Sufficient conditions for all integrals of a stationary system to be determined in the bilateraly unbounded interval

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Let us consider the system of differential equations

$$dy/dt = F(y), \tag{1}$$

where $y = (y_1, y_2, \ldots, y_n)$ is a n-dimensional vector and

$$F(y) = \big(f_1(y_1, \ldots, y_n), f_2(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n)\big)$$

is the continuous vector-function for y belonging to a domain Ω of n-dimensional Euclidean space R^n. The function $F(y)$ does not depend on t. Such a system is called a *stationary* one.

A stationary system is called a *dynamical* one if it satisfies the condition of uniqueness and if each of its solution is defined for $-\infty < t < +\infty$.

The two systems, (1) and

$$dy/dt = G(y), \tag{2}$$

are *equivalent* if there exists a scalar-function $r(y)$ which is continuous and positive for y belonging to Ω and satisfies

$$G(y) = r(y)F(y) \tag{3}$$

for every y from Ω.

The trajectories of two equivalent systems and the direction of motion on them are the same, only the velocities being changed.

The purpose of this paper, the subject of which has been suggested by T. Ważewski, is the presentation of a sufficient condition for every solution of (1) to be defined for $-\infty < t < +\infty$. This condition will be formulated and proved in section 2 (see theorem 1); before that we shall prove two lemmas (see § 1). Applying theorem 1 we shall prove in section 3 the following theorem A (see [2], p. 28, theorem of R. E. Winograd).

(*) The author wishes to express her sincere gratitude to Professor T. Ważewski for his generous help in the preparation of this paper.

THEOREM A. *Let system* (1) *satisfy the condition of uniqueness in domain Ω. There exists a dynamical system filling up all the space which is equivalent to system* (1) *in domain Ω.*

§ 1. Let H denote a closed set of points of space R^n. Put

$$\mu(y, H) = \begin{cases} 1+|y| & \text{if } H \text{ is empty,} \\ \varrho(y, H) & \text{if } H \text{ is non empty,} \end{cases} \tag{4}$$

where $|y| = \sqrt{\sum_{i=1}^{n} y_i^2}$ is the norm of vector y and $\varrho(y, H)$ is a distance of point y from the set H. The function $\mu(y, H)$ is continuous with respect to y. The equality $\mu(y, H) = 0$ is fulfilled only when y belongs to H.

The function $\mu(y, H)$ defined as above possesses the following property:

$$\text{(a)} \quad \begin{array}{llll} \text{in the case } H = 0, & \mu(y, H) \to \infty & \text{if} & |y| \to \infty, \\ \text{in the case } H \neq 0, & \mu(y, H) \to \infty & \text{if} & \varrho(y, H) \to \infty. \end{array}$$

For our purposes one can define $\mu(y, H)$ in such a manner that (a) is not satisfied — a theorem analogous to theorem 1 obtained in such a way should be less general.

LEMMA 1. *If the function $\Phi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\big)$ has a continuous derivative for $t \in (a, b)$, then*

$$\big|\overline{D}_+ \mu\big(\Phi(t), H\big)\big| \leqslant |\Phi'(t)| \quad \textit{for} \quad t \in (a, b)\,(^1). \tag{5}$$

The proof of this lemma proceeds directly from the inequality

$$|\mu(P, H) - \mu(Q, H)| \leqslant |P - Q| \quad \text{true for every } P, Q.$$

LEMMA 2. *If $\Phi(t)$ possesses a continuous derivative for $t \in (a, b)$ and if*

$$|\Phi'(t)| \leqslant K \quad \textit{for} \quad t \in (a, b), \tag{6}$$

where K is an arbitrary positive constant, then there exists

$$\lim_{t \to b} \Phi(t) \tag{7}$$

and this limit is finite.

(1) By $\overline{D}_+ L(t)$ we understand $\overline{\lim\limits_{h \to 0,\, h>0}} \dfrac{L(t+h) - L(t)}{h}$ and similarly

$$\underline{D}_+ L(t) = \underline{\lim_{h \to 0,\, h>0}} \frac{L(t+h) - L(t)}{h}.$$

Proof of lemma 2. Let t_1 belong to interval (a, b). Suppose H is composed only of one point $\Phi(t_1)$ — hence by lemma 1 and assumption (6) we get

$$|\bar{D}_+\varrho(\Phi(t), \Phi(t_1))| \leqslant |\Phi'(t)| \leqslant K \quad \text{for} \quad a < t < b;$$

therefore

$$-K \leqslant \bar{D}_+\varrho(\Phi(t), \Phi(t_1)) \leqslant K.$$

Let $L(t) = Kt+\varrho(\Phi(t), \Phi(t_1))$ and $L_1(t) = Kt-\varrho(\Phi(t), \Phi(t_1))$. By the last inequality we obtain

$$\bar{D}_+L(t) = K+\bar{D}_+\varrho(\Phi(t), \Phi(t_1)) \geqslant 0,$$

$$\bar{D}_+L_1(t) = K-\underline{D}_+\varrho(\Phi(t), \Phi(t_1)) \geqslant K-\bar{D}_+\varrho(\Phi(t), \Phi(t_1)) \geqslant 0.$$

From the theorem of Zygmund (see [3], p. 203) we obtain that $L(t)$ and $L_1(t)$ are non-decreasing in (a, b) — hence we have the inequality

$$-K(t_2-t_1) \leqslant \varrho(\Phi(t_2), \Phi(t_1)) \leqslant K(t_2-t_1)$$

where $t_2 > t_1$ and t_1, t_2 belong to (a, b). Thus (see p. 221)

$$\varrho(\Phi(t_2), \Phi(t_1)) = |\Phi(t_2)-\Phi(t_1)| \leqslant K|t_2-t_1| \quad \text{for} \quad t_1, t_2 \in (a, b).$$

On the basis of the last inequality and Cauchy's condition of convergence we obtain (7). This completes the proof of lemma 2.

§ 2. THEOREM 1. *Suppose we are given a system of differential equations*

$$dy/dt = F(y), \quad y = (y_1, y_2, \ldots, y_n),$$

where $F(y)$ is a continuous vector-function for y belonging to domain Ω (²).

Let $\lambda(z)$ be a continuous and positive function for $0 \leqslant z < \infty$ such that

$$(8) \qquad \lambda(0) = 0,$$

$$(9) \qquad \int_0^{\varepsilon} \frac{dz}{\lambda(z)} = \infty \quad (\varepsilon > 0).$$

Let every integral of each of the following equations

$$(10_1) \quad dz/dt = \lambda(z), \qquad (10_2) \quad dz/dt = -\lambda(z)$$

be defined for $-\infty < t < +\infty$ (³).

(²) We do not assume that system (1) satisfies the uniqueness condition.

(³) For example the function

$$\lambda(z) = z, \quad \lambda(z) = \frac{cz}{1+z} \quad \text{or} \quad \lambda(z) = \begin{cases} z\ln z & \text{for} \quad z \neq 0, \\ 0 & \text{for} \quad z = 0 \end{cases}$$

fulfils those conditions.

Under these assumptions a sufficient condition for every solution of (1) *to be defined for $-\infty < t < +\infty$ is that*

$$(11) \qquad |F(y)| \leqslant \lambda(\mu(y, H)) \quad \textit{for } y \textit{ belonging to } \Omega,$$

where H is the boundary of Ω and $\mu(y, H)$ is defined by (4). *(H may be an empty set.)*

Proof of theorem 1. Let Ω_1 be a set of points (t, y) such that y belongs to Ω and $-\infty < t < +\infty$ and let $y(t)$ be an integral of (1) valid in (a, b), the right-hand end of which reaches the boundary of Ω_1 (⁴).

Let us suppose, for the indirect proof, that $b < \infty$. From lemma 1 and from (11) we obtain the inequality

$$(12) \qquad |\bar{D}_+\mu(y(t), H)| \leqslant |y'(t)| = |F(y(t))| \leqslant \lambda(\mu(y(t), H))$$

for $a < t < b$, H being the boundary of Ω. By (12) we get

$$(13) \qquad -\lambda(\mu(y(t), H)) \leqslant \bar{D}_+\mu(y(t), H) \leqslant \lambda(\mu(y(t), H)).$$

Let us denote by $\varphi(t)$ the upper right-hand integral of (10_1) and by $\psi(t)$ the lower right-hand integral of (10_2) fulfilling the initial condition $\varphi(t_0) = \psi(t_0) = \mu(y(t_0), H) > 0$ where t_0 is an arbitrary number belonging to the interval (a, b). The functions $\varphi(t)$ and $\psi(t)$, due to (8) and (9), are defined and positive for $t \in (-\infty, +\infty)$. By the fundamental theorem about differential inequalities (see for example [4], p. 124, théorème 2), (13) implies

$$(14) \qquad \psi(t) \leqslant \mu(y(t), H) \leqslant \varphi(t) \quad \text{for} \quad t \in (t_0, b).$$

Since the functions $\varphi(t)$ and $\psi(t)$ are continuous and b is finite, there exist positive constants m and M such that

$$(15) \qquad 0 < m \leqslant \mu(y(t), H) \leqslant M \quad \text{for} \quad t \in (t_0, b).$$

The function $\lambda(z)$ is continuous, and consequently we can chose a constant K such that

$$(16) \qquad \lambda(z) \leqslant K \quad \text{for} \quad z \in (m, M).$$

But (15) and (16) imply

$$(17) \qquad \lambda(\mu(y(t), H)) \leqslant K \quad \text{for} \quad t \in (t_0, b);$$

therefore from (1) and (11) we have

$$|y'(t)| \leqslant K \quad \text{for} \quad t \in (t_0, b).$$

(⁴) We say that the integral $y = y(t)$ defined in interval (a, b) reaches the boundary of Ω_1, with its right-hand end if $b = \infty$ or if $b < \infty$ and there exists a sequence $t_n < b$, $t_n \to b$ such that the sequence of points $(t_n, y(t_n))$ tends to the point belonging to the boundary of Ω_1.

Hence, on account of lemma 2 we may state that there exists a limit $\lim_{t\to b} y(t) = S$ and at the same time that S is finite. Moreover, since $\mu(y, H)$ is the continuous function of y, (15) implies the inequality

$$0 < m \leqslant \mu(S, H) \leqslant M.$$

In other words S belongs to Ω. (S cannot belong to the boundary of Ω because $\mu(S, H) \neq 0$, S cannot belong to the exterior set of Ω too because it is obtained as a limit of points belonging to Ω.) The point (b, S) has a positive distance from the boundary of Ω_1, but this contradicts the assumption that $y(t)$ with its right-hand end reaches the boundary of Ω, whence $b = \infty$. Quite similarly we can prove that $a = -\infty$. Thus theorem 1 is proved.

Remark 1. Assumption (8) and (11) imply that if we put $F(y) = \Theta$ for y belonging to the complementary set of Ω, we extend system (1) over the whole space R^n; each integral of such an extended system being defined for $-\infty < t < +\infty$.

§ 3. Now we shall demonstrate how theorem 1 implies theorem A. Let $\lambda(z)$ be an arbitrary continuous and positive function defined for $0 \leqslant z < \infty$ satisfying (8) and (9) and such that each integral of (10_1) and (10_2) is defined for $-\infty < t < +\infty$.

Let

$$\mu(y, H) = \begin{cases} 1+|y| & \text{if } H \text{ is empty,} \\ \varrho(y, H) & \text{if } H \text{ is non empty} \end{cases}$$

and let $r(y) = \lambda(\mu(y, H))/(1+|F(y)|)$ where H denotes the boundary of Ω. It may easily be seen that function $r(y)$ is continuous and positive in Ω and that it satisfies the inequality

$$r(y)|F(y)| \leqslant \lambda(\mu(y, H)). \tag{18}$$

The system $dy/d\tau = G(y)$, where $G(y) = r(y)F(y)$ for y belonging to Ω and $G(y) = \Theta$ for y belonging to the complementary set of Ω, fills up all the space R^n. It is equivalent to system (1) (which satisfies by the hypothesis — see theorem A — the uniqueness condition) in domain Ω, and owing to theorem 1 and remark 1 it is dynamical.

Remark 2. The considerations of § 3 remain valid if we replace function $r(y)$ by a continuous function $g(y)$ which satisfies the inequality

$$0 < g(y) \leqslant r(y) \quad \text{for} \quad y \in \Omega. \tag{19}$$

As follows from a certain result of A. Bielecki (see [1], p. 38, théorème 1), there exists a function $g(y)$ of class C^∞ (or even an analytical one as may be deduced from the lemma 6 of [5]) which satisfies inequality (19).

Hence we have the following strenghtening of theorem A.

THEOREM B. *Let system* (1) *satisfy the condition of uniqueness in domain Ω. There exists a positive function $g(y)$ of class C^∞ (or even an analytical function $g(y)$) for $y \in \Omega$ such that the system*

$$dy/d\tau = g(y)F(y)$$

is a dynamical one.

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Reçu par la Rédaction le 25. 1. 1956