Non-local problems in the calculus of variations (II)

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§ 1. Sturm's theorem. The results obtained in the first part of this paper make it possible to extend Sturm's known theorem on differential linear homogeneous equations of the second order to analogous integro-differential equations.

This theorem, belonging to the theory of integro-differential equations, is to be applied to the investigation of the second variation in non-local problems.

Theorem 1. *If λ is not the characteristic number*[1] *of the integro-differential equation*

$$(1.1)\qquad a_2(t)q^{\cdot\cdot}(t)+a_1(t)q^{\cdot}(t)+a_0(t)q(t)=\lambda\int_\alpha^\beta b(t,t')q(t')\,dt'$$

and if $a_2(t)\neq 0$ over the closed interval $\langle\alpha,\beta\rangle$, we find that between the neighbouring two zeroes of any solution of this equation there is exactly one zero of any other solution independent of the given solution. We assume that functions $a_i(t)$, $b(t,t')$ are of class C_2.

We can even weaken this assumption.

Proof. From the first part of our paper [2] we find that considering the assumption above we can apply the theorem of existence of solutions and exactly one solution at every point, *i. e.* that we can carry exactly one solution of the equation through every point of any direction $-\infty<q^{\cdot}<\infty$ of the stripe $\alpha<t<\beta$, $-\infty<q<\infty$. It follows immediately from the form of the equation (1.1) that if the functions $q_1(t)$ and $q_2(t)$ are its solutions, function $c_1q_1+c_2q_2$ is its solution too.

(1) To the set of the characteristic numbers λ belong the characteristic values of the kernel

$$N(t,t')=\int_{t_0'}^{t'}\frac{\overline{q}_1(\tau)\overline{q}_2(t')-\overline{q}_2(\tau)\overline{q}_1(t')}{W(\tau)}b(t,\tau)\,d\tau$$

where $\overline{q}_1(t)$ and $\overline{q}_2(t)$ are two linearly independent solutions of the equation

$$a_2\overline{q}^{\cdot\cdot}+a_1\overline{q}^{\cdot}+a_0\overline{q}=0,$$

$W(\tau)$ denotes Wroński's determinant of $\overline{q}_1$, $\overline{q}_2$, and possibly any two numbers more. (See [2], p. 88-90.)

It is obvious that Wroński's determinant of any two solutions of the equation (1.1) is either identically zero or does not vanish at any point. If $q_1q_2^{\cdot}-q_2q_1^{\cdot}=0$ for $t=t_0$, though one of the derivatives $q_1^{\cdot}$ and $q_2^{\cdot}$ varies from zero, then for $t=t_0$ we have the following relation between q and q_2:

$$(1.2)\quad q_1(t_0)=\frac{q_1^{\cdot}(t_0)}{q_2^{\cdot}(t_0)}q_2(t_0)\qquad\text{or}\qquad(1.2')\quad q_2(t_0)=\frac{q_2^{\cdot}(t_0)}{q_1^{\cdot}(t_0)}q_1(t_0).$$

Let us examine the first case, the second is analogous. From the condition that only one curve of the family of which q_1 and q_2 are members goes in a given direction through every point and considering the identity

$$(1.3)\qquad q_1^{\cdot}(t_0)=\frac{q_1^{\cdot}(t_0)}{q_2^{\cdot}(t_0)}q_2^{\cdot}(t_0)$$

we get from (1.2)

$$q_1(t)\equiv\frac{q_1^{\cdot}(t_0)}{q_2^{\cdot}(t_0)}q_2(t).$$

Thus Wroński's determinant vanishes identically. If

$$(1.4)\qquad q_1^{\cdot}(t_0)=q_2^{\cdot}(t_0)=0,$$

then evidently it is possible to find such a constant $c\neq 0$ that

$$(1.5)\quad q_1(t_0)=cq_2(t_0)\qquad\text{or}\qquad(1.5')\quad q_2(t_0)=cq_1(t_0)$$

and from (1.4)

$$(1.6)\quad q_1^{\cdot}(t_0)=cq_2^{\cdot}(t_0)\qquad\text{or}\qquad(1.6')\quad q_2^{\cdot}(t_0)=cq_1^{\cdot}(t_0).$$

From equations (1.5) and (1.6) follows an identity from which we immediately get the required property of Wroński's determinant. From the above property of the examined family of functions it follows that there are two such functions q_1 and q_2 that their Wroński's determinant does not vanish at any point. For in the contrary case all the functions of this family would be linearly dependent and therefore we could not carry exactly one solution of the equation (1.1) through every point of any direction.

We can get any other solution of the equation (1.1) from a linear combination of these solutions. Hence, the equation (1.1) is identical with the differential equation

$$(1.7)\qquad\begin{vmatrix}q_1(t)&q_2(t)&q(t)\\q_1^{\cdot}(t)&q_2^{\cdot}(t)&q^{\cdot}(t)\\q_1^{\cdot\cdot}(t)&q_2^{\cdot\cdot}(t)&q^{\cdot\cdot}(t)\end{vmatrix}=0,$$

i. e. every solution of the equation (1.2) is the solution of the equation (1.7) and vice-versa. The equation (1.7) satisfies the assumptions of Sturm's classical theorem, and because of the identity of the equations (1.1) and (1.7), the theorem applies also to the solutions of the equation (1.1).

§ 2. Second Variation. We are going to study the question of the preservation of the sign by the second variation in non-local variational problems. We want to find some conditions for the occurence of the extremum of the functional

$$I = \int_{a}^{\beta}\!\!\int L\big(t, t', q(t), q(t'), q^{\cdot}(t), q^{\cdot}(t')\big)\,dt\,dt'. \tag{2.1}$$

We can obviously assume that the function $L\big(t, t', q(t), q(t'), q^{\cdot}(t), q^{\cdot}(t')\big)$ is a symmetrical function of variables t and t', *i. e.*

$$L\big(t, t', q(t), q(t'), q^{\cdot}(t), q^{\cdot}(t')\big) = L\big(t', t, q(t'), q(t), q^{\cdot}(t'), q^{\cdot}(t)\big).$$

We are able to express the second variation of functional (2.1) in the form

$$\delta^2 I = \int_{a}^{\beta}\!\!\int \{P(t, t')[\delta q(t)]^2 + Q(t, t')\,\delta q(t)\,\delta q(t') + R(t, t')[\delta q^{\cdot}(t)]^2\}\,dt\,dt' \tag{2.2}$$

where the coefficients P, Q, R are expressed by the function L and its derivatives after the substitution of the extremal function for q. P, Q, R are symmetrical functions of t and t'.

Similarly to conventional local variational problems, we can show the proper selection of variations $\delta q(t)$, so that the sign of the expression (2.2) may be identical with the sign of the function $\int_a^\beta R(t, t')\,dt'$ at any point t. Thus the necessary condition of the preservation of the sign by the second variation is the stability of the sign of the function $\int_a^\beta R(t, t')\,dt'$ over the interval $\langle a, \beta\rangle$.

We get the following theorem

THEOREM 2. *The sufficient condition of the existence of an extremum*[2] *for the functional* (2.1) *is that there be an integral of the equation*

$$\int_{a}^{\beta} \Big\{P(t, t')\,\delta q(t) + Q(t, t')\,\delta q(t') - \frac{\partial}{\partial t}[R(t, t')\,\delta q^{\cdot}(t)]\Big\}\,dt' = 0 \tag{2.3}$$

different from zero over the interval $\langle a, \beta\rangle$ and that the function $Q(t, t')$ have a sign contrary to the sign of $\int_a^\beta R(t, t')\,dt'$.

Proof. The equation (2.3) is Lagrange's equation of the functional (2.2). It is to be noticed that it is an integro-differential equation of the function $\delta q(t)$ of descriptive class in the first paragraph. Let us assume that there is an integral $z(t)$ of this equation which has no zero over the interval $\langle a, \beta\rangle$. Then every function $\delta q(t)$ is to be expressed by the form $\delta q = zw$. Let us notice that the left side of the equation (2.3) multiplied by the function $\delta q(t)$ and integrated from a to β with respect to the variable t is transformed into the second variation (2.2) in the form

$$\begin{aligned}\delta^2 I = \tfrac{1}{2}\int_{a}^{\beta}\!\!\int z(t)w(t)\Big\{&P(t, t')z(t)w(t) + Q(t, t')z(t')w(t') - \\ &- \frac{\partial}{\partial t}\big[R(t, t')\big(z^{\cdot}(t)w(t) + w^{\cdot}(t)z(t)\big)\big]\Big\}\,dt\,dt' + \\ + \tfrac{1}{2}\int_{a}^{\beta}\!\!\int z(t')w(t')\Big\{&P(t', t)z(t')w(t') + Q(t', t)z(t)w(t) - \\ &- \frac{\partial}{\partial t'}\big[R(t', t)\big(z^{\cdot}(t')w(t') + w^{\cdot}(t')z(t')\big)\big]\Big\}\,dt\,dt'.\end{aligned} \tag{2.4}$$

Now we add and subtract

$$\tfrac{1}{2}\int_{a}^{\beta}\!\!\int z(t')w(t')Q(t', t)z(t)w(t')\,dt\,dt'$$

from the first part of the above form. Then we extract $w(t)$, $w(t')$ from brackets in the first part and in the second part. An easy calculation taking into account the symmetry of the functions P, Q, R of variables t and t', makes it possible to give the expression (2.4) the form

$$\delta^2 I = \int_{a}^{\beta}\!\!\int \{R(t, t')[w^{\cdot}(t)]^2 - \tfrac{1}{2}Q(t, t')z(t)z(t')[w(t) - w(t')]^2\}\,dt\,dt'. \tag{2.5}$$

It was stated above that the necessary condition of the preservation of the positive sign by the second variation is that $\int_a^\beta R(t, t')\,dt' \geqslant 0$. If, besides, the function $Q(t, t')$ is non-positive, the second variation is non-negative. Thus our theorem is proved.

In the case of $Q(t, t')$ is zero an integration with respect to the variable t' of the expression (2.3) becomes meaningless and then the expression (2.2) is the second variation of a pure local functional.

[2] We can easily show, as in the local case, that the invariability of the sign of the second variation is a necessary and sufficient condition of the appearence of the weak extremum of the functional I. Henceforth we put "extremum" for weak extremum.

Let us give some valuation of the second variation of non-local functional.

THEOREM 3. *The sufficient condition for the functional* (2.1) *to have a minimum is that there be an integral of the pure differential equation*

$$\int_\alpha^\beta \Big\{[P(t,t')-A]\,\delta q(t) - \frac{\partial}{\partial t}[R(t,t')\,\delta q^{\cdot}(t)]\Big\}dt' = 0 \tag{2.6}$$

which does not equal zero over the interval $\langle \alpha, \beta\rangle$ *and that*

$$\int_\alpha^\beta R(t,t')\,dt' \geqslant 0$$

for every $\alpha \leqslant t \leqslant \beta$. *The expression*

$$\frac{1}{\beta-\alpha}\sqrt{\int_\alpha^\beta\!\!\int [Q(t,t')]^2\,dt\,dt'}$$

was denoted by A *in the equation* (2.6).

THEOREM 4. *The sufficient condition that the functional* (2.1) *have no minimum is that there be an integral of the equation*

$$\int_\alpha^\beta \Big\{[P(t,t')+A]\,\delta q(t) - \frac{\partial}{\partial t}[R(t,t')\,\delta q^{\cdot}(t)]\Big\}dt' = 0 \tag{2.7}$$

which equals zero at least at two points of the interval $\langle \alpha, \beta\rangle$.

We have an analogous theorem in the case of the maximum of the functional. In this case we must replace the assumption $\int_\alpha^\beta R(t,t')\,dt' \geqslant 0$ by the assumption $\int_\alpha^\beta R(t,t')\,dt' \leqslant 0$.

Proof of the theorem 3. From the assumption of the theorem we find that the square local functional

$$\int_\alpha^\beta \{\bar{P}(t)[\delta q(t)]^2 + \bar{R}(t)[\delta q^{\cdot}(t)]^2\}\,dt$$
$$= \int_\alpha^\beta\!\!\int \{[P(t,t')-A](\delta q(t))^2 + R(t,t')[\delta q^{\cdot}(t)]^2\}\,dt\,dt'$$

is positively defined, *i. e.* that the functional is greater than zero for every $q(t)$. We get the following valuation from Cauchy's inequality

$$\Big|\int_\alpha^\beta\!\!\int Q(t,t')\,\delta q(t)\,\delta q(t')\,dt\,dt'\Big| \leqslant \sqrt{\int_\alpha^\beta\!\!\int [Q(t,t')]^2\,dt\,dt' \int_\alpha^\beta\!\!\int [\delta q(t)]^2[\delta q(t')]^2\,dt\,dt'}$$
$$= A\int_\alpha^\beta\!\!\int [\delta q(t)]^2\,dt\,dt',$$

hence we get the inequality

$$\int_\alpha^\beta\!\!\int \{P(t,t')[\delta q(t)]^2 + Q(t,t')\,\delta q(t)\,\delta q(t') + R(t,t')[\delta q^{\cdot}(t)]^2\}\,dt\,dt'$$
$$\geqslant \int_\alpha^\beta\!\!\int \{[P(t,t')-A][\delta q(t)]^2 + R(t,t')[\delta q^{\cdot}(t)]^2\}\,dt\,dt',$$

which proves our theorem.

The proof of the theorem 4 is analogous.

§ 3. Some remarks about sufficient conditions. It is difficult to find useful sufficient conditions for the existence of an extremum. We are able to get an analogous function to Weierstrass' $\mathcal{E}$ function. It has the form

$$I(\bar{q})-I(q) = \int_\alpha^\beta\!\!\int \{L(t,t',\bar{q}(t),\bar{q}(t'),\bar{q}^{\cdot}(t),\bar{q}^{\cdot}(t')) - \tag{3.1}$$
$$-L(t,t',\bar{q}(t),q(t'),p(t,\bar{q}(t)),q^{\cdot}(t')) +$$
$$+[\bar{q}(t)-p(t,\bar{q}(t))]\,L_{q^{\cdot}(t)}(t,t',\bar{q}(t),p(t,\bar{q}(t)),q^{\cdot}(t'))\}\,dt\,dt'$$

where $\bar{q}(t)$ is a varied function and $q(t)$ an extremal function and $p(t,\bar{q})$ is a slope function of the field of extremal functions (see also [1]). The definiteness of the expression (3.1) is a necessary and sufficient condition for the existence of an extremum. The usefulness of the expression (3.1) for the investigation of extrema is practically very small. It is caused by the appearance of the extremal function $q(t)$ on the right side of the formula (3.1).

We can get an analogous condition to Erdman-Weierstrass' corner conditions.

$$\sum_{i=1}^{n}\int_{\gamma_{i-1}}^{\gamma_i}\{L_{q^{\cdot}(t)}(\omega_p(\gamma_p),\omega_i(t')) - L_{q^{\cdot}(t)}(\omega_{p+1}(\gamma_p),\omega_i(t'))\}\,dt' = 0, \tag{3.2}$$

$$\sum_{i=1}^{n}\int_{\gamma_{i-1}}^{\gamma_i}\{L(\omega_p(\gamma_p),\omega_i(t')) - \omega_p^{\cdot}(\gamma_p)\,L_{q^{\cdot}(t)}(\omega_p(\gamma_p),\omega_i(t'))\}\,dt' \tag{3.3}$$
$$= \sum_{i=1}^{n}\int_{\gamma_{i-1}}^{\gamma_i}\{L(\omega_{p+1}(\gamma_p),\omega_i(t')) - \omega_{p+1}^{\cdot}(\gamma_p)\,L_{q^{\cdot}(t)}(\omega_{p+1}(\gamma_p),\omega_i(t'))\}\,dt',$$

$$L(\omega_k(\gamma_p),\omega_i(t')) = L(\gamma_p,t',\omega_k(\gamma_p),\omega_i(t'),\omega_k^{\cdot}(\gamma_p),\omega_i^{\cdot}(t')).$$

We leave out the tiresome proof of formula (3.2). Classical methods of applying Erdman-Weierstrass' condition in order to get criteria for the existence of the extremum of the functional (2.1) bring no result. That is so because the solution of the integro-differential equation (1.1) is not necessarily a solution of the equation

$$a_2(t)q^{\cdot\cdot}(t)+a_1(t)q^{\cdot}(t)+a_0(t)q(t) = \lambda\int_{\gamma}^{\delta} b(t,t')q(t')\,dt'$$

if $(\alpha-\gamma)^2+(\beta-\delta)^2 \neq 0$, while the solution of a pure differential equation over an interval is a solution of this equation in any sub-interval.

§ 4. First integrals of the generalized Lagrange-equations. In the conventional local variational problems in the case of the Lagrange-function being independent of t it is always possible to construct first integrals of the Lagrange-equations, *i. e.* functions of $t, q(t), q^{\cdot}(t)$ which are constant on the extremals. The existence of first integrals may be connected with the invariance properties of the given functionals with respect to finite continuous transformations of t and $q(t)$ (cf. E. Noether, [3]). In the case of non-local variational problems the situation is essentialy different. From the invariance of the functional there follows merely the equality of certain functions at the boundary points α and β. We are going now to study this problem in some detail.

Consider the following transformation of the variables $q(t), t$ depending on the parameter σ

$$t^* = T(t, q, q^{\cdot}, \sigma), \qquad q^* = Q(t, q, q^{\cdot}, \sigma). \tag{4.1}$$

The functions Q and T are continuous functions of their arguments, possessing continuous second order derivatives with respect to all arguments $\alpha \leqslant t \leqslant \beta$, $-\infty < q$, $q^{\cdot} < +\infty$, $-A \leqslant \sigma \leqslant A$, A is a positive constant. We assume further that for $\sigma = 0$ this transformation reduces to identity $q^* = q$, $t^* = t$. To investigate transformation properties of the functional (2.1) it is sufficient to consider only infinitesimal transformation. In this case the transformation (4.1) may be written in the form

$$t^* = t_0+\delta t, \quad \delta t = T_{\sigma}(t)\delta\sigma, \quad q^* = q_0+\delta q, \quad \delta q = Q_{\sigma}(t)\delta\sigma \tag{4.2}$$

where $\delta\sigma$ is constant, $T_{\sigma}(t)$ and $Q_{\sigma}(t)$ are derivatives of T and Q with respect to σ at the point

$$t = t_0, \qquad \sigma = \sigma_0.$$

The variation of I due to the infinitesimal variations (4.1) is

$$\delta I = \int_{\alpha}^{\beta}\left\{\frac{\partial L}{\partial q}\delta_0 q + \frac{\partial L}{\partial q^{\cdot}}\delta_0 q^{\cdot} + \frac{d(L\delta t(t))}{dt}\right\}dt \tag{4.3}$$

where

$$\delta_0 q = \delta q - q^{\cdot}\delta t.$$

Integrating by parts we get

$$\delta I = \int_{\alpha}^{\beta}\left\{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial q^{\cdot}}\right\}\delta_0 q\,dt + \delta\sigma F(t)\Big|_{\alpha}^{\beta} \tag{4.4}$$

where

$$\delta\sigma F(t) = L\delta t(t) + \frac{\partial L}{\partial q^{\cdot}}\delta_0 q = H\delta t + \frac{\partial L}{\partial q^{\cdot}}\delta q, \tag{4.5}$$

$$H = L - \frac{\partial L}{\partial q^{\cdot}}q^{\cdot}. \tag{4.6}$$

Introducing (4.2) into (4.5) we get

$$F(t) = HT_{\sigma}(t) + \frac{\partial L}{\partial q^{\cdot}}Q_{\sigma}(t). \tag{4.7}$$

THEOREM 5. *If the functional* (2.1) *is invariant with respect to the group of transformations* (4.1) *and if* $q(t)$ *is an arbitrary extremal of this functional, then*

$$F(\alpha) = F(\beta). \tag{4.8}$$

The proof follows immediately from (4.4) if we observe that $\delta I = 0$ (in the case of invariance), that the integral in (4.4) vanishes on the extremals, and that $\delta\sigma$ is an arbitrary constant.

It may be noted that, in contradiction to the local case, the constancy of $F(t)$ inside $\langle\alpha, \beta\rangle$ does not follow from (4.7). Indeed, in the local case we may vary the points α and β arbitrarily without affecting the Lagrange equations, which are pure differential equations and do not depend on α and β. The constant $F(t)$ is in this case the first integral of the Lagrange equation. In the non-local case the Lagrange equation is an integro-differential equation and contains the parameters α and β as integration limits. The variation of these parameters would affect the structure of this equation and therefore also its solution. Thus from (4.7) we may not derive the constancy of $F(t)$ inside $\langle\alpha, \beta\rangle$. In fact, as we shall see immediately, in general $F(t)$ vary in this domain.

The question arises whether there exists a function-functional of t, q and $q^{\cdot}$ remaining constant on the extremals throughout the interval $\langle\alpha, \beta\rangle$. The answer to this question is given by the following theorem:

THEOREM 6. *The quantity*

$$\overline{F}(t) = F(t) - \int_{\alpha}^{t}\left\{HT^{\cdot}_{\sigma} + \frac{\partial L}{\partial t}T_{\sigma} + \frac{d}{dt}\left(\frac{\partial L}{\partial q^{\cdot}}Q_{\sigma}\right)\right\}dt \tag{4.9}$$

is constant on the extremals throughout the interval $\langle\alpha, \beta\rangle$.

The proof is obtained immediately by differentiation with respect to t, the use of the Lagrange equation and of the equation

$$H^{\cdot} = \partial L/\partial t, \tag{4.10}$$

which holds also in the non-local case.

It may be noted that the quantity (4.9) contains only first-order derivatives of q and a double integral over the domain $\langle\alpha, \beta\rangle\times\langle\alpha, t\rangle$. The problem of finding the solution of the original Lagrange equation (which is a second order integro-differential equation containing a single integral over the interval $\langle\alpha, \beta\rangle$) is thus reduced to the problem of finding the solution of the first order integro-differential equation (4.9) containing a double integral and an arbitrary constant. Thus (4.9) may be viewed as the first integral of the Lagrange equation.

It may be noticed further that the quantity $\overline{F}(t)$ is constant on the extremals whether the functional I is invariant with respect to the group (4.1) or not. In the case when it is invariant, the quantity $\overline{F}(t)$ goes over into $F(t)$ on the boundaries α and β of the considered interval. Indeed, it is seen immediately from (4.9) that $F(\alpha) = \overline{F}(\alpha)$. It may be shown that, owing to the invariance of the functional I,

$$\int_\alpha^\beta \left\{HT_\sigma^{\cdot} + \frac{\partial L}{\partial t}T_\sigma + \frac{d}{dt}\left(\frac{\partial L}{\partial q^{\cdot}}Q_\sigma\right)\right\}dt = 0. \tag{4.11}$$

Equation (4.11) follows from equation (4.3) if we express $\delta_0 q$, $\delta_0 q^{\cdot}$ and δt in this equation by means of (4.2) and if we put $\delta I = 0$. Thus in the case of invariance we also have

$$F(\beta) = \overline{F}(\beta).$$

The first integral (4.10) provide us with the generalization of the notion of the integral also in the local case. In this case the quantities H and L do not contain any integrals and according to the conventional formulation the first integral exists if $\partial L/\partial t = 0$. According to our formulation we may construct the first integral also in the case if $\partial L/\partial t \neq 0$. Indeed, integrating equation (4.10) over the interval $\langle\alpha, t\rangle$, we get

$$H(t) - \int_\alpha^t \frac{\partial L}{\partial t}dt = \text{const}. \tag{4.12}$$

Expression (4.12) could also be obtained from the general formula (4.9) by taking $T_\sigma(t) = 1$, $Q_\sigma(t) = 0$. It represents a first integral of the Lagrange equation since it contains an arbitrary constant and is a first-order integro-differential equation possessing, according to the general theorem of Part I [2], a one-parametric family of solutions.

The proof of the above theorems has been carried out in such a way that it may immediately be generalized to the case of an arbitrary number of unknown functions $q_1(t), \ldots, q_n(t)$. Generalizations to problems containing independent variables $t_1, \ldots, t_m$ are also straightforward. The application of the general procedure to physical problems (determination of constants of motion) may be found in the paper of one of us [4].

The consideration of infinite groups of transformations yield nothing new as compared with the paper of E. Noether. In the particular case of one unknown function considered here we get the corresponding parametric variational problem containing two unknown functions of one parameter. The demand of invariance with respect to a change of parametrization (infinite group) results in a relation between the two Lagrange equations for the two unknown functions, exactly as in the local case.

References

[1] S. Faedo, *Condizioni necessarie par la semicontinuità di un nuovo tipo di funzionali*, Ann. di Mat. 23 (1944), p. 69-121.

[2] A. Krzywicki, J. Rzewuski, J. Zamorski, A. Zięba, *Non-local problems in the calculus of variations (I)*, Ann. Pol. Math. 2 (1955), p. 77-96.

[3] E. Noether, *Invariante Variationsprobleme*, Nachr. Ges. Göttingen (1918), p. 235-237.

[4] J. Rzewuski, *Differential conservation laws in non-local field theories*, Nuovo Cimento 10 (1953), p. 784-802.

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