En vertu de (2.5) on a

\[(4.2) \quad g_i(x) \leq p_i(x) \quad (i = 1, 2, \ldots, n; x \in \mathcal{D}),\]

\[(4.3) \quad g_i(t) \leq p_i(t) = q_i \quad (i = 1, 2, \ldots, n).\]

Les fonctions \( f_1(t, g_1(t), \ldots, g_n(t)) \) étant continues pour \( t \in \mathcal{D} \), on a

\[(4.4) \quad p_i(x) = f_i(x, g_1(x), \ldots, g_n(x)) \quad (i = 1, 2, \ldots, n; x \in \mathcal{D}),\]

Les fonctions \( f_i(x, y_1, \ldots, y_n) \) étant croissants par rapport à \( y_1, \ldots, y_n \) (cf. l’hypothèse II) on déduit de (4.4) et (4.2) le système d’inégalités différentielles

\[(4.5) \quad p_i(x) \leq f_i(x, p_1(x), \ldots, p_n(x)) \quad (i = 1, 2, \ldots, n; x \in \mathcal{D}),\]

auquel on peut appliquer le théorème B cité au § 3. On obtiendra ainsi, en vertu de (4.3), (4.5) et (2.4), les inégalités (3.3) qui, rapprochées des inégalités (4.2), conduisent aux inégalités (2.6), c. q. f. d.

§ 5. Remarque. Le théorème I est en particulier vrai pour \( n = 1 \). Dans ce cas chacun des systèmes (2.1), (2.5) et (2.6) se réduit à une relation. Posons en particulier

\[ f_i(x, y) = \mathcal{M}(\mathcal{M}(x))y_i, \quad \xi = 0, \quad \eta = \mathcal{M}, \quad g_i(x) = |\mathcal{M}(x)|.\]

L’inégalité (1.1) intervenant dans le lemme de M. Bellman prend la forme

\[ g_i(x) \leq \eta + \int_{\xi}^{\eta} f_i(t, g_i(t))dt.\]

L’intégrale supérieure \( y_i = k_i(x) \) de l’équation

\[ y_i = f_i(x, y_i) = \mathcal{M}(\mathcal{M}(x))y_i, \]

pour laquelle \( g_i(x) = \eta = \mathcal{M} \) est de la forme \( k_i(x) = \mathcal{M} \exp \{N \mathcal{M}(\mathcal{M}(x))dt\}.\)

En vertu du théorème I on obtient l’inégalité \( g_i(x) \leq k_i(x) \) pour \( 0 \leq x \leq a \) et cette inégalité coïncide évidemment avec l’inégalité (1.2).

Travaux cités


Wave propagation in a stratified medium

by E. J. Scott (Urbana, Illinois)

In a recent article\(^{a}\) the problem of propagation of heat in a bar consisting of many parts having different thermal properties was analyzed. Equally as important in the applications is the consideration of wave propagation in a medium composed of material having different physical characteristics. Examples of such media are: (1) a transmission line consisting of segments each of which is made of a different metal, (2) a taut string having parts of various densities, and (3) contiguous slabs of different materials.

To be specific, let us suppose that a dissipationless transmission line consists of \( n \) parts \( a_{k-1} < x < a_k \) \( (k = 1, 2, \ldots, n) \), \( a_0 = 0 \) having the corresponding line constants \( a_k \). We shall consider the following boundary value problem:

\[(1) \quad \frac{\partial^2 V_k}{\partial x^2} = \frac{\partial^2 V_k}{\partial t^2}, \quad x_{k-1} < x < a_k, \quad x_0 = 0, \quad t > 0 \quad (k = 1, 2, \ldots, n);\]

\[(2) \quad V_k(0, t) = G(t), \quad V_k(a_k, t) = H(t), \quad t > 0;\]

\[(3) \quad V_k(x_k, t) = V_{k+1}(a_k, t) = 0 \quad (k = 1, 2, \ldots, n-1);\]

\[(4) \quad \frac{\partial V_k}{\partial x}(x_k, t) = \frac{\partial V_{k+1}}{\partial x}(a_k, t) \quad (k = 1, 2, \ldots, n-1);\]

\[(5) \quad V_k(x, 0) = F_k(x), \quad x_{k-1} < x < x_k, \quad a_0 = 0 \quad (k = 1, 2, \ldots, n);\]

\[(6) \quad \frac{\partial V_k}{\partial t}(x, 0) = q_k(x), \quad x_{k-1} < x < x_k, \quad a_0 = 0 \quad (k = 1, 2, \ldots, n).\]

Conditions (2) are the boundary conditions, (3) and (4) are the continuity conditions, and (5) as well as (6), the initial conditions.

We shall employ the Laplace transform to effect a solution. To that end, let \( L_1[V_k(x, t)] = \tau_k(x, p) \) \( (k = 1, 2, \ldots, n) \). Then, taking the Laplace transform of (1), we obtain

\[(7) \quad p^2 \tau_k(x, p) - p V_k(x, 0) - \frac{\partial V_k(x, 0)}{\partial t} = \frac{\partial^2 \tau_k(x, p)}{\partial x^2}, \quad x_{k-1} < x < a_k, \quad a_0 = 0 \quad (k = 1, 2, \ldots, n).\]

\(^{a}\) V. Vodžička, Kündigung der Leistung durch eine Barre mit mehreren

If the system is initially in equilibrium, \( V_\delta(x, 0) = \partial V_\delta(x, 0)/\partial t = 0 \) and (7) reduces to

\[
d^2 v_\delta(x, p)/dx^2 - (p^2/a^2) v_\delta(x, p) = 0
\]

whose solution is

\[
v_\delta(x, p) = A_k \cosh(p/a_k)x + B_k \sinh(p/a_k)x \quad (k = 1, 2, \ldots, n).
\]

The transforms of (2), (3) and (4) with respect to \( t \) are:

\[
v_\delta(0, p) = g(p), \quad v_\delta(x, p) = h(p),
\]

\[
v_\delta(x, p) = v_{k+1}(x, p), \quad d^2 v_\delta(x, p)/dx^2 = d^2 v_{k+1}(x, p)/dx^2 \quad (k = 1, 2, \ldots, n-1),
\]

where \( L[\theta(t)] = g(p) \) and \( L[H(t)] = h(p) \).

From conditions (9), (10) and (11), we obtain

\[
g(p) = A_1,
\]

\[
A_k \cosh(p/a_k)x_k + B_k \sinh(p/a_k)x_k
\]

(13)

\[
= A_{k+1} \cosh(p/a_{k+1})x_k + B_{k+1} \sinh(p/a_{k+1})x_k \quad (k = 1, 2, \ldots, n-1),
\]

(14)

\[
(a_{k+1}/a_k)[A_k \sinh(p/a_k)x_k + B_k \cosh(p/a_k)x_k]
\]

\[
= A_{k+1} \sinh(p/a_{k+1})x_k + B_{k+1} \cosh(p/a_{k+1})x_k \quad (k = 1, 2, \ldots, n-1),
\]

(15)

\[
A_k \cosh(p/a_k)x_k = B_k \sinh(p/a_k)x_k = h(p).
\]

Solving the system of equations (13) and (14) yields the result

\[
A_{k+1} = A_k \delta_k + B_k \delta_k \quad (k = 1, 2, \ldots, n-1),
\]

(16)

\[
B_{k+1} = -A_k \delta_k - B_k \delta_k \quad (k = 1, 2, \ldots, n-1),
\]

(17)

where

\[
\delta_k = \begin{vmatrix}
\cosh(p/a_k)x_k & \sinh(p/a_k)x_k \\
(a_{k+1}/a_k)\sinh(p/a_{k+1})x_k & \cosh(p/a_{k+1})x_k
\end{vmatrix}
\]

(18)

\[
A_k = \begin{vmatrix}
\sinh(p/a_k)x_k & \sinh(p/a_k)x_k \\
(a_{k+1}/a_k)\cosh(p/a_{k+1})x_k & \cosh(p/a_{k+1})x_k
\end{vmatrix}
\]

(19)

\( \delta_k \) is the determinant obtained from \( \delta_k \) by replacing the hyperbolic sine function by the hyperbolic cosine function and vice versa, and \( \delta_k \) is the determinant obtained from \( A_k \) by replacing the hyperbolic sine function by the hyperbolic cosine function and vice versa.

Thus, since \( A_1 \) is known, (16) and (17) enable us to express \( A_n \) and \( B_n \) in terms of \( B_1 \) only. These values when substituted in (15) gives a linear equation which can be solved for \( B_1 \). By retraceing one's steps the values \( A_{n-1}, B_{n-1}, A_{n-2}, B_{n-2}, \ldots, A_1, B_1, A_2 \) and \( B_2 \) are obtained. Hence, by the complex inversion integral,

\[
V_\delta(x, t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} [A_k \cosh(p/a_k)x + B_k \sinh(p/a_k)x]e^{ipt}dp \quad (k = 1, 2, \ldots, n).
\]

The procedure is perfectly general. When \( k \) is large, however, the amount of algebra involved is considerable. We shall consider in detail a problem for which \( k = 1, 2 \).

Let us suppose that initially a finite transmission line is dead and that the ends \( x_2 = 0 \) and \( x_3 \) are maintained at zero potential and \( E_k \), respectively, at.

(21)

\[
V_\delta(0, t) = 0, \quad V_\delta(x_3, t) = E_k, \quad t > 0.
\]

It follows readily that \( g(p) = 0, \ h(p) = E_k/p, \ A_1 = 0, \)

\[
A_k = \frac{E_k}{-\delta_k p}, \quad B_k = \frac{E_k}{p^2 \delta_k}, \quad 0 < x < x_2,
\]

and

\[
A_k = \frac{E_k}{-\delta_k p}, \quad B_k = \frac{E_k}{p^2 \delta_k}, \quad x_2 < x < x_3.
\]

Therefore, from (9) we have the following equations:

\[
v_\delta(x, p) = \frac{E_k \sinh(p/a_1)x}{p\sinh(p/a_1)x_k \cosh(p/a_1)x_k + (a_2/a_1) \cosh(p/a_2)x_k \sinh(p/a_2)x_k} \quad 0 < x < x_1,
\]

(22)

\[
v_\delta(x, p) = \frac{E_k \sinh(p/a_1)x}{p\sinh(p/a_1)x_k \cosh(p/a_1)x_k - (a_2/a_1) \cosh(p/a_2)x_k \sinh(p/a_2)x_k} \quad x_1 < x < x_2,
\]

(23)

\[
v_\delta(x, p) = \frac{E_k \sinh(p/a_1)x}{p\sinh(p/a_1)x_k \cosh(p/a_1)x_k + (a_2/a_1) \cosh(p/a_2)x_k \sinh(p/a_2)x_k} \quad x_2 < x < x_3.
\]
By the inversion theorem, we thus have

\begin{align}
V_1(x, t) & = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^1(p, x) e^{pt} dp, \\
V_2(x, t) & = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^2(p, x) e^{pt} dp.
\end{align}

The integrands of (24) and (25) are single-valued functions with poles at \( p = 0 \) and those values of \( p \) for which

\begin{equation}
\alpha_1 \tanh(p/\alpha_1) x_1 + \alpha_2 \tanh(p/\alpha_2) (x_2 - x_1) = 0.
\end{equation}

It can be shown readily that equation (26) has the roots \( p = 0 \) and \( p = \pm \beta_n \), where \( \beta_n \) are the real positive roots of

\begin{equation}
\alpha_1 \tan(\beta x_1/\alpha_2) + \alpha_2 \tan(\beta (x_2 - x_1)/\alpha_2) = 0.
\end{equation}

One observes that \( p = 0 \) is a simple and not a double pole.

For the integrand of equation (24), we find from the theory of residues that

\[
\text{Res}(0) = \frac{E_0 x}{\alpha_1},
\]

\[
\text{Res}(\pm \beta_n) = \frac{E_0 e^{\pm \beta x} \sin(\pm \beta x/\alpha_1)}{\pm M},
\]

where

\[M = \beta_n [x_1/\alpha_1 \cos(\beta x_1/\alpha_1) \cos(\beta (x_2 - x_1)/\alpha_2) - (x_2 - x_1)/\alpha_2 \alpha_2/\alpha_1^2 \sin(\beta x_1/\alpha_1) \sin(\beta (x_2 - x_1)/\alpha_2)].\]

Therefore,

\[
V_1(x, t) = E_0 x [x_1] + 2E_0 \sum_{n=1}^{\infty} \frac{\cos(\beta_n x) \sin(\beta_n x/\alpha_2)}{M} (0 < x < x_2).
\]

For the integrand of equation (25), we find similarly that \( \text{Res}(0) = \frac{E_0 x}{\alpha_2} \),

\[
\text{Res}(\pm \beta_n) = \frac{E_0 e^{\pm \beta x} \sin(\pm \beta x/\alpha_2) \cos(\beta (x_2 - x_1)/\alpha_2)}{N} + \frac{E_0 e^{\pm \beta x} \sin(\pm \beta x/\alpha_2) \cos(\beta (x_2 - x_1)/\alpha_2)}{N},
\]

where

\[N = \pm \beta_n [(x_1/\alpha_1 \cos(\beta x_1/\alpha_1) \cos(\beta (x_2 - x_1)/\alpha_2) - (x_2 - x_1)/\alpha_2 \alpha_2/\alpha_1^2 \sin(\beta x_1/\alpha_1) \sin(\beta (x_2 - x_1)/\alpha_2)].\]

and consequently

\[
V_2(x, t) = E_0 x [x_2] + \sum_{n=1}^{\infty} \frac{\cos(\beta_n t) \sin(\beta_n x_1/\alpha_2) \sin(\beta_n (x_2 - x_1)/\alpha_2)}{N} + \frac{\cos(\beta_n t) \cos(\beta_n x_1/\alpha_2) \sin(\beta_n (x_2 - x_1)/\alpha_2)}{N},
\]

where

\[R = \beta_n [(x_2 - x_1) \cos(\beta_n x_1/\alpha_2) \cos(\beta_n (x_2 - x_1)/\alpha_2) - (x_2 - x_1)/\alpha_2 \alpha_2/\alpha_1^2 \sin(\beta_n x_1/\alpha_2) \sin(\beta_n (x_2 - x_1)/\alpha_2)].\]

Remark. It might be mentioned that the preceding analysis applies directly to a spherically stratified medium for which there is spherical symmetry. For, the differential equation is \( \frac{\partial^2}{\partial r^2} \frac{\partial}{\partial r} = k^2 \frac{\partial^2}{\partial r^2} \frac{\partial}{\partial r} \) and this is exactly of the same form as (1).