References

[1] G. Alexits, *Über Transformierten der arithmetischen Mittel von Orthogonalreihen*, Acta Math. Ac. Hungaricae 2 (1951), p. 1-8.

[2] S. Borgen, *Über (C, 1)-Summierbarkeit von Reihen orthogonaler Funktionen*, Math. Annalen 98 (1928), p. 125-150.

[3] L. S. Bosanquet, *Note on convergence and summability factors (III)*, Proc. of the London Math. Soc. Ser. 2, 50 (1949), p. 482-496.

[4] Г. М. Фихтенгольц, *Курс дифференциального и интегрального исчисления II*, Москва 1951.

[5] S. Kaczmarz, *Über die Konvergenz der Reihen von Orthogonalfunktionen*, Math. Zeitsch. 23 (1925), p. 263-270.

[6] — *Sur la convergence et sommabilité des développements orthogonaux*, Studia Math. 1 (1929), p. 81-121.

[7] — und H. Steinhaus, *Theorie der Orthogonalreihen*, Warszawa-Lwów 1935.

[8] L. Kronecker, *Quelques remarques sur la détermination des valeurs moyennes*, Comptes Rendus Paris 103 (1886), p. 980-987.

[9] Gen-ichiro, Sunouchi and Shigeki, Yano, *Convergence and summability of orthogonal series*, Proc. Japan Acad. 26, No. 7 (1950), p. 10-16.

[10] O. Toeplitz, *Über allgemeine lineare Mittelbildungen*, Prace Mat.-Fiz. 22 (1911), p. 113-119.

[11] A. Zygmund, *Trigonometrical series*, Warszawa 1935.

[12] — *Remarque sur un théorème de M. Kaczmarz*, Math. Zeitschr. 25 (1926), p. 297-298.

[13] — *Remarque sur la sommabilité des séries*, Bull. Ac. Pol. (1926), p. 186.

[14] — *Sur l'application de la première moyenne arithmétique dans la théorie des séries de fonctions orthogonales*, Fundamenta Math. 10 (1927), p. 356-362.

Reçu par la Rédaction le 16. 7. 1956

A note on some properties of the functions $\varphi(n)$, $\sigma(n)$ and $\theta(n)$

by A. Schinzel (Warszawa) and Y. Wang (Peking)

§ 1. Introduction. A. Schinzel has proved in [4] that for every sequence a of h positive numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$ there exist natural numbers n and n' such that

$$\left|\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i\right| < \varepsilon, \qquad \left|\frac{\sigma(n'+i)}{\sigma(n'+i-1)} - a_i\right| < \varepsilon \qquad (i = 1, 2, \ldots, h)\,(^1).$$

Professor Hua Loo-Keng has pointed out that by Brun's method we can prove the existence of positive constants $c = c(a, \varepsilon)$ and $X_0 = X_0(a, \varepsilon)$ such that the number of numbers n satisfying the first of these inequalities in the interval $1 \leqslant n \leqslant X$ is greater than

$$cX/\log^{h+1}X \quad \text{for} \quad X > X_0.$$

In the present paper we give the proof of this theorem, of an analogous theorem on the function $\sigma(n)$ and of a theorem on the function $\theta(n)$(²) which is weaker but gives a positive solution of the problem put forward in paper [2] of A. Schinzel and comprises the theorem from paper [3] of A. Schinzel.

The question whether a theorem analogous to the theorems on functions φ and σ is true for the function θ remains open.

§ 2. An auxiliary theorem. Let

$$A_0 = h!\,q_1 \ldots q_s q_{01} \ldots q_{0t_0}, \qquad A_i = q_{i1} \ldots q_{it_i} \quad (1 \leqslant i \leqslant h)$$

be positive integers, where $q_1, q_2, \ldots, q_s$ are all the prime numbers in the interval $0 < x \leqslant 10(h+1)$ and q_{ij} $(0 \leqslant i \leqslant h,\ 1 \leqslant j \leqslant t_i)$ are primes greater than $10(h+1)$ such that $A_0, A_1, \ldots, A_h > 1$ are relatively prime in pairs.

(¹) $\varphi(n)$ denotes the Euler function, $\sigma(n)$ — the sum of divisors of number n.
(²) $\theta(n)$ denotes the number of divisors of n.

If $Z > A_0A_1^2 \dots A_h^2$, then let us denote by $N_Z(X)$ the number of integral solutions $(x_0, \dots, x_h)$ of the system of equations

$$(1) \qquad A_0x_0+i = iA_ix_i \quad (1 \leqslant i \leqslant h)$$

satisfying the conditions

$$(2) \qquad 1^\circ\ 1 \leqslant x_0 \leqslant X, \quad \text{and} \quad 2^\circ \text{ if } p|x_i, \text{ then } p > Z \ (0 \leqslant i \leqslant h),$$

where p denotes a prime.

THEOREM 1. *There exist positive constants c_1, depending on h only, and c_2, X_1, depending on A_i's only, such that*

$$N_{Xc_1(X)} > c_2X/\log^{h+1}X \quad (X > X_1).$$

Proof. If $Z > A_0A_1^2 \dots A_h^2$, λ is a given integer in the interval $0 \leqslant \lambda < A_0A_1^2 \dots A_h^2$ and $p_1 < p_2 < \dots < p_r$ are all prime numbers not dividing $A_0A_1 \dots A_h$ and not exceeding Z and if a_{ji} $(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h)$ are given integers satisfying the conditions $0 \leqslant a_{ji} < p_j$ $(1 \leqslant j \leqslant r,$ $0 \leqslant i \leqslant h)$ and $a_{ji_1} \neq a_{ji_2}$ for $i_1 \neq i_2$ $(1 \leqslant j \leqslant r)$, then we can define $M_Z(X)$ as the number of x satisfying the conditions

$$(3) \qquad 1 \leqslant x \leqslant X, \quad x \equiv \lambda (\bmod A_0A_1^2 \dots A_h^2), \quad x \not\equiv a_{ji} (\bmod p_j)$$
$$(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h).$$

It is evident, that theorem 1 is a consequence of the following two lemmas:

LEMMA 1. *There exist λ and $a_{ji}'s$ such that*

$$N_Z(X) \geqslant M_Z(X).$$

LEMMA 2. *There exist positive constants c_1, depending on h only, and c_2, X_1, depending on A_i's only, such that*

$$M_{Xc_1(X)} \geqslant c_2X/\log^{h+1}X \quad (X \geqslant X_1)$$

for any given λ and $a_{ji}'s$.

Proof of lemma 1. First we shall define λ and $a_{ji}'s$ as follows: By lemma 2 [4] there exists m such that

$$A_i|m+i, \quad \left(A_i, \frac{m+i}{A_i}\right) = 1, \quad A_0^2A_1^2 \dots A_h^2 > m+i > 0 \quad (0 \leqslant i \leqslant h).$$

Let $\lambda = m/A_0$. The solution of the following congruence

$$(4) \qquad A_0y+i \equiv 0 (\bmod p_j) \quad (0 \leqslant y < p_j)$$

will be denoted by a_{ji} $(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h)$. It is evident that $a_{j0} = 0$ $(1 \leqslant j \leqslant r)$. If $i_1 \neq i_2$ and $a_{ji_1} = a_{ji_2}$, then from (4) we have $p_j | i_1 - i_2$. We obtain a contradiction since $0 < |i_1-i_2| < h$ and $p_j > 10(h+1)$.

We take x satisfying (3) with these λ and $a_{ji}'s$ and define $x = x_0$. From (1)-(3) of [4]

$$(5) \qquad A_0x_0+i = iA_ix_i \quad (1 \leqslant i \leqslant h),$$

where

$$(6) \qquad (x_i, A_0A_1 \dots A_h) = 1 \quad (0 \leqslant i \leqslant h).$$

From (4), since $x \not\equiv a_{ji} (\bmod p_j)$ $(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h)$, we have

$$(7) \qquad \big(A_0x_0(A_0x_0+1) \dots (A_0x_0+h),\ p_1 \dots p_r\big) = 1.$$

From (5), (6), (7) we find that if $p|x_i$, then $p > Z$ $(0 \leqslant i \leqslant h)$. Thus we have proved that there exist λ and $a_{ji}'s$ such that from any x satisfying conditions (3) we can construct a solution of (1) satisfying (2) and different x correspond to the different solutions of (1). Thus we have proved lemma 1. In § 6 a proof of lemma 2 is given. This proof is obtained by a modification of a method elaborated by H. Rademacher [1] in the case $h = 1$. We shall precede it by lemmas and estimations in § 3-§ 5.

§ 3. Some lemmas. Let $A = A_0A_1^2 \dots A_h^2$ and write

$$(8) \qquad M_Z(X) = P(\lambda, a, A, X; p_1, \dots, p_r)$$

for given λ and a_{ji} $(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h)$. In particular, let $P(\lambda, A, X)$ denote the number of x satisfying the conditions $1 \leqslant x \leqslant X$ and $x \equiv \lambda (\bmod A)$.

LEMMA 3. *There exist integers λ_i $(0 \leqslant i \leqslant h)$ satisfying $0 \leqslant \lambda_i \leqslant Ap_r$ $(0 \leqslant i \leqslant h)$ and $\lambda_{i_1} \neq \lambda_{i_2}$ $(i_1 \neq i_2)$ such that*

$$(9) \qquad P(\lambda, a, A, X; p_1, \dots, p_r) = P(\lambda, a, A, X; p_1, \dots, p_{r-1}) - \sum_{i=0}^{h} P(\lambda_i, a, Ap_r, X; p_1, \dots, p_{r-1}).$$

Proof. By definition, $P(\lambda, a, A, X; p_1, \dots, p_r)$ is equal to the difference between the number of x satisfying the conditions

$$(10) \qquad 1 \leqslant x \leqslant X, \quad x \equiv \lambda (\bmod A), \quad x \not\equiv a_{ji} (\bmod p_j)$$
$$(1 \leqslant j \leqslant r-1,\ 0 \leqslant i \leqslant h)$$

and the number of x satisfying (10) and one of the following congruences:

$$(11) \qquad x \equiv a_{ri} (\bmod p_r) \quad (0 \leqslant i \leqslant h).$$

Since $(A, p_r) = 1$, each of the following systems of congruences is solvable and has a unique solution in the interval $0 \leqslant x < Ap_r$

$$\begin{cases} x \equiv \lambda (\mathrm{mod}\, A), \\ x \equiv a_{r0} (\mathrm{mod}\, p_r), \end{cases} \quad \begin{cases} x \equiv \lambda (\mathrm{mod}\, A), \\ x \equiv a_{r1} (\mathrm{mod}\, p_r), \end{cases} \quad \ldots \quad \begin{cases} x \equiv \lambda (\mathrm{mod}\, A), \\ x \equiv a_{rh} (\mathrm{mod}\, p_r). \end{cases}$$

Denote these solutions by $\lambda_0, \lambda_1, \ldots, \lambda_h$ respectively. Since

$$a_{ri_1} \not\equiv a_{ri_2} (\mathrm{mod}\, p_r) \qquad (i_1 \neq i_2)$$

we have $\lambda_{i_1} \neq \lambda_{i_2}$ $(i_1 \neq i_2)$. Hence $P(\lambda, a, A, X; p_1, \ldots, p_r)$ is equal to the difference between the number x satisfying (10) and the number of x satisfying one of the following conditions (as $i = 0, 1, \ldots,$):

$$(12) \qquad 1 \leqslant x \leqslant X, \quad x \equiv \lambda_i (\mathrm{mod}\, Ap_r), \quad x \equiv a_{ji} (\mathrm{mod}\, p_j)$$
$$(1 \leqslant j \leqslant r-1,\ 0 \leqslant i \leqslant h).$$

This proves lemma 3.

Briefly we write

$$(13) \qquad \begin{aligned} P(\lambda, a, D, X; p_1, \ldots, p_r) &= P(D, X; p_1, \ldots, p_r), \\ P(\lambda, D, X) &= P(D, X). \end{aligned}$$

Hence, it follows from (9) that, with the usual convention in notation, we have

$$(14) \qquad P(A, X; p_1, \ldots, p_r)$$
$$= P(A, X; p_1, \ldots, p_{r-1}) - (h+1) P(Ap_r, X; p_1, \ldots, p_{r-1}).$$

Using lemma 3 r times successively we get

LEMMA 4.

$$P(A, X; p_1, \ldots, p_r) = P(A, X) - (h+1) \sum_{a=1}^{r} P(Ap_a, X; p_1, \ldots, p_{a-1}).$$

Let

$$(15) \qquad r = r_0 \geqslant r_1 \geqslant \ldots \geqslant r_t = 1$$

be any given sequence of t positive integers.

LEMMA 5.

$$P(A, X; p_1, \ldots, p_r) \geqslant P(A, X) - (h+1) \sum_{a=1}^{r} P(Ap_a, X) +$$
$$+ (h+1)^2 \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} P(Ap_a p_{a_1}, X; p_1, \ldots, p_{a_1-1}).$$

Proof. By lemma 4, we have

$$P(A, X; p_1, \ldots, p_r) = P(A, X) - (h+1) \sum_{a=1}^{r} P(Ap_a, X) +$$
$$+ (h+1)^2 \sum_{a=1}^{r} \sum_{a_1 < a} P(Ap_a p_{a_1}, X; p_1, \ldots, p_{a_1-1}).$$

This proves lemma 5.

Using lemma 5 t times successively and observing that $r_t = 1$, we get

LEMMA 6.

$$P(A, X; p_1, \ldots, p_r) \geqslant P(A, X) - (h+1) \sum_{a=1}^{r} P(Ap_a, X) +$$
$$+ (h+1)^2 \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} P(Ap_a p_{a_1}, X) - (h+1)^3 \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} \sum_{\substack{\beta_1 \leqslant r_1 \\ \beta_1 < a_1}} P(Ap_a p_{a_1} p_{\beta_1}, X) + \ldots +$$
$$+ (h+1)^{2t} \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} \sum_{\substack{\beta_1 \leqslant r_1 \\ \beta_1 < a_1}} \ldots \sum_{\substack{a_{t-1} \leqslant r_{t-1} \\ a_{t-1} < \beta_{t-2}}} \sum_{\substack{\beta_{t-1} \leqslant r_{t-1} \\ \beta_{t-1} < a_{t-1}}} \sum_{\substack{a_t \leqslant r_t \\ a_t < \beta_{t-1}}} P(Ap_a p_{a_1} p_{\beta_1} \ldots p_{a_t}, X).$$

By the definition of $P(\lambda, D, X)$ we have

$$P(\lambda, D, X) = [X/D] + \theta_\lambda, \qquad \theta_\lambda = 0 \text{ or } 1.$$

Hence, for all λ, we have $|P(\lambda, D, X) - X/D| \leqslant 1$ and from lemma 6 we get

LEMMA 7. *For any given λ and $a_{ji}'s$*

$$M_Z(X) = P(A, X; p_1, \ldots, p_r) \geqslant X \frac{E}{A} - R$$

where

$$E = 1 - (h+1) \sum_{a=1}^{r} \frac{1}{p_a} + (h+1)^2 \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} \frac{1}{p_a p_{a_1}} -$$
$$- (h+1)^3 \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} \sum_{\substack{\beta_1 \leqslant r_1 \\ \beta_1 < a_1}} \frac{1}{p_a p_{a_1} p_{\beta_1}} + \ldots +$$
$$+ (h+1)^{2t} \sum_{a=1}^{r} \sum_{\substack{a_1 \leqslant r_1 \\ a_1 < a}} \sum_{\substack{\beta_1 \leqslant r_1 \\ \beta_1 < a_1}} \ldots \sum_{\substack{a_{t-1} \leqslant r_{t-1} \\ a_{t-1} < \beta_{t-2}}} \sum_{\substack{\beta_{t-1} \leqslant r_{t-1} \\ \beta_{t-1} < a_{t-1}}} \sum_{\substack{a_t \leqslant r_t \\ a_t < \beta_{t-1}}} \frac{1}{p_a p_{a_1} p_{\beta_1} \ldots p_{a_t}}$$

and

$$R \leqslant 1+(h+1)\sum_{a=1}^{r} 1+(h+1)^2\sum_{a=1}^{r}\sum_{\substack{a_1\leqslant r_1\\ a_1\leqslant a}} 1+(h+1)^3\sum_{a=1}^{r}\sum_{\substack{a_1\leqslant r_1\\ a_1<a}}\sum_{\substack{\beta_1\leqslant r_1\\ \beta_1<a_1}} 1+\ldots+$$

$$+(h+1)^{2t}\sum_{a=1}^{r}\sum_{\substack{a_1\leqslant r_1\\ a_1<a}}\sum_{\substack{\beta_1\leqslant r_1\\ \beta_1<a_1}}\ldots\sum_{\substack{a_{t-1}\leqslant r_{t-1}\\ \beta_{t-1}<a_{t-1}}}\sum_{\substack{\beta_{t-1}\leqslant r_{t-1}\\ \beta_{t-1}<a_{t-1}}}\sum_{\substack{a_t\leqslant r_t\\ a_t<\beta_{t-1}}} 1$$

$$\leqslant [1+(h+1)r]\prod_{n=1}^{t}[1+(h+1)r_n]^2.$$

§ 4. Estimation of R. In this section and the next we shall always assume that $c_3, c_4, \ldots$ are positive constants depending on h only. Let r_1 denote the least positive integer for which

$$\pi_1 = \prod_{r_1<s\leqslant r_0}\left(1-\frac{h+1}{p_s}\right) \geqslant \frac{1}{1{,}3}.$$

Similarly, we define r_n as the least positive integer for which

$$\pi_n = \prod_{r_n<s\leqslant r_{n-1}}\left(1-\frac{h+1}{p_s}\right) \geqslant \frac{1}{1{,}3} \quad (1\leqslant n\leqslant t-1), \tag{16}$$

$$\pi_t = \prod_{r_t\leqslant s\leqslant r_{t-1}}\left(1-\frac{h+1}{p_s}\right) \geqslant \frac{1}{1{,}3}. \tag{17}$$

(Since $1-(h+1)/p_s > 1-(h+1)/10(h+1) = 9/10 > 1/1{,}3$, we finally have $r_t = 1$.) From the definition of r_n, we have

$$\frac{9}{10}\pi_n = \left(1-\frac{h+1}{10(h+1)}\right)\pi_n < \left(1-\frac{h+1}{p_{r_n}}\right)\pi_n < \frac{1}{1{,}3} < \frac{4}{5} \quad (1\leqslant n\leqslant t-1)$$

or

$$\pi_n < \tfrac{8}{9} \quad (1\leqslant n\leqslant t-1). \tag{18}$$

Hence

$$\log[1+(h+1)r_n]^2 \leqslant c_3\log hr_n \leqslant c_4\log p_{r_n} < c_5\prod_{s=1}^{r_n}(1-1/p_s)^{-1}\prod_{p|\Delta}(1-1/p)^{-1}$$

$$< c_5\prod_{j=1}^{n}\prod_{r_j<s\leqslant r_{j-1}}\big(1-(h+1)/p_s\big)\prod_{s=1}^{r}\big(1-(h+1)/p_s\big)^{-1}\prod_{p|\Delta}\big(1-(h+1)/p\big)^{-1}$$

$$< c_5\prod_{p\leqslant Z}\big(1-(h+1)/p\big)^{-1}(8/9)^n \quad (0\leqslant n\leqslant t-1).$$

Then by lemma 7

$$\log R \leqslant \log\Big\{[1+(h+1)r_0]\prod_{n=1}^{t-1}[1+(h+1)r_n]^2(h+2)^2\Big\}$$

$$< c_6\prod_{p\leqslant Z}(1-1/p)^{-1}\sum_{n=0}^{\infty}(8/9)^n = 9c_6\prod_{p\leqslant Z}(1-1/p)^{-1} < c_7\log Z \text{ (3)}$$

i. e.,

$$R \leqslant \exp(c_7\log Z) = Z^{c_7}. \tag{19}$$

§ 5. Estimation of E. Let $S_n^{(l)}$ $(1\leqslant n\leqslant t-1)$ be the l-th elementary symmetric function of

$$\big\{(h+1)/(p_{r_n+1}), \ldots, (h+1)/(p_{r_{n-1}})\big\}$$

and let $S_t^{(l)}$ be the l-th elementary symmetric function of

$$\big\{(h+1)/p_{r_t}, \ldots, (h+1)/p_{r_{t-1}}\big\}.$$

Put

$$E = E_t, \qquad E_n = E_n^{(0)} - E_n^{(1)} + \ldots - E_n^{(2n-1)} + E_n^{(2n)} \tag{20}$$

where $E_n^{(0)} = 1$ $(1\leqslant n\leqslant t)$, $E_n^{(\nu)}$ $(1\leqslant n\leqslant t-1)$ denotes the absolute value of the sum of terms of E with exactly ν prime factors the indices of those prime factors being greater than r_n, and $E_t^{(\nu)}$ denotes the absolute value of the sum of all terms of E with exactly ν prime factors. We have

$$E_n^{(\nu)} = E_{n-1}^{(\nu)} + E_{n-1}^{(\nu-1)}S_n^{(1)} + \ldots + E_{n-1}^{(1)}S_n^{(\nu-1)} + S_n^{(\nu)} \tag{21}$$

$$(2\leqslant n\leqslant t,\ 1\leqslant \nu\leqslant 2n-1).$$

It is clear that $E_n^{(\nu)} = 0$ if $n\leqslant t-1$, $\nu\geqslant 2n$ and

$$\begin{gathered} E_t^{(2t)} \leqslant E_{t-1}^{(2t)} + E_{t-1}^{(2t-1)}S_t^{(1)} + \ldots + E_{t-1}^{(1)}S_t^{(2t-1)} + S_t^{(2t)},\\ E_t^{(\nu)} = 0 \quad (\nu > 2t). \end{gathered} \tag{22}$$

(3) The fact that $\prod_{p\leqslant x}(1-1/p)^{-1} \leqslant a\log x$, for $x\geqslant 2$, where a is an absolute positive constant, implies that

$$\prod_{h+1<p\leqslant x}\left(1-\frac{h+1}{p}\right)^{-1} \leqslant a(h)\log^{h+1}x,$$

for $x > h+1$, where $a(h)$ is a positive constant depending on h only.

From (16) and (17) we have

$$(23)\qquad \begin{aligned} S_n^{(1)} &= \sum_{r_n<s\leqslant r_{n-1}} \frac{h+1}{p_s} < -\log\pi_n < \log 1{,}3 < 0{,}3 \qquad (1\leqslant n\leqslant t-1),\\ S_t^{(1)} &= \sum_{r_t\leqslant s\leqslant r_{t-1}} \frac{h+1}{p_s} < -\log\pi_t < \log 1{,}3 < 0{,}3 .\end{aligned}$$

If $\nu>1$ and $1\leqslant n\leqslant t-1$, we get

$$\nu S_n^{(\nu)} = \sum_{p_s'q'=q} 1 \sum_{q} \frac{(h+1)^\nu}{q} = \sum_{\substack{p_s',q'\\(p_s',q')=1}} \frac{(h+1)^\nu}{p_s'q'}$$

$$\leqslant \sum_{p_s'} \frac{h+1}{p_s'} \sum_{q'} \frac{(h+1)^{\nu-1}}{q'} = S_n^{(1)} S_n^{(\nu-1)},$$

where p_s' runs over all prime numbers such that $p_{r_n}<p_s'\leqslant p_{r_{n-1}}$, q runs over all products $p_1'\dots p_\nu'$ of ν different prime numbers, q' runs over all products $p_1'\dots p_{\nu-1}'$ of $\nu-1$ different prime numbers.

Similarly, we have $\nu S_t^{(\nu)}\leqslant S_t^{(1)}S_t^{(\nu-1)}$ $(\nu>1)$. Hence, by (23), we have

$$(24)\qquad S_n^{(\nu)} < S_n^{(\nu-1)} \quad (\nu>1), \qquad S_n^{(\nu)} \leqslant \frac{(S_n^{(1)})^\nu}{\nu!} < \frac{(0{,}3)^\nu}{\nu!}$$
$$(1\leqslant n\leqslant t,\ \nu=1,2,\dots).$$

From (24) we immediately get

$$(25)\qquad \Big|\sum_{j\geqslant 2n-i} (-1)^j S_n^{(j)}\Big| \leqslant S_n^{(2n-i)} \qquad (1\leqslant n\leqslant t).$$

By (20) and (21), if $2\leqslant n\leqslant t-1$, we have

$$E_n = \sum_{\nu=0}^{2n-1} (-1)^\nu E_n^{(\nu)} = \sum_{\nu=0}^{2n-1} (-1)^\nu \sum_{i+j=\nu} E_{n-1}^{(i)} S_n^{(j)} = \sum_{i+j<2n} (-1)^{i+j} S_n^{(j)} E_{n-1}^{(i)}$$

$$= \sum_{i<2n-2} (-1)^i E_{n-1}^{(i)} \sum_{j<2n-i} (-1)^j S_n^{(j)}$$

$$= \sum_{i<2n-2} (-1)^i E_{n-1}^{(i)} \Big[\pi_n - \sum_{j\geqslant 2n-i} (-1)^j S_n^{(j)}\Big]$$

$$= \pi_n E_{n-1} - \sum_{i<2n-2} (-1)^i E_{n-1}^{(i)} \sum_{j\geqslant 2n-i} (-1)^j S_n^{(j)}.$$

Similarly we have

$$E_t = \sum_{\nu=0}^{2t} (-1)^\nu E_t^{(\nu)} = \pi_t E_{t-1} - \sum_{i<2t-2} (-1)^i E_{t-1}^{(i)} \sum_{j\geqslant 2t-i} (-1)^j S_t^{(j)} + E_t^{(2t)}.$$

Hence, from (24) and (25), we have

$$(26)\qquad E_n \geqslant \pi_n E_{n-1} - \Phi_n \qquad (2\leqslant n\leqslant t),$$

where

$$(27)\qquad \Phi_n = \sum_{i\leqslant 2n-3} E_{n-1}^{(i)} S_n^{(2n-i)} \qquad (2\leqslant n\leqslant t).$$

Hence, we get

$$(28)\qquad \begin{aligned} E &= E_t \geqslant \pi_t E_{t-1} - \Phi_t \geqslant \pi_t(E_{t-2}\pi_{t-1} - \Phi_{t-1}) - \Phi_t \\ &\geqslant \pi_2\dots\pi_t \Big\{E_1 - \frac{\Phi_2}{\pi_2} - \frac{\Phi_3}{\pi_2\pi_3} - \dots - \frac{\Phi_t}{\pi_2\pi_3\dots\pi_t}\Big\} \\ &> \prod_{s=1}^{r}\Big(1-\frac{h+1}{p_s}\Big)\Big\{1 - S_1^{(1)} - \frac{\Phi_2}{\pi_2} - \frac{\Phi_3}{\pi_2\pi_3} - \dots - \frac{\Phi_t}{\pi_2\pi_3\dots\pi_t}\Big\}.\end{aligned}$$

From (21), (22), (23), (24), we have for all $u\geqslant 0$, $t\geqslant 2$

$$2^u(0{,}3)^{-u}E_2^{(u)} \leqslant 2^u(0{,}3)^{-u} \sum_{i+j=u} E_1^{(i)} S_2^{(j)} \leqslant \sum_{i+j=u} E_1^{(i)} \frac{2^u(0{,}3)^{-i}}{j!}$$

$$\leqslant \frac{2^u}{u!} + \frac{(0{,}3)2^u(0{,}3)^{-1}}{(u-1)!} = \frac{2^u}{u!} + \frac{2^u}{(u-1)!} \leqslant 6 .$$

By induction we get

$$(29)\qquad 2^\nu(0{,}3)^{-\nu}E_n^{(\nu)} \leqslant 6e^{2n-4} \qquad (t\geqslant n\geqslant 2) \quad \text{for all } \nu .$$

We have shown that (29) is true for $n=2$. Now suppose it is true for $n\geqslant 2$; then

$$2^\nu(0{,}3)^{-\nu}E_{n+1}^{(\nu)} \leqslant 2^\nu(0{,}3)^{-\nu} \sum_{\mu=0}^{\nu} E_n^{(\mu)} S_{n+1}^{(\nu-\mu)}$$

$$< 2^\nu(0{,}3)^{-\nu} \sum_{\mu=0}^{\nu} 6e^{2n-4}2^{-\mu}(0{,}3)^\mu \frac{(0{,}3)^{\nu-\mu}}{(\nu-\mu)!}$$

$$= 6e^{2n-4} \sum_{\mu=0}^{\nu} \frac{2^{\nu-\mu}}{(\nu-\mu)!} < 6e^{2n-2}.$$

It completes the proof.

If $t \geqslant n \geqslant 3$, we get

$$(30)\qquad \Phi_n = \sum_{\nu=0}^{2n-3} E_{n-1}^{(\nu)} S_n^{(2n-\nu)} \leqslant 6e^{2n-6} \sum_{\nu=0}^{2n-3} 2^{-\nu}(0,3)^{\nu} \frac{(0,3)^{2n-\nu}}{(2n-\nu)!}$$

$$= 6e^{2n-6}\left(\frac{0,3}{2}\right)^{2n} \sum_{\nu=0}^{2n-3} \frac{2^{2n-\nu}}{(2n-\nu)!} < 6e^{2n-6}(0,2)^{2n}(e^2-5),$$

and we also easily get

$$(31)\qquad \Phi_2 = S_2^{(4)} + S_1^{(1)} S_2^{(3)} \leqslant \frac{(0,3)^4}{4!} + \frac{(0,3)^4}{3!} < 0,0018.$$

From (16), (17), (28), (30) and (31) we get

$$(32)\qquad E > \prod_{s=1}^{r}\left(1-\frac{h+1}{p_s}\right) \times$$

$$\times\left\{1-0,3-1,3\cdot 0,0018-6(e^2-5)\sum_{\nu=3}^{\infty}(0,2)^{2\nu}e^{2\nu-6}(1,3)^{\nu-1}\right\}$$

$$> 0,5\prod_{s=1}^{r}\left(1-\frac{h+1}{p_s}\right) \geqslant 0,5\prod_{p\leqslant Z}\left(1-\frac{h+1}{p}\right) > \frac{c_8}{\log^{h+1}Z}.$$

§ 6. Proof of lemma 2. From lemma 7, (19) and (32) we have

$$M_Z(X) > \frac{c_8}{A}\cdot\frac{X}{\log^{h+1}Z} - Z^{c_7}$$

for any given λ and $a_{ji}'s$. We take $c_1 = 1/(c_7+1)$ and $Z = X^{c_1}$. It is obvious that there exist c_2 and X_1, depending on $A_i's$ only, such that

$$M_{X^{c_1}}(X) > \frac{c_2 X}{\log^{h+1}X} \quad \text{for} \quad X > X_1, \quad \text{q. e. d.}$$

§ 7. Theorems on the functions φ and σ.

THEOREM 2. *For any given sequence a of h non-negative numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$, there exists a positive integer n such that*

$$(33)\qquad \left|\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i\right| < \varepsilon \quad (1 \leqslant i \leqslant h).$$

There exist positive constants $c = c(a, \varepsilon)$ and $X_0 = X_0(a, \varepsilon)$ such that in any interval $1 \leqslant n \leqslant X$ the number of n satisfying (33) is greater than $cX/\log^{h+1}X$ whenever $X > X_0$.

Proof. To begin with, by similar arguments as in the proofs of lemma 3a and of theorem 1 [4], we can choose $A_0, A_1, \ldots, A_h$, depending on $a_i's$ and ε only and satisfying the same conditions as in theorem 1, such that

$$(34)\qquad \left|\frac{\varphi(A_1)/A_1}{\varphi(A_0)/A_0} - a_1\right| < \frac{\varepsilon}{2} \quad\text{and}\quad \left|\frac{\varphi(iA_i)/iA_i}{\varphi[(i-1)A_{i-1}]/(i-1)A_{i-1}} - a_i\right| < \frac{\varepsilon}{2}$$
$$(2 \leqslant i \leqslant h).$$

For those $A_i's$ we assume that $(x_0, \ldots, x_h)$ is a solution of (1) satisfying (2) with $Z = X^{c_1}$. If we take $A_0 x_0 = n$, then $iA_i x_i = n+i$ $(1 \leqslant i \leqslant h)$. Since $(x_i, A_i) = 1$ $(0 \leqslant i \leqslant h)$, we have

$$(35)\qquad \begin{aligned} \frac{\varphi(n+1)}{\varphi(n)} &= \frac{\varphi(A_1x_1)}{\varphi(A_0x_0)} = \frac{(\varphi(A_1)/A_1)(\varphi(x_1)/x_1)A_1x_1}{(\varphi(A_0)/A_0)(\varphi(x_0)/x_0)A_0x_0} \\ &= \frac{\varphi(A_1)/A_1}{\varphi(A_0)/A_0}\cdot\frac{\varphi(x_1)/x_1}{\varphi(x_0)/x_0}\cdot\frac{n+1}{n}, \\ \frac{\varphi(n+i)}{\varphi(n+i-1)} &= \frac{\varphi(iA_i)/iA_i}{\varphi[(i-1)A_{i-1}]/(i-1)A_{i-1}}\cdot\frac{\varphi(x_i)/x_i}{\varphi(x_{i-1})/x_{i-1}}\cdot\frac{n+i}{n+i-1}. \end{aligned}$$

On account of $x_i \leqslant c_9 X$ $(0 \leqslant i \leqslant h)$ we can choose $X_2(A) = X_2(a, \varepsilon) > X_1$ such that the number of prime divisors (identical or different) of each x_i $(0 \leqslant i \leqslant h)$ does not exceed $c_{10} = [1/c_1]+2$ for $X > X_2$. Hence

$$1 \geqslant \frac{\varphi(x_i)}{x_i} = \prod_{p|x_i}\left(1-\frac{1}{p}\right) \geqslant \left(1-\frac{1}{X^{c_1}}\right)^{c_{10}} \to 1 \quad (\text{as } X \to \infty).$$

From (34) and (35) we can choose $X_3(a, \varepsilon) > X_2$ such that if $n = A_0x_0 > X_3$, then

$$\left|\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i\right| < \varepsilon \quad (1 \leqslant i \leqslant h).$$

Thus we have proved that from every solution of (1) satisfying (2) with $Z = X^{c_1}$ and such that $A_0x_0 > X_3$ we can define a positive integer $n = A_0x_0$ satisfying (33), and that evidently different solutions correspond to different n. It is clear that the number of solutions of (1) satisfying (2) with $Z = X^{c_1}$ and such that $A_0x_0 \leqslant X_3$ is less than X_3^{h+1}.

Hence, by theorem 1, there exist positive constants X_0 and c, depending on $a_i's$ and ε only, such that, if $X > X_0$ in any interval $1 \leqslant n \leqslant X$, the number of n satisfying (33) is greater than $cX/\log^{h+1}X$, q. e. d.

THEOREM 3. *In theorem 2 the function φ can be replaced by σ (clearly the constants c and X_0 must be changed).*

The proof of theorem 3 is analogous to the preceding one but based on the proofs of lemma 3b and theorem 2 [4].

§ 8. Theorems on the function θ. We now prove

THEOREM 4. *For any given positive integer h, there exists a constant $b = b(h)$ such that for any given sequence a of h integers $a_1, a_2, \ldots, a_h > 1$ there exists a positive integer n such that*

$$\theta(i)a_i < \theta(n+i) < b\theta(i)a_i \quad (1 \leqslant i \leqslant h). \tag{36}$$

There exist positive constants $c' = c'(a)$ and $X_0' = X_0'(a)$ such that in any interval $1 \leqslant n \leqslant X$ the number of n satisfying (36) is greater than $c'X/\log^{h+1}X$, whenever $X > X_0'$.

Proof. We can choose $A_0, A_1, \ldots, A_h$, depending on $a_i's$ only and satisfying the same conditions as in theorem 1, such that $\theta(A_i) = a_i$ $(1 \leqslant i \leqslant h)$. For those $A_i's$ we assume that $(x_0, \ldots, x_h)$ is a solution of (1) satisfying (2) with $Z = X^{c_1}$. If we take $A_0x_0 = n$, then $iA_ix_i = n+i$ $(1 \leqslant i \leqslant h)$. Since $(x_i, iA_i) = 1$ $(1 \leqslant i \leqslant h)$, we have

$$\theta(n+i) = \theta(iA_i)\theta(x_i) = \theta(i)a_i\theta(x_i) \quad (1 \leqslant i \leqslant h).$$

As in the proof of theorem 2, we can choose $X_2'(a)$ such that the number of prime divisors (identical or different) of each x_i $(1 \leqslant i \leqslant h)$ does not exceed c_{10} for $X > X_2'$. Hence for $X > X_2'$

$$\theta(i)a_i < \theta(n+i) < \theta(i)a_i2^{c_{10}} = b\theta(i)a_i \quad (1 \leqslant i \leqslant h).$$

Further proof is analogous to the proof of theorem 2.

From theorem 4 we can directly obtain

THEOREM 5. *For any given sequence of h numbers $a_1, \ldots, a_h$, where $a_i = 0$ or $+\infty$ $(1 \leqslant i \leqslant h)$ there exists an infinite sequence of natural numbers $n_1, n_2, \ldots$ such that*

$$\lim_{k\to\infty} \frac{\theta(n_k+i)}{\theta(n_k+i-1)} = a_i \quad (1 \leqslant i \leqslant h).$$

In the course of publication theorems 3 and 5 were proved independently by Shao Pin-Tsung (to appear in Shou Hsueh Chin-Chan (Progress of Mathematics) and Pei-Ta-Hsueh Pao (Transactions of Peking University)) (cf. [5]).

References

[1] H. Rademacher, *Beiträge zur Viggo Brunschen Methode in der Zahlentheorie*, Abh. Math. Sem. Hamburgischen Univ. 3 (1923), p. 12-30.

[2] A. Schinzel, *Quelques théorèmes sur les fonctions $\varphi(n)$ et $\sigma(n)$*, Bull. Acad. Polon. Sci., Cl. III, 2 (1954), p. 467-469.

[3] — *Sur une propriété du nombre de diviseurs*, Publ. Math. 3, Debrecen 1954, p. 261-262.

[4] — *On functions $\varphi(n)$ and $\sigma(n)$*, Bull. Acad. Polon. Sci., Cl. III, 3 (1955), p. 415-419.

[5] Shao Pin-Tsung, *On the distribution of the values of a class of arithmetical functions*, Bull. Acad. Polon. Sci., Cl. III, 4 (1956), p. 569-572.

Reçu par la Rédaction le 1. 3. 1956

Note added in the proof by A. Schinzel. Theorem 1 easily results from the following theorem of G. Ricci (see G. Ricci, *Su la congettura di Goldbach e la constante di Schnirelman*, Annali della R. Scuola Normale Superiore di Pisa 6 (2) 1937, p. 83):

Let a_1x+b_1, $a_2x+b_2, \ldots, a_fx+b_f$, where $(a_i, b_i) = 1$ $(i = 1, 2, \ldots, f)$ are f different arithmetical progressions, let D be a fixed divisor of the polynomial

$$(a_1x+b_1)(a_2x+b_2)\ldots(a_fx+b_f)$$

and put $a_1x+b_1 = d_1P_1$, $a_2x+b_2 = d_2P_2, \ldots, a_fx+b_f = d_fP_f$, where $d_1d_2\ldots d_f = D$.

The number of natural numbers $x \leqslant \xi$ such that all integers $P_1, P_2, \ldots, P_f$ have no prime factors $\leqslant \xi^{1/(1+2\tau(f))}$ is of the same order of magnitude as $\xi/\log^f \xi$.

In fact, in virtue of lemma 2 [4] there exists a natural number m such that

$$A_i | m+i, \quad (A_i, (m+i)/A_i) = 1 \quad (0 \leqslant i \leqslant h)$$

and in virtue of the formulas (1)-(3) of [4]

$$(A_0^2A_1^2\ldots A_h^2, m) = A_0, \quad (A_0^2A_1^2\ldots A_h^2, m+i) = iA_i \quad (1 \leqslant i \leqslant h).$$

Put

$$a_1 = A_0A_1^2\ldots A_h^2, \quad b_1 = m/A_0,$$
$$a_i = A_0^2A_1^2\ldots A_h^2/(i-1)A_{i-1}, \quad (1 < i \leqslant h+1).$$
$$b_i = (m+i-1)/(i-1)A_{i-1}$$

We therefore get $(a_i, b_i) = 1$ $(1 \leqslant i \leqslant h+1)$ and $(A_0, b_1b_2\ldots b_{h+1}) = 1$ whence also $((h+1)!, b_1b_2\ldots b_{h+1}) = 1$. From the last equality it follows that the polynomial

$$(a_1x+b_1)(a_2x+b_2)\ldots(a_{h+1}x+b_{h+1})$$

has no fixed divisor > 1, since such divisor D divides $(h+1)!$.

Putting in the above mentioned theorem of Ricci $d_i = 1$, $a_ix+b_i = x_{i-1}$ $(1 \leqslant i \leqslant h)$ we find that the number of natural numbers $x \leqslant \xi$ such that all the numbers x_i $(0 \leqslant i \leqslant h)$ have no prime factors $\leqslant \xi^{1/(1+2\tau(h+1))}$ is of the same order of magnitude as $\xi/\log^{h+1}\xi$. Put

$$\xi = \frac{A_0x-m}{A_0^2A_1^2\ldots A_h^2}, \quad z = \xi^{1/(1+2\tau(h+1))}.$$

As the number x_i satisfy the system of equations (1) and for $x \leqslant \xi$ the conditions (2), we get the inequality

$$N_{x^{c_1}}(X) > \frac{c_2X}{\log^{h+1}X} \quad (X > X_1)$$

where $c_2 > 0$ and X_1 are constants depending only on A_i and c_1 is an arbitrary constant $< \dfrac{1}{1+2\tau(h+1)}$, depending therefore only on h.