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On a class of double sequence transformations

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1. Introduction and definitions: A double sequence $\{s_{pq}\}$ is said to be *bounded* if $|s_{pq}| \leqslant k$ for all p and q and it is said to be *convergent*, in the Pringsheim sense, if $\lim_{p,q\to\infty} s_{pq}$ exists and is finite. If in addition to convergence $\lim_{p\to\infty} s_{pq}$ for each q and $\lim_{q\to\infty} s_{pq}$ for each p exist, then the sequence is said to be *regularly convergent* and the space of such sequences is denoted by $(\boldsymbol{rc})$. If the limits defined above are all equal, to l say, then the sequence $\{s_{pq}\}$ is said to be *perfectly convergent* to l (see, for example, Alexiewicz and Orlicz [2]). The space of perfectly convergent sequences is denoted by $\boldsymbol{P}$ and that of perfectly convergent sequences with limit zero is denoted by $\boldsymbol{P}_0$. The space $\boldsymbol{P}_0$ is called $(\boldsymbol{rcrn})$ by Hamilton [4], [5], but the notation $\boldsymbol{P}_0$ is due to Alexiewicz and Orlicz [2].

Transformations of double sequences: Let $A \equiv (a_{mnpq})$ be a four-dimensional matrix for $m, n, p, q = 0, 1, 2, \ldots$ Then the set of equations

$$t_{mn} = A_{mn}(s) = \sum_{p,q=0}^{\infty} a_{mnpq} s_{pq} \tag{1}$$

determines the transformation of $\{s_{pq}\}$ into $\{t_{mn}\}$, which we shall call the *A-transformation of* $\{s_{pq}\}$.

Defining the product of two four-dimensional matrices A and B by the equation

$$(AB)_{mnrs} = \sum_{p,q=0}^{\infty} a_{mnpq} b_{pqrs} \tag{2}$$

we see that (1) can be written in the form

$$t = As$$

where t is the matrix defined by $t_{mnrs} = 0$ $(r, s \neq 0)$, $t_{mn00} = t_{mn}$ and s, the matrix defined by $s_{mnpq} = 0$ $(p, q \neq 0)$, $s_{mn00} = s_{mn}$.

The matrix A will be said to be *completely conservative* for a certain space, not containing a divergent sequence, if all the double sequences belonging to that space are transformed into sequences of the same space, the limits concerned not necessarily being the same. If in addition the limits are also preserved we shall say that the matrix A is *completely regular* for that space.

Hamilton [4] has proved the following theorems:

THEOREM H-1. *The matrix A transforms all sequences $\epsilon \boldsymbol{P}_0$ into sequences $\epsilon \boldsymbol{P}$ if, and only if,*

(i) $\sup\limits_{m,n} \sum\limits_{p,q=0}^{\infty} |a_{mnpq}| < \infty$,

(ii) *for each fixed p and q, the double sequence $\{a_{mnpq}\} \epsilon \boldsymbol{P}$.*

THEOREM H-2. *The matrix A is completely regular for $\boldsymbol{P}_0$ if, and only if,*

(i) $\sup\limits_{m,n} \sum\limits_{p,q=0}^{\infty} |a_{mnpq}| < \infty$,

(ii) *for each fixed p and q, the double sequence $\{a_{mnpq}\} \epsilon \boldsymbol{P}_0$.*

In a subsequent paper, [5], Hamilton has established the necessary and sufficient conditions for A to be completely regular for the space $(\boldsymbol{rc})$.

In Section 2 of this paper, confining myself to a restricted class of double sequences, viz. $\boldsymbol{P}$, I have obtained the necessary and sufficient conditions for the matrix A to be completely conservative or completely regular for that space. It is also proved that the completely conservative matrices of the above type form a Banach algebra under a suitable norm.

Section 3 of the paper is a study of those matrices which are of type M for $\boldsymbol{P}$. Banach ([3], Lemma 1, p. 91) has proved that for any T-matrix A satisfying certain conditions there exists a convergent simple sequence contained in the closure of $R_c(A)$ where $R_c(A)$ is the set of A-transforms of convergent simple sequences. The analogue of the theorem for double sequences has been given by Hill [6], when the matrix of transformation is taken to be one that is completely regular for $(\boldsymbol{rc})$. We prove in this paper a theorem, very similar to these, for the class of double sequences and the matrix A is taken to be completely conservative for $\boldsymbol{P}$ and in addition satisfies certain suitable restrictions and in fact is more general than a matrix which is completely regular for $\boldsymbol{P}$.

The last section, Section 4, is a study of four-dimensional Hausdorff methods. Adams [1] has obtained the necessary and sufficient conditions for such a matrix to be completely regular for the space of bounded, convergent, double sequences. Making use of those, we obtain here the necessary and sufficient conditions in order that the method be completely regular for $\boldsymbol{P}$. This in turn yields an inclusion condition which leads to a theorem of the Mercerian type, considered earlier by the author [8].

2. It may be noted, as proved by Alexiewicz and Orlicz [2], that both $\boldsymbol{P}_0$ and $\boldsymbol{P}$ are Banach spaces under the norm $\|s\| = \operatorname{l.u.b.}\limits_{p,q} |s_{pq}|$. Also, we have

LEMMA 1. *The form of a linear continuous functional*(1) *over $\boldsymbol{P}$ is given by*

$$f(x) = c \lim x_{mn} + \sum_{m,n} c_{mn} x_{mn}, \qquad x \epsilon \boldsymbol{P}$$

where $\|f\| = |c| + \sum\limits_{m,n} |c_{mn}|$.

This is indeed an immediate corollary to the following theorem due to Hill [6]:

THEOREM OF HILL. *Every linear continuous functional defined in $(\boldsymbol{rc})$ is of the form*

$$f(x) = c \lim_{m,n} x_{mn} + \sum_n c_n \cdot \lim_m x_{mn} + \sum_m c'_m \cdot \lim_n x_{mn} + \sum_{m,n} c_{mn} x_{mn}, \qquad x \epsilon (\boldsymbol{rc})$$

where $\|f\| = |c| + \sum\limits_n |c_n| + \sum\limits_m |c'_m| + \sum\limits_{m,n} |c_{mn}|$.

Now we shall prove the following theorem:

THEOREM 1. *The matrix A is completely conservative for $\boldsymbol{P}$ if, and only if,*

(i) $\sup\limits_{m,n} \sum\limits_{p,q=0}^{\infty} |a_{mnpq}| < \infty$,

(ii) *for each fixed p and q, $\{a_{mnpq}\} \epsilon \boldsymbol{P}$, with limit l_{pq}, say,*

(iii) $\sigma_{mn} = \sum\limits_{p,q=0}^{\infty} a_{mnpq} \epsilon \boldsymbol{P}$, *with limit l, say.*

Under these conditions the limit of the transformed sequence will be

$$s\Big[l - \sum_{p,q} l_{pq}\Big] + \sum_{p,q} l_{pq} s_{pq}, \qquad where \qquad s = \lim s_{pq}.$$

Proof. Since $\boldsymbol{P}_0 \subset \boldsymbol{P}$ the necessity of conditions (i) and (ii) follows from Hamilton's Theorem H-1.

Now, taking the sequence $s_{pq} = 1$ for all p and q (so that $s = 1$), we see that $t_{mn} = \sum\limits_{p,q} a_{mnpq}$, which, by hypothesis, should be $\epsilon \boldsymbol{P}$ and thus condition (iii) is also necessary.

To prove that the conditions are sufficient, let $\{s_{pq}\} \epsilon \boldsymbol{P}$ with limit s. Then take the sequence $s'_{pq} = s$, for all p and q. Now, the sequence $\{s_{pq} - s'_{pq}\} \epsilon \boldsymbol{P}_0$. Therefore, by Theorem H-1, it follows that $\{A(s_{pq} -$

(1) For definition, see Banach ([3], p. 16).

$-s'_{pq})\} \epsilon \boldsymbol{P}$ since the matrix A satisfies the conditions (i) and (ii) of the theorem. Also $A(s'_{pq}) = s\sum_{p,q} a_{mnpq}$ is also $\epsilon \boldsymbol{P}$, by condition (iii) above, and therefore

$$A(s_{pq}-s'_{pq})+A(s'_{pq}) = A(s_{pq}) \epsilon \boldsymbol{P}.$$

This proves that A is completely conservative for the space $\boldsymbol{P}$. To prove the last part of the theorem, putting $s'_{pq} = s_{pq}-s$ and $t'_{mn} = \sum_{p,q} a_{mnpq}s'_{pq}$, we get $s'_{pq} \epsilon \boldsymbol{P}_0$ and

$$\lim_m t'_{mn} = \lim_m \sum_{p,q} a_{mnpq}s'_{pq} = \sum_{p,q} l_{pq}s'_{pq}.$$

The same is true of $\lim_n t'_{mn}$ and $\lim_{m,n} t'_{mn}$, since $t'_{mn} \epsilon \boldsymbol{P}_0$. Now,

$$t_{mn} = \sum_{p,q} a_{mnpq}s_{pq} = \sum_{p,q} a_{mnpq}(s'_{pq}+s) = t'_{mn}+s\sum_{p,q} a_{mnpq}.$$

So, we have

$$\lim_m t_{mn} = \sum_{p,q} l_{pq}s'_{pq}+sl = \sum_{p,q} l_{pq}s_{pq}+s\Big[l-\sum_{p,q} l_{pq}\Big].$$

THEOREM 2. *The matrix A is completely regular for $\boldsymbol{P}$ if, and only if,*

(i) $\sup_{m,n} \sum_{p,q} |a_{mnpq}| < \infty$,

(ii) *for each fixed p and q, $\{a_{mnpq}\} \epsilon \boldsymbol{P}_0$,*

(iii) $\sigma_{mn} = \sum_{p,q} a_{mnpq} \epsilon \boldsymbol{P}$, *with limit* 1.

Proof. Since $\boldsymbol{P}_0 \subset \boldsymbol{P}$, it follows from Theorem H-2 that the conditions (i) and (ii) are necessary. Taking, as in Theorem 1, the sequence $s_{pq} = 1$ for all p and q, we see that condition (iii) is also necessary.

That the conditions are sufficient follows from the last part of Theorem 1, since by the hypothesis of this theorem $l_{pq} = 0$ for all p and q and $l = 1$.

I shall now prove some properties of the matrices considered hitherto.

THEOREM 3. *The matrices $A \equiv (a_{mnpq})$ which are completely regular for $\boldsymbol{P}_0$ form a Banach algebra under the norm*

$$\|A\| = \text{l. u. b.}_{m,n} \sum_{p,q} |a_{mnpq}|.$$

Proof. It is easily seen that the sum $A+B$ of any two members of the family is also of the same type. We shall next prove that the product AB of two members of the family is also a member of the family. Let the matrix $C \equiv (c_{mnpq})$ denote the product AB. Then $c_{mnpq} = \sum_{r,s} a_{mnrs}b_{rspq}$. Therefore

$$\sum_{p,q} |c_{mnpq}| = \sum_{p,q} \Big|\sum_{r,s} a_{mnrs}b_{rspq}\Big| \leqslant \sum_{r,s} |a_{mnrs}| \sum_{p,q} |b_{rspq}|$$

and consequently the elements of the matrix C satisfy the condition (i) of Theorem H-2.

Also, for each fixed p and q, $c_{mnpq} = \sum_{r,s} a_{mnrs}b_{rspq}$ and therefore the sequence $\{c_{mnpq}\}$ may be regarded as the transform of the sequence $\{b_{rspq}\}$, considered as a double sequence for varying r and s, by the matrix A which is completely regular for $\boldsymbol{P}_0$. Therefore, for each fixed p and q, the sequence $\{c_{mnpq}\} \epsilon \boldsymbol{P}_0$ since $\{b_{rspq}\}$ has the same property, since the matrix B is by hypothesis completely regular for $\boldsymbol{P}_0$ and therefore satisfies the condition (ii) of Theorem H-2. Thus the product $C = AB$ is also a member of the family. Now it is enough to prove that the space is complete under the norm defined above.

Let $A^{(r)} \equiv (a^r_{mnpq})$ be a sequence of matrices which are completely regular for $\boldsymbol{P}_0$. Then we must show that $\|A^r-A^s\| \leqslant \varepsilon$, $r,s \geqslant k$, is a necessary and sufficient condition for the existence of a matrix A, completely regular for $\boldsymbol{P}_0$ and such that $\|A^r-A\| \leqslant \varepsilon$, for all large r. It is easily verified that this condition is necessary. To prove that the condition is sufficient, we have, by hypothesis,

$$\sum_{p,q} |a^r_{mnpq}-a^s_{mnpq}| \leqslant \varepsilon, \quad r,s \geqslant k, \tag{3}$$

independent of m and n. Therefore, $a^r_{mnpq} \to a_{mnpq}$, say, as $r \to \infty$. Letting $r \to \infty$, we obtain from the equation (3) above

$$\sum_{p,q} |a_{mnpq}-a^s_{mnpq}| \leqslant \varepsilon, \quad s \geqslant k.$$

Therefore $\|A-A^s\| \leqslant \varepsilon$, for $s \geqslant k$. Further, we have

$$\sum_{p,q} |a_{mnpq}-a^k_{mnpq}| \geqslant \sum_{p,q} |a_{mnpq}| - \sum_{p,q} |a^k_{mnpq}|$$

and consequently,

$$\sum_{p,q} |a_{mnpq}| \leqslant \sum_{p,q} |a^k_{mnpq}|+\varepsilon \leqslant M'$$

since A^k is completely regular for $\boldsymbol{P}_0$. Also we have

$$a_{mnpq}-\varepsilon \leqslant a^k_{mnpq} \leqslant a_{mnpq}+\varepsilon.$$

Therefore letting $m\to\infty$, $n\to\infty$ and $m,n\to\infty$ we see that A is completely regular for $\boldsymbol{P}_0$ and this completes the proof of the theorem.

THEOREM 4. *The matrices A which transform sequences $\epsilon\boldsymbol{P}_0$ into those $\epsilon\boldsymbol{P}$ form a B-algebra under the norm in Theorem* 3.

The proof is similar to that of Theorem 3.

THEOREM 5. *The matrices A which are completely conservative for $\boldsymbol{P}$ form a B-algebra under the norm in Theorem* 3.

After the proof of Theorem 3, it suffices to prove, in the notation of the proof of Theorem 3, that $\sigma_{mn}=\sum_{p,q}a_{mnpq}\epsilon\boldsymbol{P}$. Let $\sigma_{mn}^{(s)}=\sum_{p,q}a_{mnpq}^{s}$. Then

$$|\sigma_{mn}-\sigma_{m_1,n_1}|\leqslant|\sigma_{mn}-\sigma_{mn}^{k}|+|\sigma_{mn}^{k}-\sigma_{m_1,n_1}^{k}|+|\sigma_{m_1,n_1}^{k}-\sigma_{m_1,n_1}|\leqslant\varepsilon+\varepsilon+\varepsilon=3\varepsilon$$

and hence the result.

3. We start with the following definitions:

The *range* $R_P(A)$ of a matrix A completely conservative for $\boldsymbol{P}$ is the set of „points" $y=Ax$ where $x\epsilon\boldsymbol{P}$. It is obvious that $R_P(A)\subset\boldsymbol{P}$.

The matrix $A\equiv(a_{mnpq})$ is said to be of *type* M if the relations $\sum_{i,j}|a_{ij}|<\infty$ and $\sum_{i,j}a_{ij}a_{ijpq}=0$, $p,q=0,1,2,\ldots$ always imply that $a_{ij}\equiv0$.

With these definitions, we prove the following theorem:

THEOREM 6. *Let $A\equiv(a_{mnpq})$ be completely conservative for $\boldsymbol{P}$ and let its elements satisfy the condition,*

$$\chi(A)=\lim_{m,n}\sum_{p,q}a_{mnpq}-\sum_{p,q}\lim_{m,n}a_{mnpq}\neq0$$

and let $y=\{y_{mn}\}\epsilon\boldsymbol{P}$ be such that $\sum_{i,j}a_{ij}y_{ij}=0$ whenever $\sum_{i,j}|a_{ij}|<\infty$, and $\sum_{i,j}a_{ij}a_{ijpq}=0$, for $p,q=0,1,2,\ldots$ Then for every $\varepsilon>0$ there exists an $x=\{x_{mn}\}\epsilon\boldsymbol{P}$ such that

$$|A_{ij}(x)-y_{ij}|\leqslant\varepsilon,\quad \textit{for}\quad i,j=0,1,2,\ldots$$

Proof. The conclusion of the theorem simply implies that $y\epsilon\overline{R_P(A)}$. Therefore by Lemma 1, p. 51 of Banach [3], it is enough to prove that every linear continuous functional vanishing over $R_P(A)$ will vanish at y. By Lemma 1 of this paper any linear continuous functional over $\boldsymbol{P}$ is of the form

$$b(s)=c\lim_{m,n}s_{mn}+\sum_{m,n}c_{mn}s_{mn},\qquad s\epsilon\boldsymbol{P}$$

where $\|b\|=|c|+\sum_{m,n}|c_{mn}|$. Since $R_P(A)\subset\boldsymbol{P}$ any linear continuous functional over $R_P(A)$ is of the form

$$b(t)=b(As)=c\lim_{m,n}(As)_{mn}+\sum_{m,n}c_{mn}(As)_{mn},\qquad s\epsilon\boldsymbol{P},$$

and by Theorem 1

$$=c\Big[ls+\sum_{p,q}l_{pq}(s_{pq}-s)\Big]+\sum_{m,n}c_{mn}\sum_{p,q}a_{mnpq}s_{pq}$$

$$=cs\Big[l-\sum_{p,q}l_{pq}\Big]+\sum_{p,q}s_{pq}\Big[cl_{pq}+\sum_{m,n}c_{mn}a_{mnpq}\Big]$$

(the interchange in the order of summation in the above expression being justified by the boundedness condition on the matrix as also on the sequence $\{s_{pq}\}$, besides the convergence of $\sum_{m,n}|c_{mn}|$). Therefore

$$(4)\qquad b(t)=cs\chi(A)+\sum_{pq}s_{pq}b_{pq}$$

where

$$(5)\qquad b_{pq}=cl_{pq}+\sum_{m,n}c_{mn}a_{mnpq}.$$

Now suppose that the value of the functional is zero in $R_P(A)$. Taking for $\{s_{pq}\}$ the set of sequences $\{\delta_{ij}^{mn}\}$ where δ_{ij}^{mn} is the Kronecker delta, we find from the equation (4) above that $b_{pq}=0$, for $p,q=0,1,2,\ldots$ Therefore, we have $b(t)=cs\chi(A)$. But $\chi(A)\neq0$. Taking now for $\{s_{pq}\}$ the sequence $s_{pq}\equiv1$, we get $c=0$. Therefore from the equation (5) we get

$$\sum_{m,n}c_{mn}a_{mnpq}=0,\quad \text{for}\quad p,q=0,1,2,\ldots$$

This, along with $\sum_{m,n}|c_{mn}|<\infty$, implies by hypothesis that $\sum_{m,n}c_{mn}y_{mn}=0$ for $y\epsilon\boldsymbol{P}$, and this completes the proof of the theorem.

THEOREM 7 (Converse of Theorem 6). *Let A be completely conservative for $\boldsymbol{P}$ and let $y\epsilon\boldsymbol{P}$ be such that, for every $\varepsilon>0$, there exists an $x\epsilon\boldsymbol{P}$ such that*

$$|A_{ij}(x)-y_{ij}|\leqslant\varepsilon,\quad \textit{for all}\ i,j=0,1,2,\ldots$$

Then for any $\{a_{ij}\}$ such that

$$\sum_{i,j}|a_{ij}|<\infty\quad \textit{and}\quad \sum_{i,j}a_{ij}a_{ijpq}=0,\quad p,q=0,1,2,\ldots$$

we have $\sum_{i,j}a_{ij}y_{ij}=0$.

Proof. Let $\{a_{ij}\}$ be given as above. Then $\alpha(s)=\sum_{i,j}a_{ij}s_{ij},\ s\in \boldsymbol{P}$ is a linear continuous functional in $\boldsymbol{P}$ and $\sum_{i,j}|a_{ij}|=\|\alpha\|$. Also, since $A(x)=\{A_{ij}(x)\}$ is in $\boldsymbol{P}$, we have,

$$(6)\qquad |\alpha(Ax-y)|=\Big|\sum_{i,j}a_{ij}\Big[\sum_{pq}a_{ijpq}x_{pq}-y_{ij}\Big]\Big|\leqslant\|\alpha\|\cdot\|Ax-y\|\leqslant\|\alpha\|\cdot\varepsilon,$$

by hypothesis. Now,

$$\sum_{i,j}\sum_{p,q}|a_{ij}||a_{ijpk}||x_{pq}|=\sum_{i,j}|a_{ij}|\sum_{p,q}|a_{ijpq}|\cdot|x_{pq}|\leqslant\sum_{i,j}|a_{ij}|\sum_{p,q}|a_{ijpq}|\cdot\|x\|.$$

Therefore $\alpha(Ax)=\sum_{i,j}a_{ij}\sum_{p,q}a_{ijpq}x_{pq}=\sum_{p,q}x_{pq}\sum_{i,j}a_{ij}a_{ijpq}=0$, by hypothesis (the interchange in the order of summation being justified as before). Therefore now equation (6) yields $\big|\sum_{i,j}a_{ij}y_{ij}\big|\leqslant\|\alpha\|\cdot\varepsilon$ and consequently, $\sum_{i,j}a_{ij}y_{ij}=0$.

Remark. It may be noted that in the hypothesis of the theorem we have not assumed that $\chi(A)\neq 0$, as is the case in the previous theorem, and thus the theorem is to be looked upon as a generalized converse of Theorem 6. Also in the theorem $\boldsymbol{P}$ can be replaced by $\boldsymbol{P}_0$, throughout.

In the light of Theorems 6 and 7, we have the following theorems:

THEOREM 8. *The matrix A which is completely regular for $\boldsymbol{P}$, is of type M, if, and only if, $R_P(A)$ is dense in $\boldsymbol{P}$.*

THEOREM 9. *If the matrices A and B, which are completely regular for $\boldsymbol{P}$ are both of type M, then so is their product AB.*

THEOREM 10. *If the product $C=AB$ of two matrices A and B, completely regular for $\boldsymbol{P}$ is of type M, then A must be of type M.*

The above theorems may be looked upon as extensions of the corresponding theorems which I proved for simple sequences in an earlier paper [7].

Next, we prove the

THEOREM 11. *Let the matrix A be completely conservative for $\boldsymbol{P}$ and let $\chi(A)\neq 0$. Then for every bounded sequence $x_0=\{s_{mn}\}$ such that $Ax_0\in\boldsymbol{P}$ and for every $\varepsilon>0$, there exists a double sequence $x\in\boldsymbol{P}$ such that $|A_{ij}(x)-A_{ij}(x_0)|\leqslant\varepsilon$ for all $i,j=0,1,2,\ldots$*

Proof. For $i,j=0,1,2,\ldots$, put $y_{ij}=A_{ij}(x_0)$. Then by hypothesis $\{y_{ij}\}\in\boldsymbol{P}$. Let $\{a_{ij}\}$ be a sequence such that $\sum_{i,j}|a_{ij}|<\infty$ and $\sum_{i,j}a_{ij}a_{ijpq}=0$, for $p,q=0,1,2,\ldots$ Then

$$\sum_{i,j}a_{ij}y_{ij}=\sum_{i,j}a_{ij}\sum_{p,q}a_{ijpq}s_{pq}=\sum_{p,q}s_{pq}\sum_{i,j}a_{ij}a_{ijpq}$$

by the definition of $\{y_{ij}\}$ (the order of summation being interchangeable on account of the hypothesis and the boundedness of $\{s_{pq}\}$ and since $\sum_{i,j}|a_{ij}|<\infty$). Now the result follows at once from Theorem 6.

4. In this section we shall study the conditions of regularity for a four-dimensional *Hausdorff matrix*, for the space $\boldsymbol{P}$. We start with the following definitions:

The four-dimensional matrix $H\equiv(h_{mnpq})$ (²) defined by

$$h_{mnpq}=\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}\quad(m\geqslant p,\ n\geqslant q),\qquad h_{mnpq}=0\ \ (m<p,\ n<q)$$

is called a *Hausdorff matrix*, generated by μ_{mn} and denoted by (H,μ).

Let $\chi(u,v)$ be a function of bounded variation (in the sense of Hardy and Krause) in the square U $(0\leqslant u\leqslant 1,\ 0\leqslant v\leqslant 1)$. Then the quantities

$$\mu_{mn}=\int_0^1\int_0^1 u^m v^n d_u d_v\chi(u,v)$$

are called *moment constants*. If, in addition, the function $\chi(u,v)$ satisfies the following conditions of continuity

$$\chi(u,0)=\chi(u,+0),\qquad \lim_{v\to 0}\chi(u,v)=\chi(u,+0),\qquad 0\leqslant u\leqslant 1,$$

$$\chi(0,v)=\chi(+0,v),\qquad \lim_{u\to 0}\chi(u,v)=\chi(+0,v),\qquad 0\leqslant v\leqslant 1,$$

$$\chi(0,0)=\chi(u,0)=\chi(0,v)=0,\qquad \chi(1,1)=1,$$

so that $\mu_{00}=1$, then μ_{mn} are called *regular moment constants*.

With these definitions Adams [1] has proved that the matrix (H,μ) is regular (*i. e.* completely regular, in the sense of this paper) for the class of bounded, convergent double sequences if, and only if, μ_{mn} are regular moment constants.

The theorem below gives the necessary and sufficient conditions so that the Hausdorff matrix may be completely regular for $\boldsymbol{P}$.

THEOREM 12. *The matrix (H,μ) is completely regular for $\boldsymbol{P}$ if, and only if, μ_{mn} are regular moment constants.*

Proof. The proof of the above theorem of Adams [1], stated earlier shows that only if the continuity conditions on $\chi(u,v)$ are satisfied is

$$\lim_{m,n}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}=0\qquad\text{when}\qquad p=q=0.$$

(²) The differences Δ are defined by

$$\Delta^{ij}a_{mn}=\sum_{r=0}^{i}\sum_{s=0}^{j}\binom{i}{r}\binom{j}{s}(-1)^{r+s}a_{m+r,n+s}.$$

Therefore the conditions are necessary after the theorem of Adams. To prove that they are sufficiennt, we note that μ_{mn} are regular implies, by Adams [1], that

(i) $\lim\limits_{m,n}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}=0$, for all p,q,

(ii) $\lim\limits_{m}$, $\lim\limits_{n}$ and $\lim\limits_{m,n}\sum\limits_{p=0}^{m}\sum\limits_{q=0}^{n}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}=\mu_{00}=1$,

(iii) $\sum\limits_{p=0}^{m}\sum\limits_{q=0}^{n}\binom{m}{p}\binom{n}{q}|\Delta^{m-p,n-q}\mu_{pq}|\leqslant M$, independent of m,n.

Therefore we have only to prove that the hypotheses imply that

$$\lim_{m}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}=0$$

for $p,q=0,1,2,\ldots$

$$\lim_{n}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}=0.$$

Now for all u and v in the square U, we have

$$\binom{m}{p}u^{p}(1-u)^{m-p}\leqslant\sum_{p=0}^{m}\binom{m}{p}u^{p}(1-u)^{m-p}=1$$

together with a similar relation in n,q and v. Therefore, for $0<\delta<1$ we have,

$$\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}\leqslant\int_{0}^{\delta}\int_{0}^{1}d_{u}d_{v}\chi(u,v)+\binom{m}{p}(1-\delta)^{m-p}\int_{\delta}^{1}\int_{0}^{1}d_{u}d_{v}\chi(u,v).$$

Therefore,

$$\overline{\lim_{m}}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}\leqslant\chi(\delta,1)-\chi(0,1)+\chi(0,0)=0,\quad\text{for all } p,q$$

and similarly,

$$\overline{\lim_{n}}\binom{m}{p}\binom{n}{q}\Delta^{m-p,n-q}\mu_{pq}\leqslant 0\quad\text{for all } p,q$$

and hence the theorem.

The above theorem in turn yields the following (where we assume that none of the μ_{mn} and μ'_{mn} vanish):

THEOREM 13. *Let $\lambda\equiv(H,\mu)$ and $\lambda'\equiv(H,\mu')$ be two Hausdorff methods, not necessarily regular (completely) for $\boldsymbol{P}$. Then λ includes λ' — in the sense that for any sequence $\{s_{pq}\}$ such that $\lambda s\in\boldsymbol{P}$ with limit s, say, implies that $\lambda' s$ is also $\in\boldsymbol{P}$ with the same limit s — if, and only if, μ'_{mn}/μ_{mn} are regular moment constants.*

The proof of the theorem is obvious after Theorem 12, and the equation $\lambda' s=\lambda'(\lambda^{-1}\lambda)s=\lambda'\lambda^{-1}(\lambda s)$, which is true on account of the boundedness conditions on the matrices and the sequence involved.

COROLLARY. *The two methods λ and λ' are equivalent for the class $\boldsymbol{P}$ if, and only if, both μ'_{mn}/μ_{mn} and μ_{mn}/μ'_{mn} are regular moment constants.*

In the light of Theorem 12 and that of Adams stated earlier, we see that (H,μ) is completely regular for $\boldsymbol{P}$ if, and only if, it is so for the space of bounded convergent double sequences. This, in turn, yields at once the following theorem, proved earlier by me [8] for the class of bounded convergent double sequences, and this is the two dimensional analogue of the classical theorem of Mercer.

THEOREM 14. *Let $\{s_{pq}\}$ be a double sequence and $\alpha,\beta>0$. Let*

$$t_{pq}=\alpha\beta s_{pq}+\frac{\alpha(1-\beta)}{q+1}\sum_{j=0}^{q}s_{pj}+\frac{\beta(1-\alpha)}{p+1}\sum_{i=0}^{p}s_{iq}+\frac{(1-\alpha)(1-\beta)}{(p+1)(q+1)}\sum_{i=0}^{p}\sum_{j=0}^{q}s_{ij}.$$

Then if $\{t_{pq}\}\in\boldsymbol{P}$ with limit l say, then the same is true of $\{s_{pq}\}$ and conversely.

In conclusion, I have very great pleasure in thanking Professor W. Orlicz for his kindness in going through the manuscript. My warm thanks are due, in no less measure, to Professor A. Alexiewicz for his helpful criticism.

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