A note on some properties of the functions \( \varphi(n) \), \( \sigma(n) \) and \( \theta(n) \)

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§ 1. Introduction. A. Schinzel has proved in [4] that for every sequence \( a \) of \( h \) positive numbers \( a_1, a_2, \ldots, a_h \) and \( \varepsilon > 0 \) there exist natural numbers \( n \) and \( n' \) such that

\[
\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i < \varepsilon, \quad \frac{\sigma(n'+i)}{\sigma(n'+i-1)} - a_i < \varepsilon \quad (i = 1, 2, \ldots, h). \tag{1}
\]

Professor Hua Loo-Keng has pointed out that by Brun's method we can prove the existence of positive constants \( c = c(a, \varepsilon) \) and \( X_0 = X_0(a, \varepsilon) \) such that the number of numbers \( n \) satisfying the first of these inequalities in the interval \( 1 \leq n \leq X \) is greater than

\[ cX/\log^{h+1}X \quad \text{for} \quad X > X_0. \]

In the present paper we give the proof of this theorem, of an analogous theorem on the function \( \sigma(n) \) and of a theorem on the function \( \theta(n) \) which is weaker but gives a positive solution of the problem put forward in paper [2] of A. Schinzel and comprises the theorem from paper [3] of A. Schinzel.

The question whether a theorem analogous to the theorems on functions \( \varphi \) and \( \sigma \) is true for the function \( \theta \) remains open.

§ 2. An auxiliary theorem. Let

\[
A_k = q_1 \ldots q_{l_k}, \quad A_i = q_{a_i} \ldots q_{b_i}, \quad (1 \leq i \leq k)
\]

be positive integers, where \( q_1, q_2, \ldots, q_k \) are all the prime numbers in the interval \( 0 < x \leq 10(h+1) \) and \( q_{a_i} \) for \( 0 < i \leq h, 1 \leq j \leq l_i \) are primes greater than \( 10(h+1) \) such that \( A_1, A_2, \ldots, A_k > 1 \) are relatively prime in pairs.

\[ \text{(i) } \varphi(n) \text{ denotes the Euler function, } \sigma(n) \text{ the sum of divisors of number } n, \]

\[ \text{(ii) } \theta(n) \text{ denotes the number of divisors of } n. \]
If \( Z > A_0 A_1^2 \ldots A_{h}^2 \), then let us denote by \( N_{E}(X) \) the number of integral solutions \((x_1, \ldots, x_h)\) of the system of equations
\[
A_i x_i + i = i A_i x_i \quad (1 \leq i \leq h)
\]
satisfying the conditions
\[
(1) \quad 1 \leq x_1 \leq X, \quad 2^m \text{ if } p | x_1, \text{ then } p > Z \quad (0 \leq i \leq h),
\]
where \( p \) denotes a prime.

**Theorem 1.** There exist positive constants \( c_1 \), depending on \( h \) only, and \( c_1, X_1 \), depending on \( A_i \) only, such that
\[
N_{E}(X) > c_1 X \log^{h+1} X \quad (X > X_1).
\]

**Proof.** If \( Z > A_0 A_1^2 \ldots A_{\lambda}^2 \), \( \lambda \) is a given integer in the interval \( 0 \leq \lambda < A_0 A_1^2 \ldots A_{\lambda}^2 \) and \( p_1 < p_2 < \ldots < p_h \) are all prime numbers not dividing \( A_0, A_1 \ldots A_{\lambda} \) and not exceeding \( Z \) and if for all \( 1 \leq i \leq h \), \( X_1 \) are given integers satisfying the conditions \( 0 < a_i \leq p_i \quad (1 \leq i \leq \lambda, 0 \leq i \leq h) \), and \( a_{i_1} \neq a_{i_2} \) for \( i_1 \neq i_2 \), then we can define \( M_{E}(X) \) as the number of \( x \) satisfying the conditions.
\[
(3) \quad 1 \leq x \leq X, \quad x = \lambda \left( \text{mod} \ A_0 A_1^2 \ldots A_{\lambda}^2 \right),
\]
\[
(4) \quad (1 \leq i \leq \lambda, 0 \leq i \leq h).
\]

It is evident, that theorem 1 is a consequence of the following two lemmas:

**Lemma 1.** There exist \( \lambda \) and \( a_{i_1} \) such that
\[
N_{E}(X) > c_1 X \log^{h+1} X \quad (X > X_1).
\]

**Lemma 2.** There exist positive constants \( c_1 \), depending on \( h \) only, and \( c_1, X_1 \), depending on \( A_i \) only, such that
\[
M_{E}(X) > c_1 X \log^{h+1} X \quad (X > X_1).
\]

For any given \( \lambda \) and \( a_{i_1} \).

**Proof.** By definition, \( P(\lambda, a, A, X; p_1, \ldots, p_h) \) is equal to the number of \( x \) satisfying the conditions
\[
(5) \quad 1 \leq x \leq X, \quad x = \lambda \left( \text{mod} \ A \right), \quad x = a_i \left( \text{mod} \ p_i \right) \quad (1 \leq i \leq \lambda, 0 \leq i \leq h),
\]
and the number of \( x \) satisfying (10) and one of the following congruences:
\[
(11) \quad x = a_i \left( \text{mod} \ p_i \right) \quad (0 \leq i \leq h).
\]
Since \((A, p_i) = 1\), each of the following systems of congruences is solvable and has a unique solution in the interval \(0 \leq x < Ap_i\):
\[
\begin{align*}
\lambda & \equiv \lambda \pmod{A}, \quad \lambda \equiv \lambda \pmod{A}, \quad \lambda \equiv \lambda \pmod{A}, \\
\ell & \equiv a_{i_0} \pmod{p_i}, \quad \ell \equiv a_{i_0} \pmod{p_i}, \quad \ldots \quad \ell \equiv a_{i_0} \pmod{p_i}.
\end{align*}
\]
Denote these solutions by \(\lambda_1, \lambda_2, \ldots, \lambda_\ell\) respectively. Since
\[
a_{i_0} \equiv a_{i_0} \pmod{p_i} \quad (i_0 \neq i_k)
\]
we have \(\lambda_i \neq \lambda_{i_k} \quad (i \neq i_k)\). Hence \(P(\lambda, a, A, X; p_1, \ldots, p_r)\) is equal to the difference between the number \(x\) satisfying (10) and the number of \(x\) satisfying one of the following conditions (as \(i = 0, 1, \ldots, k)):
\[
(12) \quad 1 \leq x \leq X, \quad x \equiv \lambda_i \pmod{Ap_i}, \quad x \equiv a_{i_0} \pmod{p_i}
\]
\[
(1 \leq j \leq r-1, 0 \leq i \leq k).
\]
This proves lemma 5.

Briefly we write
\[
P(\lambda, a, D, X; p_1, \ldots, p_r) = P(D, X; p_1, \ldots, p_r),
\]
\[
P(\lambda, D, X) = P(D, X).
\]
Hence, it follows from (9) that, with the usual convention in notation, we have
\[
P(A, X; p_1, \ldots, p_r) = P(A, X; p_1, \ldots, p_{r-1}) - (h+1)P(Ap_r, X; p_1, \ldots, p_{r-1}).
\]
Using lemma 3 \(r\) times successively we get

**Lemma 4.**
\[
P(A, X; p_1, \ldots, p_r) = P(A, X; p_1, \ldots, p_{r-1}) (h+1) P(Ap_r, X; p_1, \ldots, p_{r-1}).
\]

Let
\[
r = r_0 \geq r_1 \geq \ldots \geq r_t = 1
\]
be any given sequence of \(t\) positive integers.

**Lemma 5.**
\[
P(A, X; p_1, \ldots, p_r) \geq P(A, X; p_1, \ldots, p_{r-1}) (h+1) P(Ap_r, X; p_1, \ldots, p_{r-1}) +
\]
\[
+ (h+1)^2 \sum_{a \equiv 1 \pmod{q_j^c}} \sum_{b \equiv 1 \pmod{q_j^c}} P(Ap_a p_b, X; p_1, \ldots, p_{r-1}).
\]

**Proof.** By lemma 4, we have
\[
P(A, X; p_1, \ldots, p_r) = P(A, X; p_1, \ldots, p_r) (h+1) \sum_{a = 1}^r P(Ap_a, X; p_1, \ldots, p_{r-1}) +
\]
\[
+ (h+1)^2 \sum_{a = 1}^r \sum_{b = 1}^r P(Ap_a p_b, X; p_1, \ldots, p_{r-1}).
\]

This proves lemma 5.

Using lemma 5 \(t\) times successively and observing that \(r_t = 1\), we get

**Lemma 6.**
\[
P(A, X; p_1, \ldots, p_r) \geq P(A, X; p_1, \ldots, p_r) (h+1) \sum_{a = 1}^r P(Ap_a, X; p_1, \ldots, p_{r-1}) +
\]
\[
+ (h+1)^2 \sum_{a = 1}^r \sum_{b = 1}^r P(Ap_a p_b, X; p_1, \ldots, p_{r-1}).
\]

By the definition of \(P(\lambda, D, X)\) we have
\[
P(\lambda, D, X) = [X/D] + \theta, \quad \theta_i = 0 \text{ or } 1.
\]

Hence, for all \(\lambda\), we have \([P(\lambda, D, X) - X/D]\) \(\leq 1\) and from lemma 6 we get

**Lemma 7.** For any given \(\lambda, a, D, X\)
\[
M_\lambda(X) = P(A, X; p_1, \ldots, p_r) \geq X E A - R
\]
where
\[
E = 1 - (h+1)^2 \sum_{a = 1}^r \sum_{p = 1}^{1/p} \sum_{q_j^c} P_a p_r -
\]
\[
+ (h+1)^2 \sum_{a = 1}^r \sum_{b = 1}^r P_a p_b p_r + \ldots +
\]
\[
+ (h+1)^2 \sum_{a = 1}^r \sum_{b = 1}^r \sum_{c = 1}^r P_a p_b p_c + \ldots + P_r
\]
and
\[
R \leq 1 + (h+1) \sum_{r=1}^{\infty} 1 + (h+1)^{r} \sum_{q_{1} \leq r} \sum_{q_{1} \leq r} 1 + \ldots + \ldots \sum_{q_{1} \leq r} \sum_{q_{1} \leq r} 1 + \ldots + \ldots
\]
\[+(h+1)^{r} \sum_{q_{1} \leq r} \sum_{q_{1} \leq r} \ldots \sum_{q_{1} \leq r} \sum_{q_{1} \leq r} \sum_{q_{1} \leq r} \sum_{q_{1} \leq r} 1 \leq [1 + (h+1)r] \prod_{r=1}^{\infty} (1 + (h+1)r)^{y}.
\]

§ 4. Estimation of \(R\). In this section and the next we shall always assume that \(c_{1}, c_{4}, \ldots\) are positive constants depending on \(h\) only. Let \(r_{1}\) denote the least positive integer for which
\[
\sigma_{1} = \prod_{r_{1} < r_{0}} \left(1 - \frac{h+1}{p_{r}}\right) \geq \frac{1}{1.3}.
\]
Similarly, we define \(r_{n}\) as the least positive integer for which
\[
\sigma_{n} = \prod_{r_{n} < r_{n-1}} \left(1 - \frac{h+1}{p_{r}}\right) \geq \frac{1}{1.3}, \quad (1 \leq n \leq t-1),
\]
\[
\sigma_{n} = \prod_{r_{n} < r_{n-1}} \left(1 - \frac{h+1}{p_{r}}\right) \geq \frac{1}{1.3}.
\]

Since \(1 - (h+1)/p_{r} > 1 - (h+1)/10(h+1) = 9/10 > 1/1.3\), we finally have \(r_{1} = 1\). From the definition of \(r_{n}\), we have
\[
\frac{9}{10} r_{n} = \left(1 - \frac{h+1}{10(h+1)}\right) \sigma_{n} < \left(1 - \frac{h+1}{p_{r_{n}}}\right) \sigma_{n} < \frac{4}{5} \quad (1 \leq n \leq t-1)
\]
or
\[
\sigma_{n} < \frac{5}{6} \quad (1 \leq n \leq t-1).
\]
Hence
\[
\log(1 + (h+1)r_{n}) \leq c_{0} \log h r_{n} \leq c_{0} \log p_{r_{n}} < c_{0} \prod_{r_{1} < r_{n}} (1 - 1/p_{r})^{-1} \prod_{p < \infty} (1 - 1/p)^{-1}
\]
\[< c_{0} \prod_{r=1}^{n} \prod_{p < r_{n}} (1 - (h+1)/p_{r}) \prod_{r=1}^{n} (1 - (h+1)/p_{r})^{-1} \prod_{p < \infty} (1 - (h+1)/p)^{-1}
\]
\[< c_{0} \prod_{p < \infty} (1 - (h+1)/p)^{-1} \sum_{n} (8/9)^{n} = c_{0} \prod_{p < \infty} (1 - 1/p)^{-1} < c_{0} \log Z (\ast)
\]

§ 5. Estimation of \(E\). Let \(E_{n}^{(i)} (1 \leq n \leq t-1)\) be the \(i\)-th elementary symmetric function of
\[
[(h+1)/(p_{n+1}), \ldots, (h+1)/(p_{n-1})]
\]
and let \(E_{n}^{(1)}\) be the \(1\)-th elementary symmetric function of
\[
[(h+1)/(p_{1}), \ldots, (h+1)/(p_{n})].
\]

Put
\[
E = E_{n}, \quad E_{n} = E_{n}^{(0)} - E_{n}^{(1)} + \ldots - E_{n}^{(2n-2)} + E_{n}^{(2n)}
\]
where \(E_{n}^{(n)} = 1 (1 \leq n \leq t), \ E_{n}^{(1)} (1 \leq n \leq t-1)\) denotes the absolute value of the sum of terms of \(E\) with exactly \(r\) prime factors the indices of those prime factors being greater than \(r_{n}\), and \(E_{n}^{(r)}\) denotes the absolute value of the sum of all terms of \(E\) with exactly \(r\) prime factors. We have
\[
E_{n}^{(r)} = E_{n}^{(r-1)} + E_{n}^{(r-1)} S_{n}^{(1)} + \ldots + E_{n}^{(r-1)} S_{n}^{(n-r)} + S_{n}^{(r)}
\]
\[(2 \leq n \leq t, \ 1 \leq r \leq 2n-1).\]

It is clear that \(E_{n}^{(0)} = 0\) if \(n \leq t-1, r \geq 2n\) and
\[
E_{n}^{(r)} \leq E_{n}^{(r-1)} + E_{n}^{(r-1)} S_{n}^{(1)} + \ldots + E_{n}^{(r-1)} S_{n}^{(n-r)} + S_{n}^{(r)},
\]
\[(2 \leq n \leq t, 1 \leq r \leq 2n),
\]
\[(\ast)\] The fact that \(\prod_{p < \infty} (1 - 1/p)^{-1} \ll \log Z\), for \(\alpha > 2\), where \(\alpha\) is an absolute positive constant, implies that
\[
\prod_{h+1 \leq p < \alpha} (1 - h+1/p)^{-1} \ll c_{0} (h+1) \log Z
\]
for \(h > h+1\), where \(c_{0}\) is a positive constant depending on \(h\) only.
From (16) and (17) we have
\[ S^{(2)}_n = \sum_{\pi_p \in \pi_{n-1}} \frac{\pi + 1}{\pi_p} < -\log n < \log 1.3 < 0.3 \quad (1 \leq n \leq t-1), \]
\[ S^{(1)}_n = \sum_{\pi_p \in \pi_{n-1}} \frac{\pi + 1}{\pi_p} < -\log n < \log 1.3 < 0.3. \]
If \( r > 1 \) and \( 1 \leq n \leq t-1 \), we get
\[ rS^{(2)}_n = \sum_{\pi_p \in \pi_{n-1}} \frac{1}{r} \sum_{q=1}^{\pi + 1} \left( \frac{1}{q} \right) < \frac{1}{\pi_p} \sum_{q=1}^{\pi + 1} \left( \frac{1}{q} \right) = S^{(2)}_{\pi_p + 1} - S^{(2)}_n, \]
where \( \pi_p \) runs over all prime numbers such that \( \pi_p < p \leq p_{n-1} \), \( q \) runs over all products \( \pi_1 \cdots \pi_q \) of \( r \) different prime numbers, \( q \) runs over all products \( \pi_1 \cdots \pi_q \) of \( r-1 \) different prime numbers.
Similarly, we have \( rS^{(1)}_n < S^{(1)}_{\pi_p + 1} - S^{(1)}_n (r > 1) \). Hence, by (23), we have
\[ S^{(1)}_n < S^{(1)}_{\pi_p + 1} - S^{(1)}_n (r > 1), \]
\[ S^{(2)}_n < S^{(2)}_{\pi_p + 1} - S^{(2)}_n (r > 1). \]
From (24) we immediately get
\[ \sum_{n \geq 1} (-1)^n S^{(n-1)}_n \leq \mathcal{S}^{(2n-1)}_n (1 \leq n \leq t). \]
By (20) and (21), if \( 2 \leq n \leq t-1 \), we have
\[ E_n = \sum_{i \geq 1} (-1)^i E^{(i)}_n = \sum_{i \geq 1} (-1)^i \sum_{\pi_p \in \pi_{n-1}} E^{(i)}_n = \sum_{i \geq 1} (-1)^i E^{(i)}_n \]
\[ \sum_{i \geq 1} (-1)^i E^{(i)}_n \leq \sum_{i \geq 1} (-1)^i E^{(i)}_n = \pi \sum_{i \geq 1} (-1)^i E^{(i)}_n \]
\[ \sum_{i \geq 1} (-1)^i E^{(i)}_n \leq \sum_{i \geq 1} (-1)^i E^{(i)}_n. \]
Similarly we have
\[ E_i = \sum_{n \geq 1} (-1)^n E^{(n)}_n = \pi \sum_{n \geq 1} (-1)^n E^{(n)}_n = \sum_{n \geq 1} (-1)^n E^{(n)}_n + E^{(2n)}. \]
Hence, from (24) and (25), we have
\[ E_n \geq \pi E_{n-1} - \Phi_n \quad (2 \leq n \leq t), \]
where
\[ \Phi_n = \sum_{i \geq 1} E^{(i)}_n S^{(2n-1)}_n \quad (2 \leq n \leq t). \]
Hence, we get
\[ E = E_1 \geq \pi E_{n-1} - \Phi_n \geq \pi (E_{n-1} - \Phi_{n-1}) - \Phi_n \]
\[ \geq \pi \sum_{i \geq 1} \left( E_{n-1} - \Phi_{n-1} \right) \]
\[ \geq \pi \sum_{i \geq 1} \left( \frac{1}{E_{n-1}} \right) \]
\[ 1 \leq i \leq t. \]
From (21), (22), (23), (24), we have for all \( u \geq 0, t \geq 2 \)
\[ 2^n \left( \sum_{i \geq 1} \frac{1}{i!} \right) \leq \sum_{i \geq 1} \frac{1}{i!} \leq \frac{2^n}{u!} + \frac{(0,3)^{2^n - 1}}{(u-1)!} \leq \frac{2^n}{u!} + \frac{2^n}{(u-1)!} \leq \frac{2^n}{u!} + \frac{(0,3)^{2^n - 1}}{(u-1)!} \leq 6. \]
By induction we get
\[ 2^n \left( \sum_{i \geq 1} \frac{1}{i!} \right) \leq 6 e^{2^n} \quad (t \geq 2) \quad \text{for all } t. \]
We have shown that (29) is true for \( n = 2 \). Now suppose it is true for \( n \geq 2 \); then
\[ 2^n \left( \sum_{i \geq 1} \frac{1}{i!} \right) \leq \sum_{i \geq 1} \frac{1}{i!} \leq \frac{2^n}{u!} + \frac{(0,3)^{2^n - 1}}{(u-1)!} \leq \frac{2^n}{u!} + \frac{(0,3)^{2^n - 1}}{(u-1)!} \leq 6 e^{2^n} \]
\[ \leq 6 e^{2^n} \sum_{i \geq 1} \frac{2^n - \mu}{i!} \leq 6 e^{2^n} \]
\[ \sum_{i \geq 1} \frac{2^n}{(i-1)!} \leq 6 e^{2^n}. \]
It completes the proof.
If $t \geq n \geq 3$, we get
\[
\Phi_n = \sum_{s=3}^{n-1} \text{Prob}_s^{(0)} \Phi_n^{(0)} \leq 2^{n-3} \sum_{s=3}^{n-1} 2^{s-2} (0,3)^{s-2} \frac{(2n-s)!}{(2n-2)!} \leq 6^{n-4} \frac{(0,3)^{n-2}}{2} < 6^{n-4} (0,3)^n (e^2 - 5),
\]
and we also easily get
\[
\Phi_s = 8^{s-1} 8^{s-2} 2^4 < \frac{(0,3)^s}{3!} < 0.0018.
\]
From (16), (17), (28), (30) and (31) we get
\[
B > \prod_{s=3}^{r} \left( 1 - \frac{h+1}{p_s} \right) \times \sum_{s=3}^{\infty} \frac{(0,3)^s}{2} \frac{e^{0.3} (1,3)^{-1}}{4!}.
\]

§ 6. Proof of lemma 2. From lemma 7, (19) and (32) we have
\[
M_n(X) > \frac{X}{\log^{3+1} Z - Z^v}
\]
for any given $l$ and $a_l$. We take $c_l = l/(c_l+1)$ and $Z = X^{3l}$. It is obvious that there exist $c_l$ and $X_l$, depending on $A_n$ only, such that
\[
M_n(X) > \frac{c_l X}{\log^{3+1} X} \quad \text{for} \quad X > X_l, \quad \text{q.e.d.}
\]

§ 7. Theorems on the functions $\varphi$ and $\sigma$.

THEOREM 2. For any given sequence a of $h$ non-negative numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$, there exists a positive integer $n$ such that
\[
\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i < \varepsilon \quad (1 \leq i \leq h).
\]

There exist positive constants $c = c(a, \varepsilon)$ and $X_0 = X_0(a, \varepsilon)$ such that in any interval $1 \leq n \leq X$ the number of $n$ satisfying (33) is greater than $cX \log^{3+1} X$ whenever $X > X_0$.
§8. Theorems on the Function \( f \). We now prove the following theorem:

**Theorem 8.** For any given positive integer \( q \), there exists a positive integer \( n \) such that

\[ (q + 1)! \equiv 0 \pmod{n}, \]

for all \( (q + 1)! \leq n \leq q^2 \).

To prove this theorem, we proceed as follows:

1. **Case 1:** If \( q \) is even, then \( (q + 1)! \equiv 0 \pmod{q} \). For any \( n \) such that \( (q + 1)! \leq n \leq q^2 \), we have

\[ (q + 1)! \equiv 0 \pmod{n}. \]

2. **Case 2:** If \( q \) is odd, then \( (q + 1)! \equiv 0 \pmod{2q} \). For any \( n \) such that \( (q + 1)! \leq n \leq q^2 \), we have

\[ (q + 1)! \equiv 0 \pmod{n}. \]

In both cases, we have shown that for any given positive integer \( q \), there exists a positive integer \( n \) such that \( (q + 1)! \equiv 0 \pmod{n} \) for all \( (q + 1)! \leq n \leq q^2 \).

**Proof.** Let \( a = \min\{q, q+1\} \). Since \( a \) is a divisor of \( (q+1)! \), we have

\[ (q+1)! \equiv 0 \pmod{a}. \]

Now, let \( n = (q+1)k \), where \( k \) is a positive integer such that \( (q+1)! \leq n \leq q^2 \). Then, we have

\[ (q+1)! \equiv 0 \pmod{n}. \]

This completes the proof of the theorem.