On Gołąb's contribution to Simpson's formula

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Recently S. Gołąb has raised certain interesting questions regarding Simpson's formula of approximate quadrature and has used ingenious methods to solve them. Our object is to raise some more problems in the same strain and to give their answers.

1. Let $f(x)$ be a function having the expansion in the neighbourhood of $x = a$, given by

$$f(a+h) = f(a)+a_p h^p + a_q h^q + a_r h^r + \ldots + g(h) \tag{1}$$

where $a_p, a_q, a_r, \ldots \neq 0$ and p, q, r, s are positive integers, $1 \leqslant p < q < r < s$, and $g(h)$ is of order h^{s+1}. Let us denote

$$P(h) = \int_a^{a+h} f(x)\,dx, \tag{2}$$

$$\overline{P}(h) = h[\lambda_0 f(a)+\lambda_1 f(a+\Theta_1 h)+\lambda_2 f(a+\Theta_2 h)] \tag{3}$$

where Θ_1, Θ_2 are numbers such that $0 < \Theta_1 \leqslant \frac{1}{2} < \Theta_2 \leqslant 1$. When $\Theta_1 = \frac{1}{2}$ and $\Theta_2 = 1$, we get Simpson's formula.

We then propose the following problem:

Determine the values of $\lambda_0, \lambda_1, \lambda_2$ so that for given Θ_1 and Θ_2 $(0 < \Theta_1 \leqslant \frac{1}{2} < \Theta_2 \leqslant 1)$, the remainder $R(h) = P(h) - \overline{P}(h)$ is of the greatest order of smallness with respect to h.

In order to do so, we put (1) in (2) and in (3) and then compare the coefficients of the powers of h on both sides. We then have

$$\lambda_0+\lambda_1+\lambda_2 = 1, \tag{4}$$

$$\lambda_1(p+1)\Theta_1^p+\lambda_2(p+1)\Theta_2^p = 1, \tag{5}$$

$$\lambda_1(q+1)\Theta_1^q+\lambda_2(q+1)\Theta_2^q = 1. \tag{6}$$

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From (5) and (6), we have

$$\lambda_1 = \frac{(p+1)\Theta_2^p-(q+1)\Theta_2^q}{\left(\frac{\Theta_2}{\Theta_1}\right)^p - \left(\frac{\Theta_2}{\Theta_1}\right)^q} \cdot \frac{\Theta_1^{-p-q}}{(p+1)(q+1)}, \tag{7}$$

$$\lambda_2 = \frac{(q+1)\Theta_1^{-p}-(p+1)\Theta_1^{-q}}{\left(\frac{\Theta_2}{\Theta_1}\right)^p - \left(\frac{\Theta_2}{\Theta_1}\right)^q} \cdot \frac{1}{(p+1)(q+1)}. \tag{8}$$

Obviously $(\Theta_2/\Theta_1)^p-(\Theta_2/\Theta_1)^q \neq 0$ if $\Theta_1 < \Theta_2 \leqslant 1$. We have thus proved the following theorem:

If Θ_1 and Θ_2 are given numbers $(0 < \Theta_1 < \Theta_2 \leqslant 1)$ and λ_1, λ_2 are given by (7) and (8) and λ_0 by (4), then $R(h)$ is of the order of smallness of h^{r+1}.

2. Naturally the following problem now arises:

Can we further raise the order of smallness of $R(h)$ by a proper choice of Θ_2?

In order to do so, we must also have, with (4) (5) and (6), the following relation:

$$\lambda_1(r+1)\Theta_1^r+\lambda_2(r+1)\Theta_2^r = 1. \tag{9}$$

The condition of consistency of (5), (6) and (9) then requires that

$$\Delta(\Theta) \equiv \begin{vmatrix} (p+1)\Theta_1^p & (p+1)\Theta^p & 1 \\ (q+1)\Theta_1^q & (q+1)\Theta^q & 1 \\ (r+1)\Theta_1^r & (r+1)\Theta^r & 1 \end{vmatrix} = 0.$$

Putting $q-p = \alpha$, $r-q = \beta$ so that $\alpha \geqslant 1$, $\beta \geqslant 1$, we have

$$\Delta(\Theta) \equiv (r+1)\Theta^{\alpha+\beta}\{(p+1)-(q+1)\Theta_1^\alpha\} - \\ -(q+1)\Theta^\alpha\{(p+1)-(r+1)\Theta_1^{\alpha+\beta}\}+(p+1)\Theta_1^\alpha\{(q+1)-(r+1)\Theta_1^\beta\} = 0. \tag{10}$$

We shall now show that this equation has a unique root Θ_0, lying in $(\Theta_1, 1)$. We have

$$\Delta(0) = (p+1)\Theta_1^\alpha\{(q+1)-(r+1)\Theta_1^\beta\} > 0, \qquad \Delta(\Theta_1) = 0. \tag{11}$$

Also

$$\Delta(1) = \alpha(r+1)\Theta_1^{\alpha+\beta}-(q+1)(\alpha+\beta)\Theta_1^\alpha+(p+1)\beta. \tag{12}$$

The right side is a function of Θ_1, and has been studied in detail by Gołąb and Olech who have shown that there exists a value Θ' in (0,1) such that $\Delta(1) > 0$ for $\Theta_1 < \Theta'$, where $\Theta' \geqslant \frac{1}{2}$. Also

$$\Delta'(1) = \Theta_1^{\alpha+\beta}[(r+1)(\alpha+\beta)(1/\Theta_1)^{\beta}\{(p+1)(1/\Theta_1)^{\alpha}-(q+1)\}- \\ -\alpha(q+1)\{(p+1)(1/\Theta_1)^{\alpha+\beta}-(r+1)\}],$$

$$\Delta'(\Theta_1) = \Theta_1^{2\alpha+\beta-1}[(r+1)(\alpha+\beta)\{(1/\Theta_1)^{\alpha}(p+1)-(q+1)\}- \\ -\alpha(q+1)\{(1/\Theta_1)^{\alpha+\beta}(p+1)-(r+1)\}].$$

We shall now show that

$$\Delta'(1) > 0 \quad \text{and} \quad \Delta'(\Theta_1) < 0. \tag{13}$$

The proof of (13) depends on the following inequality:

$$\frac{r+1}{q+1}\cdot\frac{\alpha+\beta}{\alpha}\cdot A^{\beta} > \frac{A^{\alpha+\beta}(p+1)-(r+1)}{A^{\alpha}(p+1)-(q+1)} > \frac{r+1}{q+1}\cdot\frac{\alpha+\beta}{\alpha} \quad \text{for} \quad A \geqslant 2,\ \alpha \geqslant 1. \tag{14}$$

In other words

$$A^{\beta}K(\alpha) > K(\alpha+\beta) > K(\alpha) \tag{14'}$$

$$\text{where} \quad K(\alpha) = \left(A^{\alpha}-1-\frac{\alpha}{p+1}\right)\Big/\alpha\left(1+\frac{\alpha}{p+1}\right).$$

Since β is a positive integer, it is sufficient to prove the result for $\beta = 1$. The right-hand side of the inequality (14′) is true if

$$A^{\alpha}\left\{A\alpha\left(1+\frac{\alpha}{p+1}\right)-(\alpha+1)\left(1+\frac{\alpha+1}{p+1}\right)\right\} > -\left(1+\frac{\alpha}{p+1}\right)\left(1+\frac{\alpha+1}{p+1}\right). \tag{14''}$$

If $\alpha = 1$, we see that

$$A\left\{A\left(1+\frac{1}{p+1}\right)-2\left(1+\frac{2}{p+1}\right)\right\} > -\left(1+\frac{1}{p+1}\right)\left(1+\frac{2}{p+1}\right)$$

$$\text{if} \quad \left(A-\frac{p+3}{p+2}\right)^2 > -\frac{p+3}{(p+1)(p+2)^2}$$

which is always true whatever A may be.

When $\alpha > 1$, we see that the inequality (14″) is true if

$$A > \frac{\alpha+1}{\alpha}\cdot\frac{1+\dfrac{\alpha+1}{p+1}}{1+\dfrac{\alpha}{p+1}}.$$

The greatest value of the right hand-side is 15/8, so that the inequality is certainly true if $A \geqslant 2$.

In order to prove the left hand-side of the inequality (14′), we must show that

$$A^{\alpha+1}\alpha\left(1+\frac{\alpha}{p+1}\right)-\alpha\left(1+\frac{\alpha}{p+1}\right)\left(1+\frac{\alpha+1}{p+1}\right) \\ < A^{\alpha+1}(\alpha+1)\left(1+\frac{\alpha+1}{p+1}\right)-A(\alpha+1)\left(1+\frac{\alpha}{p+1}\right)\left(1+\frac{\alpha+1}{p+1}\right),$$

i. e.

$$\{A^{\alpha+1}-(A-1)(\alpha+1)-1\}\left(1+\frac{2\alpha+1}{p+1}\right) > \frac{\alpha(\alpha+1)[(A-1)(\alpha+1)+1]}{(p+1)^2}.$$

Now the left-hand side is certainly greater than

$$\frac{\alpha(\alpha+1)}{2}(A-1)^2\left(1+\frac{2\alpha+1}{p+1}\right).$$

But

$$(A-1)^2\left(1+\frac{2\alpha+1}{p+1}\right) > \frac{2(A-1)(\alpha+1)+2}{(p+1)^2},$$

i. e.

$$(A-1)^2+(A-1)^2\frac{2\alpha+1}{p+1} > \frac{(A-1)(2\alpha+1)}{(p+1)^2}+\frac{A+1}{(p+1)^2},$$

since $(A-1)^2 > \frac{1}{4}(A+1)$, A being greater than or equal to 2. We have thus proved the inequality (14).

It may be remarked that for $A \geqslant 2$, $\alpha \geqslant 2$ the right side of the inequality (14′) can be further improved, and we have

$$K(\alpha+\beta) > (A/2)^{\beta}K(\alpha).$$

On examining the expressions for $\Delta'(1)$ and $\Delta'(\Theta_1)$, we at once get (13). Also

$$\Delta'(\Theta) = \Theta^{\alpha-1}\Theta_1^{\alpha+\beta}\left[(r+1)(\alpha+\beta)\left(\frac{\Theta}{\Theta_1}\right)^{\alpha+\beta}\left\{(p+1)\left(\frac{1}{\Theta_1}\right)^{\alpha}-(q+1)\right\}- \\ -\alpha(q+1)\left\{(p+1)\left(\frac{1}{\Theta_1}\right)^{\alpha+\beta}-(r+1)\right\}\right],$$

which vanishes at $\bar{\Theta}$ given by

$$\bar{\Theta} = \Theta_1\left[\frac{\alpha(q+1)\left\{(p+1)\left(\dfrac{1}{\Theta_1}\right)^{\alpha+\beta}-(r+1)\right\}}{(\alpha+\beta)(r+1)\left\{(p+1)\left(\dfrac{1}{\Theta_1}\right)^{\alpha}-(q+1)\right\}}\right]^{1/\beta}$$

and by (14), $\Theta_1 < \bar{\Theta} < 1$.

Hence from (11), (12), (13) we see that there exists a unique value $\Theta_0 > \overline{\Theta}$ for which $\Delta(\Theta)$ vanishes and the order of $R(h)$ is less than that of h^{s+1} for this choice of Θ.

3. We shall now show that the order of smallnes of $R(h)$ is exactly equal to $s+1$. For if it is of order $s+2$, we must further have $\lambda_1(s+1)\Theta_1^s+\lambda_2(s+1)\Theta_2^s = 1$. Combining this with (5) and (6), we must have

$$\begin{vmatrix} (p+1)\Theta_1^p & (p+1)\Theta^p & 1 \\ (q+1)\Theta_1^q & (q+1)\Theta^q & 1 \\ (s+1)\Theta_1^s & (s+1)\Theta^s & 1 \end{vmatrix} = 0$$

simultaneously with (10). Putting $\gamma = s-q$, and simplyfying, we get

$$(s+1)\Theta^{a+\gamma}\Theta_1^{-\gamma}\{(p+1)\Theta_1^{-a}-(q+1)\} -(q+1)\Theta^a\{(p+1)\Theta_1^{-a-\gamma}-(s+1)\}+(p+1)\{(q+1)\Theta_1^{-\gamma}-(s+1)\} = 0. \tag{15}$$

We shall now show that (10) *and* (15) *cannot have a common root* Θ_0 *such that* $\Theta_1 < \Theta_0 < 1$.

Put $\gamma = x$ in (15) and put

$$\Phi(x) \equiv (x+q+1)\left\{\left(\frac{\Theta_0}{\Theta_1}\right)^x - \mu\right\} - \lambda\left(\frac{1}{\Theta_1}\right)^x$$

where

$$\lambda = \frac{\left\{\left(\frac{\Theta_0}{\Theta_1}\right)^a - 1\right\}(p+1)(q+1)}{\Theta_0^a\{(p+1)\Theta_1^{-a}-(q+1)\}} \quad \text{and} \quad \mu = \frac{(p+1)-\Theta_0^a(q+1)}{\Theta_0^a\{(p+1)\Theta_1^{-a}-(q+1)\}}.$$

Then the vanishing of $\Phi(x)$ for $x = \beta$ and for $x = \gamma$ implies that (10) and (15) have a common root Θ_0. But $\Phi(0) = 0 = -\Phi(-a)$. It is easy to see that $\lambda > 0$ and since $\Phi(0) = 0$, we have $(q+1)(1-\mu) = \lambda > 0$, so that $\mu < 1$. Also

$$\Phi''(x) = \left(\frac{\Theta}{\Theta_1}\right)^x\left[(x+q+1)\left(\log\frac{\Theta_0}{\Theta_1}\right)^2 + 2\log\frac{\Theta_0}{\Theta_1} - \lambda\left(\log\frac{1}{\Theta_1}\right)^2\cdot\left(\frac{1}{\Theta_1}\right)^x\right].$$

Then $\Phi''(x)$ will vanish at the points where the straight line

$$y = (x+q+1)\left(\log\frac{\Theta_0}{\Theta_1}\right)^2 + 2\log\frac{\Theta_0}{\Theta_1}$$

intersects the exponential curve

$$y = \lambda\left(\log\frac{1}{\Theta_1}\right)^2\cdot\left(\frac{1}{\Theta_1}\right)^x.$$

It is easy to see that these two cannot intersect in more than two points so that $\Phi''(x)$ cannot vanish at more than two points. Hence $\Phi(x)$ cannot have more than four zeros. Now $\Phi(x) = (x+q+1)\Psi(x)$, where

$$\Psi(x) = \left(\frac{\Theta_0}{\Theta_1}\right)^x - \mu - \frac{\lambda(1/\Theta_1)^x}{x+q+1}.$$

Since $\Phi(x) \neq 0$ at $x = -q-1$, it is enough to examine the nature of the zeros of $\Psi(x)$. Further, we know that $\Phi(-a) = \Phi(0) = \Phi(\beta) = 0$, so that $\Psi(x)$ vanishes also at $x = -a$, 0 and β, and hence by Rolle's theorem $\Psi'(x)$ must vanish at two points — one lying in $(-a, 0)$ and the other in $(0, \beta)$. Now

$$\Psi'(x) = \left(\frac{\Theta_0}{\Theta_1}\right)^x \frac{(x+q+1)B-1}{(x+q+1)^2}\left[\frac{(x+q+1)^2A}{(x+q+1)B-1} - \lambda\left(\frac{1}{\Theta_0}\right)^x\right]$$

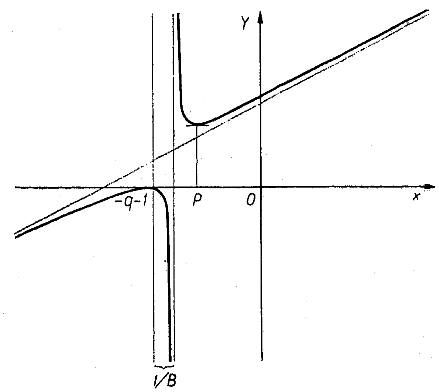
where we have put, for the sake of brevity, $A = \log(\Theta_0/\Theta_1)$ and $B = \log(1/\Theta_1)$. Obviously $B > A > 0$ and if $0 < \Theta_1 < 1/e$, $B \geqslant 1$. Since $\Psi'(x) \neq 0$ at $x = -q-1+1/B$, the zeros of $\Psi'(x)$ are the points of intersection of the curves

$$y = \lambda\left(\frac{1}{\Theta_0}\right)^x, \quad \text{i. e.} \quad y = \lambda e^{Cx} \quad \text{where} \quad C = \log\frac{1}{\Theta_0} \tag{16}$$

and

$$y = \frac{(x+q+1)^2A}{(x+q+1)B-1}. \tag{17}$$

Now (17) represents a hyperbola shown in the figure:



The asymptotes are

$$(x+q+1)B-1 = 0$$

and

$$By-A(x+q+1) = \frac{A}{B}.$$

In the figure the abscissa of the point P where the upper branch of the hyperbola turns upwards is $-q-1+2/B$. The only points of intersection of the curves (16) and (17) can lie to the right of the vertical asymptote of the hyperbola and the number of such points can be at most three. Indeed, one can easily prove the following lemma:

The transcendental equation

$$a^x = \frac{(ax+\beta)^2}{\gamma x+\delta}$$

where $a>1$, $\gamma>0$ *and* $a\delta-\beta\gamma\neq 0$ *can have at most three real different roots.*

We have already observed that $\Psi'(x)$ must vanish at two points R and S (say), R lying between the lines $x=-a$ and $x=0$ and S lying between the lines $x=0$ and $x=\beta$.

Since $a=q-p<q+1-2/B$, the point $(-a,0)$ lies to the right of P and so at the point R the tangent to the curve (16) must be below that of the hyperbola. For if it were otherwise, the curves being strictly increasing to the right of P, the curve (16) would newer cut the curve (17) again, which is contrary to what we have proved above. Hence at the point S the curve (16) crosses the hyperbola (17) from below, and so there is no other point of intersection of these curves after S. We have thus proved that (16) and (17) must intersect in 3 points of which one is to the right of OY, one lies between O and P and the third to the left of P.

Hence $\Psi'(x)$ vanishes at 2 points to the left of OY and one to the right of OY. If $\Psi(\gamma)=0$, we should be led to the impossible conclusion that $\Psi'(x)$ vanishes at 2 points to the right of OY. Hence the only positive integral zeros of $\Psi(x)$ and thus also of $\Phi(x)$ are 0 and β. We have thus proved that (10) and (15) cannot have a common root Θ_0.

References

[1] S. Gołąb, *Contribution à la formule simpsonienne de quadrature approchée*, Annales Polonici Math. 1 (1) (1954), p. 166-175.

[2] S. Gołąb et C. Olech, *Contribution à la théorie de la formule simpsonienne des quadratures approchées*, Annales Polonici Math. 1 (1) (1954), p. 176-183.

Etude de la solution fondamentale de l'équation elliptique et des problèmes aux limites

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1. Introduction. Soit l'équation linéaire aux dérivées partielles du second ordre de la forme générale

$$(1)\qquad \Psi(u) = \sum_{a,\beta=1}^{n} a_{a\beta}(x_1,\ldots,x_n)\frac{\partial^2 u}{\partial x_a \partial x_\beta} + \sum_{a=1}^{n} b_a(x_1,\ldots,x_n)\frac{\partial u}{\partial x_a} + c(x_1,\ldots,x_n)u = 0$$

où les coefficients $a_{a\beta}(x_1,\ldots,x_n)$, $b_a(x_1,\ldots,x_n)$, $c(x_1,\ldots,x_n)$ sont des fonctions des n coordonnées rectangulaires $(x_1,\ldots,x_n)$, déterminées dans un domaine borné et mesurable Ω dans l'espace euclidien à n dimensions.

Une prémière méthode d'étude de la solution fondamentale de l'équation (1) dans le cas elliptique a été donnée par E. Levi [2] dans le cas des coefficients admettants des dérivées secondes. Elle a été développée par W. Sternberg [7] dans le cas $n=3$, mais encore avec la même hypothèse sur les dérivées, ensuite approfondie et généralisée par M. Gevrey [1] pour les coefficients vérifiant la condition de Hölder et n quelconque. Les recherches de M. Gevrey sont basées sur la *méthode des cylindres* de la dérivation des intégrales généralisées. Cette méthode est correcte dans l'étude des dérivées premières du potentiel, mais elle présente des lacunes dans celle des dérivées secondes, bien que M. Gevrey ait obtenu résultats positifs.

Dans le présent travail nous exposerons une autre méthode d'étude des dérivées des intégrales généralisées et nous compléterons l'étude de la solution fondamentale. Ensuite nous étudierons deux problèmes aux limites.

2. Le cas des coefficients constants. Nous rappelons l'étude du cas des coefficients $a_{a\beta}$ *constants* pour l'équation

$$(2)\qquad \hat{D}u = \sum_{a,\beta=1}^{n} a_{a\beta}\frac{\partial^2 u}{\partial x_a \partial x_\beta} = 0$$