varrho^{(q)}_{m_1\ldots m_n}(f) \tag{5.1}$$

$$\leqslant M_n^q 2^{-q\mu(H)/2}\Big\{\int_0^2\cdots\int_0^2 [\omega^H(x_{l_1}, \ldots, x_{l_{n-s}}; 2^{-\nu_1}, \ldots, 2^{-\nu_n})]^{p-r}\times$$

$$\times\Big[\int_0^2\cdots\int_0^2 |F^H(f; x_1, \ldots, x_n; 2^{-\nu_1-1}, \ldots, 2^{-\nu_n-1})|^r dx_{k_1}\ldots dx_{k_s}\Big] dx_{l_1}\ldots dx_{l_{n-s}}\Big\}^{1/(p-1)}.$$

Proof. According to lemma 3, the operation $A\triangle H$ defines a one-to-one correspondence from E to E. Hence

$$\varrho^{(q)}_{m_1\ldots m_n}(f) = \sum_{A\subset E} |a^{A\triangle H}_{m_1\ldots m_n}(f)|^q. \tag{5.2}$$

For $(m_1, \ldots, m_n) \in A_{\nu_1\ldots\nu_n}$ and $i \in H$ we have $2^{\nu_i-1} \leqslant m_i < 2^{\nu_i}$, whence $|\sin m_i\pi 2^{-\nu_i-1}|^q \geqslant 2^{-q/2}$. Applying this inequality and formula (5.2) we obtain

$$\varrho^{(q)}_{m_1\ldots m_n}(f) \leqslant 2^{-q\mu(H)/2}\sum_{A\subset E}|(-1)^{\mu(\bar{A}\cap H)} 2^{\mu(H)}\prod_{i\in H}\sin m_i\pi 2^{-\nu_i-1} a^{A\triangle H}_{m_1\ldots m_n}(f)|^q.$$

Thus, formula (4.1) yields

$$\varrho^{(q)}_{m_1\ldots m_n}(f) \leqslant 2^{-q\mu(H)/2}\sum_{A\subset E}|a^{A}_{m_1\ldots m_n}[F^H(f)]|^q, \tag{5.3}$$

where $h_i = 2^{-\nu_i-1}$ for $i = 1, 2, \ldots, n$.

Applying (5.3) and subsequently the inequality (2.5) to function F^H we obtain the inequality

$$\sum_{(m_1,\ldots,m_n)\in A_{\nu_1\ldots\nu_n}} \varrho^{(q)}_{m_1\ldots m_n}(f)$$

$$\leqslant M_n^q 2^{-q\mu(H)/2}\Big[\int_0^2\cdots\int_0^2 |F^H(f; x_1, \ldots, x_n; 2^{-\nu_1-1}, \ldots, 2^{-\nu_n-1})|^p dx_1\ldots dx_n\Big]^{q/p},$$

which implies (5.1).

LEMMA 5. *With the same notation and assumptions as in lemma 4, we have for $\beta_1, \ldots, \beta_n \geqslant 0$ and $0<\gamma<2$:*

$$(5.4)\quad \sum_{(m_1,\ldots,m_n)\in\Delta_{\nu_1\ldots\nu_n}} (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$$
$$\leqslant M_n^\gamma 2^{\sum_{i=1}^n\left(\beta_i+\frac{q-\gamma}{q}\right)\nu_i-\frac{\gamma\mu(H)}{2}} \Big\{\int_0^2\ldots\int_0^2 [\omega^H(x_{l_1},\ldots,x_{l_{n-s}};2^{-\nu_1},\ldots,2^{-\nu_n})]^{p-r}\times$$
$$\times\Big[\int_0^2\ldots\int_0^2|F^H(f;x_1,\ldots,x_n;2^{-\nu_1-1},\ldots,2^{-\nu_n-1})|^r dx_{k_1}\ldots dx_{k_s}\Big]dx_{l_1}\ldots dx_{l_{n-s}}\Big\}^{\gamma/p}.$$

Proof. Applying the Hölder inequality $\sum uv\leqslant(\sum u^k)^{1/k}(\sum v^l)^{1/l}$ with $k=q/(q-\gamma)$, $l=q/\gamma$, $u=(m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}$ and $v=[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$ to the sum on the left-hand side of inequality (5.4), we obtain

$$(5.5)\quad \sum_{(m_1,\ldots,m_n)\in\Delta_{\nu_1\ldots\nu_n}} (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$$
$$\leqslant\Big\{\sum_{(m_1,\ldots,m_n)\in\Delta_{\nu_1\ldots\nu_n}}[(m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}]^{q/(q-\gamma)}\Big\}^{1-\gamma/q}\times$$
$$\times\Big\{\sum_{(m_1,\ldots,m_n)\in\Delta_{\nu_1\ldots\nu_n}}\varrho^{(q)}_{m_1\ldots m_n}(f)\Big\}^{\gamma/q}.$$

However, it is easy to see that

$$(5.6)\quad \sum_{(m_1,\ldots,m_n)\in\Delta_{\nu_1\ldots\nu_n}}[(m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}]^{q/(q-\gamma)}\leqslant 2^{\sum_{i=1}^n\left(\frac{q\beta_i}{q-\gamma}+1\right)\nu_i}.$$

Applying inequality (5.6) and lemma 4 to the right-hand side of inequality (5.5) we obtain (5.4).

6. We shall prove three theorems on the absolute convergence of multiple Fourier series.

THEOREM 1. *Let $\mathcal{H}$ be a class of non-empty subsets of the set $E=(1,2,\ldots,n)$*(6)*. Let us assume that the function $f(x_1,\ldots,x_n)$ defined in the whole n-dimensional Euclidean space satisfies the following conditions:*

1° the function is periodic with period 2 in each variable,

2° it is integrable in the region $(0\leqslant x_i\leqslant 2;\ i=1,2,\ldots,n)$ with the p-th power for a certain $1<p\leqslant 2$,

3° for every $H\in\mathcal{H}$ there exists a number r_H such that $1\leqslant r_H\leqslant p$ and $V^H_{r_H}(f)<\infty$.

(6) The class $\mathcal{H}$ may be empty.

We further assume, for certain $\beta_1,\ldots,\beta_n\geqslant 0$ and $0<\gamma<2$, the convergence of the series

$$(a)\quad \sum_{\nu_{k_1}=1}^\infty\ldots\sum_{\nu_{k_s}=1}^\infty 2^{\sum_{i=1}^s(\beta_{k_i}+1-\gamma)\nu_{k_i}}\times$$
$$\times\Big\{\int_0^2\ldots\int_0^2[\omega^H(x_{l_1},\ldots,x_{l_{n-s}};2^{-\nu_1},\ldots,2^{-\nu_n})]^{p-r_H}dx_{l_1}\ldots dx_{l_{n-s}}\Big\}^{\gamma/p}\quad \text{for } H\in\mathcal{H},$$

$$(b)\quad \sum_{\nu_{k_1}=1}^\infty\ldots\sum_{\nu_{k_s}=1}^\infty 2^{\sum_{i=1}^s[\beta_{k_i}+1-(1-1/p)\gamma]\nu_{k_i}}[\omega^H_p(2^{-\nu_1},\ldots,2^{-\nu_n})]^\gamma\quad \text{for } H\notin\mathcal{H},\ H\neq 0,$$

where $H=(k_1,\ldots,k_s)$, $\bar H=(l_1,\ldots,l_{n-s})$. Then the series (2.3) is convergent.

Proof. According to lemma 1, the convergence of the series (2.3) is equivalent to that of the series

$$(6.1)\quad \sum_{m_1,\ldots,m_n=0}^\infty (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$$

with arbitrary $q\geqslant 2$, or to that of the series

$$(6.2)\quad \sum_{\substack{m_1,\ldots,m_n=0\\(m_1,\ldots,m_n)\neq 0}}^\infty (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$$

obtained from (6.1) by dropping the term with indices $m_1=\ldots=m_n=0$.

We denote by $\sum_{0\neq H\subset E}$ summation extended over all non-empty subsets H of the set E. Then, putting $H=\underset{i}{E}\{\nu_i\neq 0\}=(k_1,\ldots,k_s)$, and applying lemma 5 (with $r_H=p$ for $H\notin\mathcal{H}$) we obtain

$$\sum_{\substack{m_1,\ldots,m_n=0\\(m_1,\ldots,m_n)\neq 0}}^\infty (m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$$
$$=\sum_{0\neq H\subset E}\sum_{\nu_{k_1}=1}^\infty\ldots\sum_{\nu_{k_s}=1}^\infty\sum_{(m_1,\ldots,m_n)\in\Delta_{\nu_1\ldots\nu_n}}(m_1+1)^{\beta_1}\ldots(m_n+1)^{\beta_n}[\varrho^{(q)}_{m_1\ldots m_n}(f)]^{\gamma/q}$$
$$\leqslant M_n^\gamma\sum_{0\neq H\subset E}2^{-\gamma\mu(H)/2}\sum_{\nu_{k_1}=1}^\infty\ldots\sum_{\nu_{k_s}=1}^\infty 2^{\sum_{i=1}^s\left(\beta_{k_i}+\frac{q-\gamma}{q}\right)\nu_{k_i}}\times$$
$$\times\Big\{\int_0^2\ldots\int_0^2[\omega^H(x_{l_1},\ldots,x_{l_{n-s}};2^{-\nu_1},\ldots,2^{-\nu_n})]^{p-r_H}\times$$
$$\times\Big[\int_0^2\ldots\int_0^2|F^H(f;x_1,\ldots,x_n;2^{-\nu_1-1},\ldots,2^{-\nu_n-1})|^{r_H}dx_{k_1}\ldots dx_{k_s}\Big]dx_{l_1}\ldots dx_{l_{n-s}}\Big\}^{\gamma/p}.$$

This inequality proves that the series (6.2) and therefore also (2.3) are convergent if the series

$$(6.3)\quad \sum_{\nu_{k_1}=1}^{\infty}\cdots\sum_{\nu_{k_s}=1}^{\infty} 2^{\sum_{i=1}^{s}\left(\beta_{k_i}+\frac{q-\gamma}{q}\right)\nu_{k_i}} \Big\{\int_0^2\cdots\int_0^2 [\omega^H(x_{l_1},\ldots,x_{l_{n-s}};2^{-\nu_1},\ldots,2^{-\nu_n})]^{p-r_H}\times$$

$$\times\Big[\int_0^2\cdots\int_0^2 |F^H(f;x_1,\ldots,x_n;2^{-\nu_1-1},\ldots,2^{-\nu_n-1})|^{r_H}dx_{k_1}\ldots dx_{k_s}\Big]dx_{l_1}\ldots dx_{l_{n-s}}\Big\}^{\gamma/p}$$

are convergent for every $H\neq 0$, $H\subset E$.

Suppose $H\notin\mathcal{H}$, $H\neq 0$. Then inequality (3.2) with $r_H=p$ yields

$$\Big[\int_0^2\cdots\int_0^2 |F^H(f;x_1,\ldots,x_n;2^{-\nu_1-1},\ldots,2^{-\nu_n-1})|^{r_H}dx_1\ldots dx_n\Big]^{1/p} \leqslant \omega_p^H(2^{-\nu_1},\ldots,2^{-\nu_n})$$

and the convergence of the series (b) implies that of the series (6.3).

Suppose $H\in\mathcal{H}$. Then lemma 2 yields

$$\int_0^2\cdots\int_0^2 |F^H(f;x_1,\ldots,x_n;2^{-\nu_1-1},\ldots,2^{-\nu_n-1})|^{r_H}dx_{k_1}\ldots dx_{k_s}\leqslant [V_{r_H}^H(f)]^{r_H}2^{-\sum_{i=1}^{s}\nu_{k_i}}.$$

This, together with the convergence of the series (a) implies that of the series (6.3).

THEOREM 1'. *Let the function $f(x_1,\ldots,x_n)$ satisfy for a class $\mathcal{H}$ of non-empty subsets of the set $E=(1,2,\ldots,n)$ the assumptions 1°, 2° and 3° of theorem 1. Further assume, for certain $\beta_1,\ldots,\beta_n\geqslant 0$ and $0<\gamma<2$, the convergence of the series*

$$(\mathrm{a}_1)\quad \sum_{\nu_{k_1}=1}^{\infty}\cdots\sum_{\nu_{k_s}=1}^{\infty}\prod_{i\in H}\nu_i^{\beta_i-\gamma}\times$$

$$\times\Big\{\int_0^2\cdots\int_0^2[\omega^H(x_{l_1},\ldots,x_{l_{n-s}};\nu_1^{-1},\ldots,\nu_n^{-1})]^{p-r_H}dx_{l_1}\ldots dx_{l_{n-s}}\Big\}^{\gamma/p}\quad\text{for}\quad H\in\mathcal{H},$$

$$(\mathrm{b}_1)\quad \sum_{\nu_{k_1}=1}^{\infty}\cdots\sum_{\nu_{k_s}=1}^{\infty}\prod_{i\in H}\nu_i^{\beta_i-(1-1/p)\gamma}[\omega_p^H(\nu_1^{-1},\ldots,\nu_n^{-1})]^{\gamma}\quad\text{for}\quad H\notin\mathcal{H},\ H\neq 0,$$

where $H=(k_1,\ldots,k_s)$, $\bar{H}=(l_1,\ldots,l_{n-s})$. Then the series (2.3) is convergent.

The proof of this theorem is obtained similarly to that of theorem 1, after a suitable modification of lemma 5.

THEOREM 2. *Let the function $f(x_1,\ldots,x_n)$ satisfy for a class $\mathcal{H}$ of non-empty subsets of the set $E=(1,2,\ldots,n)$*(7) *the assumptions 1°, 2° and 3°*

(7) See (6).

of theorem 1. Let us further assume, for certain $\beta_1,\ldots,\beta_n\geqslant 0$ and $0<\gamma<2$, that the following conditions are satisfied:

(a_2) *for $H=(k_1,\ldots,k_s)\in\mathcal{H}$ we have*

$$(6.4)\quad \omega^H(x_{l_1},\ldots,x_{l_{n-s}};h_1,\ldots,h_n)\leqslant K_H(x_{l_1},\ldots,x_{l_{n-s}})h_{k_1}^{\alpha_1^H}\ldots h_{k_s}^{\alpha_s^H},$$

where

$$(l_1,\ldots,l_{n-s})=\bar{H},\qquad \int_0^2\cdots\int_0^2[K_H(x_{l_1},\ldots,x_{l_{n-s}})]^{p-r_H}dx_{l_1}\ldots dx_{l_{n-s}}<\infty$$

and

$$(6.5)\quad \alpha_i^H>\frac{p(\beta_{k_i}+1-\gamma)}{\gamma(p-r_H)}\quad\text{for}\quad r_H\neq p,\qquad \beta_{k_i}<\gamma-1\quad\text{for}\quad r_H=p;$$

(b_2) *for $H=(k_1,\ldots,k_s)\notin\mathcal{H}$ and $H\neq 0$ we have*

$$(6.6)\quad \omega_p^H(h_1,\ldots,h_n)\leqslant K_H h_{k_1}^{\alpha_1^H}\ldots h_{k_s}^{\alpha_s^H},$$

where K_H is a constant and

$$(6.7)\quad \alpha_i^H>\frac{p(\beta_{k_i}+1-\gamma)+\gamma}{\gamma p}.$$

Then the series (2.3) is convergent.

Proof. According to inequality (6.4), the series (a) in theorem 1 is majorized by the series

$$\sum_{\nu_{k_1}=1}^{\infty}\cdots\sum_{\nu_{k_s}=1}^{\infty}2^{\sum_{i=1}^{s}[\beta_{k_i}+1-\gamma-\alpha_i^H\gamma(1-r_H/p)]\nu_{k_i}},$$

the convergence of which is assured by inequality (6.5). Similarly (6.6) implies that the series (b) in theorem 1 is majorized by the series

$$\sum_{\nu_{k_1}=1}^{\infty}\cdots\sum_{\nu_{k_s}=1}^{\infty}2^{\sum_{i=1}^{s}[\beta_{k_i}+1-(1-1/p)\gamma-\alpha_i^H\gamma]\nu_{k_i}},$$

which, according to (6.7), is convergent. Then the assumptions in theorem 1 are satisfied, thus implying the convergence of the series (2.3).

7. It should be noted that the above theorems constitute generalizations of results concerning the absolute convergence of double Fourier series given in papers [2], [7] and [10]. Putting $n=p=2$, $\beta_1=\beta_2=0$, $\mathcal{H}=0$ in theorem 1', we obtain theorem III in paper [7]. Putting $n=p=2$, $\beta_1=\beta_2=0$, $\gamma=1$, $\mathcal{H}=0$ in theorem 2, we obtain the condition

$a_i^H > \frac{1}{2}$ (see [2]). Taking for $\mathfrak{N}$ in theorem 2 the class of all non-empty subsets of the set $E = (1, 2)$, $n = p = 2$, $r_H = \gamma = 1$, we obtain the condition of Žak, $a_i^H > 2\beta_{k_i}$ (see [10]). The counter-examples given in these papers and concerning the strength of the results do not exhaust this problem. The author intends to take up this problem once more together with investigations concerning conditions of type (6.4) and (6.6).

Theorems 1, 1′ and 2 generalize many known theorems holding for single Fourier series to multiple Fourier series, *e. g.* all theorems in §§ 6.3, 6.31-6.34 and 6.6.5-6 in [9] (8).

The results obtained may be generalized to certain classes of almost periodic functions of n variables, by the method used in [5].

References

[1] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. 40 (1936), p. 175-207.

[2] З. Г. Челидзе, *Об абсолютной сходимости двойных рядов Фурье*, Доклады Акад. Наук СССР 54 (1946), p. 117-120.

[3] E. W. Hobson, *Theory of functions of a real variable I*, Cambridge 1927.

[4] S. Minakshisundaram and O. Szász, *On absolute convergence of multiple Fourier series*, Trans. Amer. Math. Soc. 61 (1947), p. 36-53.

[5] J. Musielak, *O bezwzględnej zbieżności szeregów Fouriera pewnych funkcji prawie okresowych*, Zeszyty Nauk. Uniw. im. A. Mickiewicza 1 (1957), p. 9-17.

[6] — *Some conditions sufficient for the absolute convergence of multiple Fourier series*, Bull. Acad. Polon. Sci., Cl. III, 5(1957), p. 251-254.

[7] G. E. Reves and O. Szász, *Some theorems on double trigonometric series*, Duke Math. Journ. 9 (1942), p. 693-705.

[8] F. Riesz, *Über eine Verallgemeinerung der Parsevalschen Formel*, Math. Zeitschr. 18 (1923), p. 117-124.

[9] A. Zygmund, *Trigonometrical series*, Warszawa-Lwów 1935.

[10] И. Е. Жак, *К абсолютной сходимости двойных рядов Фурье*, Сообщ. Ак. Наук Груз. ССР 12 (1951), p. 129-133.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 23. 6. 1956

(8) It is easily seen that in theorem 6.31 in [9] it is sufficient to assume that $f(x)$ is of finite *r-th variation* for a certain $r < 2$ (see also [5], theorem 5).

ANNALES
POLONICI MATHEMATICI
V. (1958)

Sur un problème mixte pour l'équation du type hyperbolique

par G. Majcher (Kraków)

I. Cas de la dépendance linéaire entre la fonction $u(x, y)$ et ses dérivées sur la courbe Γ

§ 1. Enoncé du problème. Considérons une équation du type hyperbolique à deux variables indépendantes, réduite à la forme canonique

$$H[u] \equiv u''_{xy} + a(x, y)u'_x + b(x, y)u'_y + c(x, y)u = f(x, y). \tag{1}$$

Soit D un domaine limité par la demi-droite caractéristique $y = 0$, $x \geqslant 0$ de l'équation (1), par la droite $x = x_0$, $x_0 > 0$ et par une courbe Γ issue de l'origine et représentée par l'équation:

$$x = \theta(y), \quad y \geqslant 0 \quad (\text{ou} \quad y = \tau(x), \quad x \geqslant 0),$$

où $\theta'(y) > 0$, $\theta(0) = 0$.

Nous nous proposons de trouver une intégrale $u(x, y)$ de l'équation (1) *qui soit de classe C^1 dans la fermeture $\bar{D}$ du domaine D, admette une dérivée partielle u''_{xy} continue dans cet ensemble et satisfasse aux conditions aux limites*

$$A(y)u'_x(x, y) + B(y)u'_y(x, y) + C(y)u(x, y) = g(y), \tag{2}$$

pour $x = \theta(y)$, $y \geqslant 0$ *et*

$$u(x, 0) = h(x) \quad \textit{pour} \quad x \geqslant 0. \tag{3}$$

Dans la suite le problème posé sera appelé brièvement *problème* (M) (1).

Nous démontrerons dans cette partie du travail l'existence de la solution du problème (M).

§ 2. Existence de la solution du problème (M). 1. Supposons vérifiées les hypothèses suivantes:

(1) Ce problème a été posé par M. Krzyżański.