En vertu de (2.5) on a

(4.2) $$g_i(x) \leqslant p_i(x) \quad (i = 1, 2, \ldots, n;\ x \in D),$$

(4.3) $$g_i(\xi) \leqslant p_i(\xi) = \eta_i \quad (i = 1, 2, \ldots, n).$$

Les fonctions $f_i\big(t, g_1(t), \ldots, g_n(t)\big)$ étant continues pour $t \in D$, on a

(4.4) $$p_i'(x) = f_i\big(x, g_1(x), \ldots, g_n(x)\big) \quad (i = 1, 2, \ldots, n;\ x \in D).$$

Les fonctions $f_i(x, y_1, \ldots, y_n)$ étant croissant par rapport à $y_1, \ldots, y_n$ (cf. l'hypothèse H) on déduit de (4.4) et (4.2) le système d'inégalités différentielles

(4.5) $$p_i'(x) \leqslant f_i\big(x, p_1(x), \ldots, p_n(x)\big) \quad (i = 1, 2, \ldots, n;\ x \in D),$$

auquel on peut appliquer le théorème B cité au § 3. On obtiendra ainsi, en vertu de (4.3), (4.5) et (2.4), les inégalités (3.3) qui, rapprochées des inégalités (4.2), conduisent aux inégalités (2.6), c. q. f. d.

**§ 5.** Remarque 1. Le théorème I est en particulier vrai pour $n = 1$. Dans ce cas chacun des systèmes (2.1), (2.5) et (2.6) pris séparément se réduit à une relation. Posons en particulier

$$f_1(x, y_1) = NM|h(x)|y_1, \quad \xi = 0, \quad \eta_1 = M, \quad g_1(x) = |G(x)|.$$

L'inégalité (1.1) intervenant dans le lemme de M. Bellman prend la forme

$$g_1(x) \leqslant \eta_1 + \int_{\xi}^{x} f_1\big(t, g_1(t)\big)\, dt.$$

L'intégrale supérieure $y_1 = k_1(x)$ de l'équation $y_1' = f(x, y_1) = NM|h(x)|y_1$ pour laquelle $g_1(\xi) = \eta_1 = M$ est de la forme $k_1(x) = M\exp\big(NM\int_0^x |h(t)|\, dt\big)$.

En vertu du théorème I on obtient l'inégalité $g_1(x) \leqslant k_1(x)$ pour $0 \leqslant x < a$ et cette inégalité coïncide évidemment avec l'inégalité (1.2).

### Travaux cités

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## Wave propagation in a stratified medium

by E. J. Scott (Urbana, Illinois)

In a recent article[1]) the problem of propagation of heat in a bar consisting of many parts having different thermal properties was analyzed. Equally as important in the applications is the consideration of wave propagation in a medium composed of material having different physical characteristics. Examples of such media are: (1) a transmission line consisting of segments each of which is made of a different metal, (2) a taut string having parts of various densities, and (3) contiguous slabs of different materials.

To be specific, let us suppose that a dissipationless transmission line consists of $n$ parts $x_{k-1} < x < x_k$ $(k = 1, 2, \ldots, n,\ x_0 = 0)$ having the corresponding line constants $a_k$. We shall consider the following boundary value problem:

(1) $$\frac{\partial^2 V_k}{\partial t^2} = a_k^2 \frac{\partial^2 V_k}{\partial x^2}, \quad x_{k-1} < x < x_k, \quad x_0 = 0, \quad t > 0 \quad (k = 1, 2, \ldots, n);$$

(2) $$V_1(0, t) = G(t), \quad V_n(x_n, t) = H(t), \quad t > 0;$$

(3) $$V_k(x_k, t) = V_{k+1}(x_k, t) \quad (k = 1, 2, \ldots, n-1);$$

(4) $$\frac{\partial V_k(x_k, t)}{\partial x} = \frac{\partial V_{k+1}(x_k, t)}{\partial x} \quad (k = 1, 2, \ldots, n-1);$$

(5) $$V_k(x, 0) = F_k(x), \quad x_{k-1} < x < x_k, \quad x_0 = 0 \quad (k = 1, 2, \ldots, n);$$

(6) $$\frac{\partial V_k(x, 0)}{\partial t} = \varphi_k(x), \quad x_{k-1} < x < x_k, \quad x_0 = 0 \quad (k = 1, 2, \ldots, n).$$

Conditions (2) are the boundary conditions, (3) and (4) are the continuity conditions, and (5) as well as (6), the initial conditions.

We shall employ the Laplace transform to effect a solution. To that end, let $L_t\{V_k(x, t)\} = v_k(x, p)$ $(k = 1, 2, \ldots, n)$. Then, taking the Laplace transform of (1), we obtain

(7) $$p^2 v_k(x, p) - pV_k(x, 0) - \partial V_k(x, 0)/\partial t = a_k^2 d^2 v_k(x, p)/dx^2,$$
$$x_{k-1} < x < x_k, \quad x_0 = 0 \quad (k = 1, 2, \ldots, n).$$

[1]) V. Vodička, *Conduction de la chaleur dans une barre formée de plusiers parties en materiaux différents*, Prace Mat. Fiz. 48 (1952), p. 45-52.

If the system is initially in equilibrium, $V_k(x, 0) = \partial V_k(x, 0)/\partial t = 0$ and (7) reduces to

$$d^2 v_k(x, p)/dx^2 - (p^2/a_k^2)v_k(x, p) = 0 \tag{8}$$

whose solution is

$$v_k(x, p) = A_k \cosh(p/a_k)x + B_k \sinh(p/a_k)x \quad (k = 1, 2, \ldots, n). \tag{9}$$

The transforms of (2), (3) and (4) with respect to $t$ are:

$$v_1(0, p) = g(p), \quad v_n(x_n, p) = h(p), \tag{10}$$

$$v_k(x_k, p) = v_{k+1}(x_k, p), \quad dv_k(x_k, p)/dx = dv_{k+1}(x_k, p)/dx \quad (k = 1, 2, \ldots, n-1), \tag{11}$$

where $L\{G(t)\} = g(p)$ and $L\{H(t)\} = h(p)$.

From conditions (9), (10) and (11), we obtain

$$g(p) = A_1, \tag{12}$$

$$A_k \cosh(p/a_k)x_k + B_k \sinh(p/a_k)x_k = A_{k+1} \cosh(p/a_{k+1})x_k + B_{k+1} \sinh(p/a_{k+1})x_k \quad (k = 1, 2, \ldots, n-1), \tag{13}$$

$$(a_{k+1}/a_k)[A_k \sinh(p/a_k)x_k + B_k \cosh(p/a_k)x] = A_{k+1} \sinh(p/a_{k+1})x_k + B_{k+1} \cosh(p/a_{k+1})x_k \quad (k = 1, 2, \ldots, n-1), \tag{14}$$

$$A_n \cosh(p/a_n)x_n + B_n \sinh(p/a_n)x_n = h(p). \tag{15}$$

Solving the system of equations (13) and (14) yields the result

$$A_{k+1} = A_k \delta_k + B_k \Delta_k \quad (k = 1, 2, \ldots, n-1), \tag{16}$$

$$B_{k+1} = -A_k \bar{\Delta}_k - B_k \bar{\delta}_k \quad (k = 1, 2, \ldots, n-1), \tag{17}$$

where

$$\delta_k = \begin{vmatrix} \cosh(p/a_k)x_k & \sinh(p/a_{k+1})x_k \\ (a_{k+1}/a_k)\sinh(p/a_k)x_k & \cosh(p/a_{k+1})x_k \end{vmatrix}, \tag{18}$$

$$\Delta_k = \begin{vmatrix} \sinh(p/a_k)x_k & \sinh(p/a_{k+1})x_k \\ (a_{k+1}/a_k)\cosh(p/a_k)x_k & \cosh(p/a_{k+1})x_k \end{vmatrix}, \tag{19}$$

$\bar{\delta}_k$ is the determinant obtained from $\delta_k$ by replacing the hyperbolic sine function by the hyperbolic cosine function and *vice versa*, and $\bar{\Delta}_k$ is the determinant obtained from $\Delta_k$ by replacing the hyperbolic sine function by the hyperbolic cosine function and *vice versa*.



Thus, since $A_1$ is known, (16) and (17) enable us to express $A_n$ and $B_n$ in terms of $B_1$ only. These values when substituted in (15) gives a linear equation which can be solved for $B_1$. By retracing one's steps the values $A_{n-1}, B_{n-1}, A_{n-2}, B_{n-2}, \ldots, A_3, B_3, A_2$ and $B_2$ are obtained. Hence, by the complex inversion integral,

$$V_k(x, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} [A_k \cosh(p/a_k)x + B_k \sinh(p/a_k)x]e^{pt}\,dp \quad (k = 1, 2, \ldots, n). \tag{20}$$

The procedure is perfectly general. When $k$ is large, however, the amount of algebra involved is considerable. We shall consider in detail a problem for which $k = 1, 2$.

Let us suppose that initially a finite transmission line is dead and that the ends $x_0 = 0$ and $x_2$ are maintained at zero potential and $E_0$, respectively, *i. e.*

$$V_1(0, t) = 0, \quad V_2(x_2, t) = E_0, \quad t > 0. \tag{21}$$

It follows readily that $g(p) = 0$, $h(p) = E_0/p$, $A_1 = 0$,

$$A_2 = \frac{E_0 \Delta_1}{p[\Delta_1 \cosh(px_2/a_2) - \bar{\delta}_k \sinh(px_2/a_2)]},$$

$$B_1 = \frac{E_0}{p[\Delta_1 \cosh(px_2/a_2) - \bar{\delta}_k \sinh(px_2/a_2)]},$$

and

$$B_2 = \frac{-\bar{\delta}_k E_0}{p[\Delta_1 \cosh(px_2/a_2) - \bar{\delta}_k \sinh(px_2/a_2)]}.$$

Therefore, from (9) we have the following equations:

$$v_1(x, p) = \frac{E_0 \sinh(p/a_1)x}{p\{\sinh(p/a_1)x_1 \cosh(p/a_2)(x_2 - x_1) + (a_2/a_1)\cosh(p/a_1)x_1 \sinh(p/a_2)(x_2 - x_1)\}}, \quad 0 < x < x_1, \tag{22}$$

$$v_2(x, p) = \frac{E_0\{\sinh(p/a_1)x_1 \cosh(p/a_2)(x - x_1) + (a_2/a_1)\cosh(p/a_1)x_1 \sinh(p/a_2)(x - x_1)\}}{p\{\sinh(p/a_1)x_1 \cosh(p/a_2)(x_2 - x_1) + (a_2/a_1)\cosh(p/a_1)x_1 \sinh(p/a_2)(x_2 - x_1)\}}, \quad x_1 < x < x_2. \tag{23}$$

By the inversion theorem, we thus have

(24) $$V_1(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v_1(x,p)e^{pt}\,dp,$$

(25) $$V_2(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v_2(x,p)e^{pt}\,dp.$$

The integrands of (24) and (25) are single-valued functions with poles at $p=0$ and those values of $p$ for which

(26) $$a_1\tanh(p/a_1)x_1 + a_2\tanh(p/a_2)(x_2-x_1) = 0.$$

It can be shown readily that equation (26) has the roots $p=0$ and $p=\pm i\beta_n$, where $\beta_n$ are the real positive roots of

(27) $$a_1\tan(\beta x_1/a_1) + a_2\tan\{\beta(x_2-x_1)/a_2\} = 0.$$

One observes that $p=0$ is a simple and not a double pole.

For the integrand of equation (24), we find from the theory of residues that

$$\operatorname{Res}(0) = E_0 x/x_2,$$

$$\operatorname{Res}(\pm i\beta_n) = \frac{E_0 e^{\pm i\beta_n t}\sin(\pm\beta_n x/a_1)}{\pm M},$$

where

$$M = \beta_n\{(x_2/a_1)\cos(\beta_n x_1/a_1)\cos[\beta_n(x_2-x_1)/a_2] -$$
$$-[(x_2-x_1)/a_2 + x_1 a_2/a_1^2]\sin(\beta_n x_1/a_1)\sin[\beta_n(x_2-x_1)/a_2]\}.$$

Therefore,

(28) $$V_1(x,t) = E_0 x/x_2 + 2E_0\sum_{n=1}^{\infty}\frac{\cos(\beta_n t)\sin(\beta_n x/a_1)}{M} \qquad (0<x<x_1).$$

For the integrand of equation (25), we find similarly that $\operatorname{Res}(0) = E_0 x/x_2$,

$$\operatorname{Res}(\pm i\beta_n) = \frac{E_0 e^{\pm i\beta_n t}\sin(\pm\beta_n x_1/a_1)\cos[\beta_n(x-x_1)/a_2]}{N} +$$
$$+\frac{E_0 e^{\pm i\beta_n t}(a_2/a_1)\cos(\beta_n x_1/a_1)\sin[\pm\beta_n(x-x_1)/a_2]}{N},$$

where

$$N = \pm\beta_n\{(x_1/c_1)\cos(\beta_n x_1/a_1)\cos[\beta_n(x_2-x_1)/a_2] -$$
$$-[(x_2-x_1)/a_2 + x_1 c_2/a_1^2]\sin(\beta_n x_1/a_1)\sin[\beta_n(x_2-x_1)/a_2]\}$$



and consequently

(29) $$V_2(x,t) = E_0 x/x_2 +$$
$$+2E_0\sum_{n=1}^{\infty}\left(\frac{\cos(\beta_n t)\sin(\beta_n x_1/a_1)\cos[\beta_n(x-x_1)/a_2]}{S} + \right.$$
$$\left. + \frac{\cos(\beta_n t)(a_2/a_1)\cos(\beta_n x_1/a_1)\sin[\beta_n(x-x_1)/a_2]}{S}\right),$$

where

$$S = \beta_n\{(x_2/a_1)\cos(\beta_n x_1/a_1)\cos[\beta_n(x_2-x_1)/a_2] -$$
$$-[(x_2-x_1)/a_2 + a_1 x_1/a_1^2]\sin(\beta_n x_1/a_1)\sin[\beta_n(x_2-x_1)/a_2]\} \quad (x_1<x<x_2).$$

**Remark.** It might be mentioned that the preceding analysis applies directly to a spherically stratified medium for which there is spherical symmetry. For, the differential equation is $\partial^2(rU)/\partial t^2 = k^2\partial^2(rU)/\partial r^2$ and this is exactly of the same form as (1).