References

[1] R. Bellman, *On the existence and boundedness of solutions of non-linear partial differential equations of parabolic type*, Transactions of the American Mathematical Society 64 (1948), p. 21-44.

[2] W. Mlak, *Differential inequalities of parabolic type*, Annales Polonici Mathematici, that volume, p. 349-354.

[3] R. Narasimhan, *On the asymptotic stability of solutions of parabolic differential equations*, Journal of Rational Mechanics and Analysis 3.3 (1954), p. 303-313.

[4] G. Prodi, *Questioni di stabilita per equazioni non lineari alle derivate parziali di tipo parabolico*, Academia Nazionale dei Lincei, Rendiconti classe di scienze fisiche, matematiche e naturali 8.10 (1951), p. 365-370.

[5] J. Szarski, *Sur la limitation et l'unicité des solutions d'un système non-linéaire d'équations paraboliques aux dérivées partielles du second ordre*, Annales Polonici Mathematici 2.2 (1955), p. 237-249.

Differential inequalities of parabolic type

by W. Mlak (Kraków)*

In connection with the stability problem of solutions of parabolic equations some theorems concerning certain differential inequalities have been discussed (see [1] and [2]).

In this paper we discuss some generalizations of the theorems about differential inequalities of the form

$$\frac{\partial z_s}{\partial t} < F_s\left(x,\, t,\, z_1, \ldots, z_n, \frac{\partial z_s}{\partial x_i}, \frac{\partial^2 z_s}{\partial x_i \partial x_k}\right) \quad (s = 1, 2, \ldots, n).$$

1. Notation and definitions. We investigate a hypercylinder of the form $G\times(0, T\rangle$ $(T > 0)$ where G is an open bounded region lying in the space E^m of points $(x_1, \ldots, x_m)$. We write $B = G\times(0, T\rangle$; Γ being the boundary of G we write $C = \Gamma\times(0, T\rangle$; $\bar{B}$ denotes the closure of B, $\bar{C} = \Gamma\times\langle 0, T\rangle$.

Suppose that $F(x_1, \ldots, x_m, t, u_1, \ldots, u_n, q_1, \ldots, q_m, p_{11}, \ldots, p_{mm})$, written shortly as $F(x, t, u, q_i, p_{ik})$, satisfies the following condition: For every system of numbers $\bar{r}_{ik}$ $(i, k = 1, \ldots, m)$, r_{ik} $(i, k = 1, \ldots, m)$ such that the quadratic form

$$\sum_{i,k=1}^{m} (\bar{r}_{ik} - r_{ik})\, \xi_i \xi_k$$

is non-negative for arbitrary $\xi_1, \ldots, \xi_m$, the following inequality holds:

$$F(x, t, u, q_i, \bar{r}_{ik}) \geqslant F(x, t, u, q_i, r_{ik}).$$

$F(x, t, u, q_i, p_{ik})$ is then called the *elliptic function with regard to* p_{ik}. A system of equations

$$\frac{\partial u_s}{\partial t} = F_s\left(x,\, t,\, u_1, \ldots, u_n, \frac{\partial u_s}{\partial x_i}, \frac{\partial^2 u_s}{\partial x_i \partial x_k}\right) \quad (s = 1, 2, \ldots, n)$$

is called *parabolic* if every F_s is elliptic.

* I want to express my thanks to J. Szarski for valuable remarks concerning this paper.

2. The following definition is introduced: a system of functions $H_s(z_1, \ldots, z_r, \tau)$ $(s = 1, \ldots, r)$[1]) *satisfies the condition* (W) with regard to $z_1, \ldots, z_r$ if for every s $(s = 1, \ldots, r)$ and for $\bar{u}_i \leqslant \bar{\bar{u}}_i$, $i \neq s$, $\bar{u}_s = \bar{\bar{u}}_s$ we have the inequality

$$H_s(\bar{u}_1, \ldots, \bar{u}_r, \tau) \leqslant H_s(\bar{\bar{u}}_1, \ldots, \bar{\bar{u}}_s, \tau).$$

Suppose now that $F_s(x, t, u, q_i, q_{ik})$ are defined for $x \in G$, $0 < t \leqslant T$, and arbitrary $u_1, \ldots, u_n, q_i, p_{ik}$. We formulate the generalization of the Westphal-Prodi theorem.

THEOREM 1. *Suppose that the functions $F_s(x, t, u, q_i, p_{ik})$ $(s = 1, \ldots, n)$ are elliptic with regard to p_{ik} and satisfy the condition* (W) *with regard to $u_1, \ldots, u_n$. Let $u_1(x, t), \ldots, u_n(x, t)$ and $v_1(x, t), \ldots, v_n(x, t)$ be continuous in $\bar{B}$ and satisfy the inequalities*

$$(*) \qquad u_\nu(x, t) < v_\nu(x, t) \quad \text{for} \quad (x, t) \in \bar{G} + C \quad (\nu = 1, 2, \ldots, n).$$

We assume that for $(x, t) \in B$, u_ν and v_ν possess continuous derivatives

$$\frac{\partial^2 u_\nu}{\partial x_i \partial x_k}, \quad \frac{\partial^2 v_\nu}{\partial x_i \partial x_k}.$$

Suppose that for every $P = (x, t) \in B$ and every s $(1 \leqslant s \leqslant n)$ for which the condition $u_s(x, t) = v_s(x, t)$ holds, the derivatives

$$\left(\frac{\partial v_s}{\partial t}\right)_P, \quad \left(\frac{\partial v_s}{\partial t}\right)_P$$

exist and the following inequalities are satisfied:

$$(\alpha) \qquad \left(\frac{\partial u_s}{\partial t}\right)_P \leqslant F_s\left(x, t, u_1(x, t), \ldots, u_n(x, t), \left(\frac{\partial u_s}{\partial x_i}\right)^P, \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)_P\right),$$

$$(\beta) \qquad \left(\frac{\partial v_s}{\partial t}\right)_P > F_s\left(x, t, v_1(x, t), \ldots, v_n(x, t), \left(\frac{\partial v_s}{\partial x_i}\right)_P, \left(\frac{\partial^2 v_s}{\partial v_i \partial x_k}\right)_P\right).$$

Under our assumptions the inequalities $u_\nu(x, t) < v_\nu(x, t)$ $(\nu = 1, \ldots, n)$ hold for $(x, t) \in B$.

Proof. We prove our theorem by reductio ad absurdum. Suppose that the set

$$E = \sum_{\nu=1}^{n} \underset{(x,t)}{E}\{(x, t) \in \bar{B}, v_\nu(x, t) \leqslant u_\nu(x, t)\}$$

is non-empty. Denote by E_t the projection of E on the t-axis and put $\xi = \inf E_t$. We have $\xi > 0$ and for $0 \leqslant t < \xi$

$$(1) \qquad u_\nu(x, t) < v_\nu(x, t), \quad x \in \bar{G}, \quad \nu = 1, 2, \ldots, n.$$

[1]) τ denotes here a sequence of variables different from $z_1, \ldots, z_r$

Therefore

$$(2) \qquad u_\nu(x, \xi) \leqslant v_\nu(x, \xi), \quad x \in \bar{G}, \quad \nu = 1, 2, \ldots, n.$$

At least one of the functions $z_\nu(x) = v_\nu(x, \xi) - u_\nu(x, \xi)$ has in G a minimum equal to zero. If it were not so, then according to $(*)$ we should have $z_\nu(x) > 0$ for $x \in \bar{G}$, $\nu = 1, \ldots, n$, and this contradicts the definition of ξ. Hence there exists s $(1 \leqslant s \leqslant n)$ and a point $\bar{P} = (\bar{x}, \xi)$ such that $\bar{x} \in G$ and

$$(3) \qquad u_s(\bar{x}, \xi) = v_s(\bar{x}, \xi)$$

and in $\bar{x}$ the function $z_s(x)$ has a minimum. By (2) we have

$$(4) \qquad u_i(\bar{x}, \xi) \leqslant v_i(\bar{x}, \xi) \quad (i \neq s).$$

According to our assumptions, the derivatives

$$\left(\frac{\partial u_s}{\partial t}\right)_{\bar{P}}, \quad \left(\frac{\partial v_s}{\partial t}\right)_{\bar{P}}$$

exist and

$$(5) \qquad \left(\frac{\partial u_s}{\partial t}\right)_{\bar{P}} \leqslant F_s\left(\bar{x}, \xi, u_1(\bar{P}), \ldots, u_n(\bar{P}), \left(\frac{\partial u_s}{\partial x_i}\right)_{\bar{P}}, \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)_{\bar{P}}\right),$$

$$(6) \qquad \left(\frac{\partial v_s}{\partial t}\right)_{\bar{P}} > F_s\left(\bar{x}, \xi, v_1(\bar{P}), \ldots, v_n(\bar{P}), \left(\frac{\partial v_s}{\partial x_i}\right)_{\bar{P}}, \left(\frac{\partial^2 v_s}{\partial x_i \partial x_k}\right)_{\bar{P}}\right).$$

Since $z_s(x)$ has a minimum in $\bar{x}$, the quadratic form

$$\sum_{i,k=1}^{m} \left[\frac{\partial^2 (v_s - u_s)}{\partial x_i \partial x_k}\right]_{\bar{P}} \xi_i \xi_k$$

is non-negative for arbitrary $\xi_1, \ldots, \xi_m$. But F_s is elliptic — therefore, in view of

$$\left(\frac{\partial v_s}{\partial x_i}\right)_{\bar{P}} = \left(\frac{\partial u_s}{\partial x_i}\right)_{\bar{P}},$$

we have

$$(7) \qquad F_s\left(\bar{x}, \xi, v_1(\bar{P}), \ldots, v_n(\bar{P}), \left(\frac{\partial v_s}{\partial x_i}\right)_{\bar{P}}, \left(\frac{\partial^2 v_s}{\partial x_i \partial x_k}\right)_{\bar{P}}\right)$$
$$\geqslant F_s\left(x, \xi, v_1(\bar{P}), \ldots, v_n(\bar{P}), \left(\frac{\partial u_s}{\partial x_i}\right)_{\bar{P}}, \left(\frac{\partial^2 v_s}{\partial x_i \partial x_k}\right)_{\bar{P}}\right).$$

According to (3) and (4) we get, because of the condition (W),

$$(8) \qquad F_s\left(\bar{x}, \xi, v_1(\bar{P}), \ldots, v_n(\bar{P}), \left(\frac{\partial u_s}{\partial x_i}\right)_{\bar{P}}, \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)_{\bar{P}}\right)$$
$$\geqslant F_s\left(\bar{x}, \xi, u_1(\bar{P}), \ldots, u_n(\bar{P}), \left(\frac{\partial u_s}{\partial x_i}\right)_{\bar{P}}, \left(\frac{\partial^2 u_s}{\partial x_i \partial x_k}\right)_{\bar{P}}\right).$$

By (5), (6), (7) and (8) we have

$$\left(\frac{\partial v_s}{\partial t}\right)_{\overline{P}} > \left(\frac{\partial u_s}{\partial t}\right)_{\overline{P}}. \tag{9}$$

On the other hand, by (1) and (3) we have

$$\left(\frac{\partial v_s}{\partial t}\right)_{\overline{P}} \leqslant \left(\frac{\partial u_s}{\partial t}\right)_{\overline{P}}.$$

We obtain a contradiction of (9), hence E is empty.

3. We can now formulate the second theorem concerning strong differential inequalities. We introduce the following assumption (A): for every $\overline{x}\in\Gamma$ and every ν $(1\leqslant\nu\leqslant n)$ there exists a straight line l_ν such that an open segment $(\overline{x},\overline{\overline{x}}_\nu)$ of l_ν lies in G and the derivatives of the form

$$\frac{du_\nu}{dl_\nu} = \lim_{\substack{x\to\overline{x}\\ x\in l\\ x\in G}} \frac{u_\nu(\overline{x},t)-u_\nu(x,t)}{|\overline{x}-x|}, \qquad \frac{dv_\nu}{dl_\nu} = \lim_{\substack{x\to\overline{x}\\ x\in l\\ x\in G}} \frac{v_\nu(\overline{x},t)-v_\nu(x,t)}{|\overline{x}-x|}$$

exist.

THEOREM 2. *Suppose that the functions $F_s(x,t,u,q_i,p_{ik})$ $(s=1,\ldots,n)$ are elliptic and satisfy the condition* (W) *with respect to $u_1,\ldots,u_n$. We assume that the functions $\varphi_s(x,t,z_1,\ldots,z_n)$ $(s=1,\ldots,n)$ satisfy the condition* (W) *with regard to $z_1,\ldots,z_n$.*

The functions $u_1(x,t),\ldots,u_n(x,t);v_1(x,t),\ldots,v_n(x,t)$ are continuous in $\overline{B}$; for $(x,t)\in B$ they posses continuous derivatives $\partial^2u_\nu/\partial x_i\partial x_k$, $\partial^2v_\nu/\partial x_i\partial x_k$.

Let

$$u_\nu(x,0) < v_\nu(x,0), \quad x\in\overline{G}, \quad \nu=1,2,\ldots,n. \tag{10}$$

We assume that the assumption (A) *is satisfied. For every $(x,t)\in C$, $\nu=1,\ldots,n$, let*

$$\frac{du_\nu}{dl_\nu} \leqslant \varphi_\nu\big(x,t,u_1(x,t),\ldots,u_n(x,t)\big), \tag{11}$$

$$\frac{dv_\nu}{dl_\nu} > \varphi_\nu\big(x,t,v_1(x,t),\ldots,v_n(x,t)\big)\ ^{2)}. \tag{12}$$

[2]) Observe that the direction l_ν is for both derivatives du_ν/dl_ν, dv_ν/dl_ν the same. It depends on the point (x,t) and on ν.

We assume that for every $P=(x,t)\in B$ and every s $(1\leqslant s\leqslant n)$ for which $u_s(x,t)=v_s(x,t)$, the derivatives

$$\left(\frac{\partial u_s}{\partial t}\right)_P, \quad \left(\frac{\partial v_s}{\partial t}\right)_P$$

exist and the inequalities (α) *and* (β) *hold.*

Under our assumptions the inequalities $u_\nu(x,t)<v_\nu(x,t)$ $(\nu=1,\ldots,n)$ hold for $(x,t)\in\overline{B}$.

Proof. We prove the theorem by reductio ad absurdum. Applying the same arguments and using the same notation as in the proof of theorem 1 we find that at least one of the functions

$$z_\nu(x) = v_\nu(x,\xi)-u_\nu(x,\xi)$$

has in $\overline{G}$ a minimum equal to zero. These minima are not reached in $x\in\Gamma$. Indeed if it were so, then there would exist $\overline{x}\in\Gamma$ and $s(1\leqslant s\leqslant n)$ such that $z_s(x)$ has a minimum in $\overline{x}$. We have $u_i(\overline{x},\xi)\leqslant v_i(\overline{x},\xi)$, $i\neq s$, $u_s(\overline{x},\xi)=v_s(\overline{x},\xi)$. Therefore

$$\varphi_s\big(\overline{x},\xi,u_1(\overline{x},\xi),\ldots,u_n(\overline{x},\xi)\big) \leqslant \varphi_s\big(\overline{x},\xi,v_1(\overline{x},\xi),\ldots,v_n(\overline{x},\xi)\big).$$

By the boundary inequalities (11), (12) we obtain

$$\left(\frac{du_s}{dl_s}\right)_{(\overline{x},\xi)} < \left(\frac{dv_s}{dl_s}\right)_{(\overline{x},\xi)}.$$

But $z_s(x)$ has a minimum in $\overline{x}$, hence

$$\left(\frac{dv_s}{dl_s}\right)_{(\overline{x},\xi)} \leqslant \left(\frac{du_s}{dl_s}\right)_{(\overline{x},\xi)}.$$

Thus we conclude that the function $z_s(x)$ has a minimum equal to zero in G. Now, applying the same arguments as in the proof of theorem 1, by (α), (β) we come to a contradiction of the definition of ξ. This completes the proof.

Remark 1. It may easily be shown that our theorems remain true for infinite hypercylinders of the form $G\times(0,\infty)$.

Remark 2. In both theorems we do not assume that the inequalities (α), (β) hold for every $(x,t)\in B$ and $s=1,\ldots,n$. If they were satisfied everywhere in B, for every $s=1,\ldots,n$, then the assumptions of theorem 1 would be satisfied. Therefore, in this case theorem 1 and theorem 2 remain true. The situation is analogous to the situation encountered in the theory of ordinary or partial differential inequalities of the first order (see *e.g.* [3]).

References

[1] G. Prodi, *Questioni di stabilita per equazioni non lineari alle derivate parziali di tipo parabolico*, Academia dei Lincei, Rendiconti classe di scienze fisiche, matematiche e naturali 8, 10 (1951), p. 365-370.

[2] R. Narasimhan, *On the asymptotic stability of solutions of parabolic differential equations*, Journal of Rational Mechanics and Analysis 3.3 (1954), p. 303-313.

[3] J. Szarski, *Systèmes d'inégalités différentielles aux dérivées partielles du premier ordre, et leurs applications*, Annales Polonici Mathematici 1. 1 (1954), p. 149-165.

Remarque concernant le travail de W. Pogorzelski: „Sur le système d'équations intégrales à une infinité de fonctions inconnues"

(Vol. II, I, 1955, pages 106-117)

Les lignes 13, 14, 15, 16, 17 à la page 107 doivent être remplacées par la phrase suivante:

„On peut faire correspondre à tout nombre positif ε un nombre positif $\eta(\varepsilon)$ et un nombre naturel $N(\varepsilon)$ tels que

$$|F_n(x_0, y_0, u_1^0, u_2^0, \ldots) - F_n(x, y, u_1, u_2, \ldots)| < \varepsilon$$

si

$$|x_0 x| < \eta, \quad |y_0 y| < \eta, \quad |u_\nu^0 - u_\nu| < \eta \quad (\nu = 1, 2, 3, \ldots, N).$$

Cette définition de la continuité est équivalente à la définition de la continuité au sens de la métrique (6)".