Differential inequalities of parabolic type

by W. MIŁAK (Kraków)*

In connection with the stability problem of solutions of parabolic equations some theorems concerning certain differential inequalities have been discussed (see [1] and [2]).

In this paper we discuss some generalizations of the theorems about differential inequalities of the form

$$\frac{\partial^2 z}{\partial t^2} < F_s\left(x, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial^2 z}{\partial x_i \partial x_j}\right) \quad (s = 1, 2, \ldots, n).$$

1. Notation and definitions. We investigate a hypercylinder of the form $G \times (0, T)$ ($T > 0$) where $G$ is an open bounded region lying in the space $\mathbb{R}^m$ of points $(x_1, \ldots, x_m)$. We write $\mathcal{B} = G \times (0, T)$; $\mathcal{B}$ being the boundary of $G$ we write $\mathcal{C} = \mathcal{B} \times (0, T)$. $\mathcal{B}$ denotes the closure of $\mathcal{B}$.

Suppose that $F(x, \ldots, x_m, t, u_1, \ldots, u_n, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_i \partial x_j})$, written shortly as $F(x, t, u, p_\alpha)$, satisfies the following condition: For every system of numbers $r_\alpha (i, k = 1, \ldots, m)$ such that the quadratic form

$$\sum_{i, j=1}^{m} (r_\alpha - t_\alpha) \xi_i \xi_j$$

is non-negative for arbitrary $\xi_1, \ldots, \xi_m$, the following inequality holds:

$$F(x, t, u, q, p_\alpha) \geq F(x, t, u, q, t_\alpha).$$

$F(x, t, u, q, p_\alpha)$ is then called the elliptic function with regard to $p_\alpha$.

A system of equations

$$\frac{\partial u}{\partial t} = F_s\left(x, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_i \partial x_j}\right) \quad (s = 1, 2, \ldots, n)$$

is called parabolic if every $F_s$ is elliptic.

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2. The following definition is introduced: a system of functions $H_s(x_1, \ldots, x_n, t)$ ($s = 1, \ldots, r$) satisfies the condition $(W)$ with regard to $u_1, \ldots, u_s$ if for every $s (s = 1, \ldots, r)$ and for $u_i \leq u_i, i \neq s$, $u_s = u_s$, we have the inequality

$$H_s(u_1, \ldots, u_s, t) \leq H_s(u_1, \ldots, u_s, t).$$

Suppose now that $F_s(x, t, u, \xi, \eta, \nu)$ are defined for $\varepsilon \in (0, T)$, and arbitrary $u_1, \ldots, u_s, \xi, \eta, \nu$. We formulate the generalization of the Westphal-Prodi theorem.

**Theorem 1.** Suppose that the functions $F_s(x, t, u, \xi, \eta, \nu)$ ($s = 1, \ldots, n$) are elliptic with regard to $p_{\alpha \beta}$ and satisfy the condition $(W)$ with regard to $u_1, \ldots, u_n$. Let $u_1(x, t), \ldots, u_n(x, t)$ and $v_1(x, t), \ldots, v_n(x, t)$ be continuous in $\bar{B}$ and satisfy the inequalities

$$u_s(x, t) < v_s(x, t) \quad (s = 1, 2, \ldots, n).$$

We assume that for $(s, t) \in B$, $u_s$ and $v_s$ possess continuous derivatives

$$\frac{\partial}{\partial x_i} u_s, \frac{\partial}{\partial x_i} v_s,$$

Suppose that for every $P = (x, t) \in B$ and every $s (1 \leq s \leq n)$ for which the condition $u_s(x, t) = v_s(x, t)$ holds, the derivatives

$$\left( \frac{\partial v_s}{\partial t} \right)_{p}, \left( \frac{\partial u_s}{\partial t} \right)_{p}$$

exist and the following inequalities are satisfied:

(a) $\left( \frac{\partial u_s}{\partial t} \right)_{p} \leq F_s \left( x, t, u_1(x, t), \ldots, u_n(x, t), \frac{\partial u_s}{\partial x_i} \right)_{p}$;

(b) $\left( \frac{\partial v_s}{\partial t} \right)_{p} \geq F_s \left( x, t, v_1(x, t), \ldots, v_n(x, t), \frac{\partial v_s}{\partial x_i} \right)_{p}$.

Under our assumptions the inequalities $u_s(x, t) < v_s(x, t) (s = 1, \ldots, n)$ hold for $(x, t) \in B$.

**Proof.** We prove our theorem by reducing to absurdity. Suppose that the set

$$E = \bigcup_{s=1}^{r} \{E_s(x, t), t \in E_s \}$$

is non-empty. Denote by $E_{\xi}$ the projection of $E$ on the $t$-axis and put $\xi = \inf E_{\xi}$. We have $\xi > 0$ and for $0 \leq t < \xi$

$$u_s(x, t) < v_s(x, t) \quad (s = 1, 2, \ldots, n).$$

Therefore

$$u_s(x, t) \leq v_s(x, t), \quad x \in \bar{B}, \quad t = 1, 2, \ldots, n.$$

At least one of the functions $z_s(x) = v_s(x, t) - u_s(x, t)$ has at $\theta$ a minimum equal to zero. If it was not, then according to (a) we should have $z_s(x) > 0$ for $x \in \bar{B}$, $t = 1, 2, \ldots, n$, and this contradicts the definition of $\xi$. Hence there exists $s (1 \leq s \leq n)$ and a point $P = (x, t) \in \bar{B}$ such that $z_s(x) = 0$ and

$$v_s(x, t) = u_s(x, t) \quad (t \neq s).$$

According to our assumptions, the derivatives

$$\left( \frac{\partial u_s}{\partial t} \right)_{p}, \left( \frac{\partial v_s}{\partial t} \right)_{p}$$

exist and

(a) $\left( \frac{\partial u_s}{\partial t} \right)_{p} \leq F_s \left( \xi, t, u_1(P), \ldots, u_n(P), \frac{\partial u_s}{\partial x_i} \right)_{p}$;

(b) $\left( \frac{\partial v_s}{\partial t} \right)_{p} \geq F_s \left( \xi, t, v_1(P), \ldots, v_n(P), \frac{\partial v_s}{\partial x_i} \right)_{p}$.

Since $z_s(x)$ has a minimum at $\xi$, the quadratic form

$$\sum_{i=1}^{n} \left( \frac{\partial^2 z_s}{\partial x_i} \right)_{p}^2 \xi^2$$

is non-negative for arbitrary $\xi_1, \xi_2, \ldots, \xi_n$. But $F_s$ is elliptic — therefore, in view of

$$\left( \frac{\partial u_s}{\partial t} \right)_{p} = \left( \frac{\partial u_s}{\partial x_i} \right)_{p},$$

we have

(a) $\left( \frac{\partial u_s}{\partial t} \right)_{p} \leq F_s \left( \xi, t, u_1(P), \ldots, u_n(P), \frac{\partial u_s}{\partial x_i} \right)_{p}$;

(b) $\left( \frac{\partial v_s}{\partial t} \right)_{p} \geq F_s \left( \xi, t, v_1(P), \ldots, v_n(P), \frac{\partial v_s}{\partial x_i} \right)_{p}.$

According to (a) and (b) we get, because of the condition $(W)$

(a) $\left( \frac{\partial u_s}{\partial t} \right)_{p} \leq F_s \left( \xi, t, u_1(P), \ldots, u_n(P), \frac{\partial u_s}{\partial x_i} \right)_{p}$;

(b) $\left( \frac{\partial v_s}{\partial t} \right)_{p} \geq F_s \left( \xi, t, v_1(P), \ldots, v_n(P), \frac{\partial v_s}{\partial x_i} \right)_{p}.$
By (5), (6), (7) and (8) we have

\[ \left( \frac{\partial u_2}{\partial t} \right)_F \geq \left( \frac{\partial u_1}{\partial t} \right)_F. \]

On the other hand, by (1) and (3) we have

\[ \left( \frac{\partial v_2}{\partial t} \right)_F \leq \left( \frac{\partial v_1}{\partial t} \right)_F. \]

We obtain a contradiction of (9), hence \( E \) is empty.

3. We can now formulate the second theorem concerning strong differential inequalities. We introduce the following assumption (A): for every \( \bar{x} \in \Gamma \) and every \( v (1 \leq r \leq n) \) there exists a straight line \( l_r \) such that an open segment \( (\bar{x}, \bar{x}_r) \) of \( l_r \) lies \( \in G \) and the derivatives of the form

\[ \frac{du_r}{dl_r} = \lim_{x \to \bar{x}_r} \frac{u_r(\bar{x}, t) - u_r(\bar{x}, t)}{|\bar{x} - x|}, \quad \frac{dv_r}{dl_r} = \lim_{x \to \bar{x}_r} \frac{v_r(\bar{x}, t) - v_r(\bar{x}, t)}{|\bar{x} - x|} \]

exist.

**Theorem 2.** Suppose that the functions \( P_r(x, t, u, q, p) \) are elliptic and satisfy the condition (W) with respect to \( u_1, \ldots, u_n \). We assume that the functions \( q_r(x, t, u_1, \ldots, u_n) \) satisfy the condition (W) with regard to \( u_1, \ldots, u_n \).

The functions \( u_1(x, t), \ldots, u_n(x, t); v_1(x, t), \ldots, v_n(x, t) \) are continuous in \( B \); for \( (x, t) \in \Gamma \) they possess continuous derivatives \( \partial^2 u_r/\partial x^2 \partial u_r, \partial^2 v_r/\partial x^2 \partial v_r \).

Let

\[ u_r(x, 0) = v_0(x, 0), \quad x \in \Gamma, \quad r = 1, 2, \ldots, n. \]

We assume that the assumption (A) is satisfied. For every \( (x, t) \in \Gamma \), \( r = 1, \ldots, n \), let

\[ \frac{du_r}{dl_r} \leq q_r(x, t, u_r(x, t), \ldots, u_n(x, t)), \]

\[ \frac{dv_r}{dl_r} > q_r(x, t, v_1(x, t), \ldots, v_n(x, t)) \forall. \]

We assume that for every \( P = (x, t) \in B \) and every \( s (1 \leq s \leq n) \) for which \( u_s(x, t) = v_s(x, t) \), the derivatives

\[ \left( \frac{\partial u_s}{\partial x^r} \right)_F \leq \left( \frac{\partial v_s}{\partial x^r} \right)_F \]

exist and the inequalities (a) and (b) hold.

Under our assumptions the inequalities \( u_s(x, t) < v_s(x, t) (r = 1, \ldots, n) \) hold for \( (x, t) \in B \).

Proof. We prove the theorem by reductio ad absurdum. Applying the same arguments and using the same notation as in the proof of theorem 1 we find that at least one of the functions

\[ z_s(x) = v_s(x, \xi) - u_s(x, \xi) \]

has in \( \bar{G} \) a minimum equal to zero. These minima are not reached in \( x \in \Gamma \).

Indeed if it were so, then there would exist \( \bar{x} \in \Gamma \) and \( s (1 \leq s \leq n) \) such that \( z_s(\bar{x}) \) has a minimum in \( \bar{x} \).

We have \( u_s(\bar{x}, \xi) < v_s(\bar{x}, \xi), \quad t \neq s \), \( v_s(\bar{x}, \xi) = u_s(\bar{x}, \xi) \).

Therefore

\[ \varphi_s(\bar{x}, \xi, u_1(\bar{x}, \xi), \ldots, u_n(\bar{x}, \xi)) \leq v_s(\bar{x}, \xi, v_1(\bar{x}, \xi), \ldots, v_n(\bar{x}, \xi)). \]

By the boundary inequalities (11), (12) we obtain

\[ \left( \frac{du_s}{dl_s} \right)_{\bar{x}, 0} \leq \left( \frac{dv_s}{dl_s} \right)_{\bar{x}, 0}. \]

But \( z_s(x) \) has a minimum in \( \bar{x} \), hence

\[ \left( \frac{du_s}{dl_s} \right)_{\bar{x}, 0} \leq \left( \frac{du_s}{dl_s} \right)_{\bar{x}, 0}. \]

Thus we conclude that the function \( z_s(x) \) has a minimum equal to zero in \( G \). Now, applying the same arguments as in the proof of theorem 1, by (a), (b) we come to a contradiction of the definition of \( \xi \). This completes the proof.

**Remark 1.** It may easily be shown that our theorems remain true for infinite hypercylinders of the form \( G \times (0, \infty) \).

**Remark 2.** In both theorems we do not assume that the inequalities (a), (b) hold for every \( (x, t) \in B \) and \( s = 1, \ldots, n \). If they were satisfied everywhere in \( B \), for every \( x = 1, \ldots, n \), then the assumptions of theorem 1 would be satisfied. Therefore, in this case theorem 1 and theorem 2 remain true. The situation is analogous to the situation encountered in the theory of ordinary or partial differential inequalities of the first order (see e.g. [3]).
Remarque concernant le travail de W. Pogorzelski:
"Sur le système d'équations intégrales à une infinité de fonctions inconnues"

(Vol. II, 1, 1955, pages 106-117)

Les lignes 13, 14, 15, 16, 17 à la page 107 doivent être remplacées par la phrase suivante:

"On peut faire correspondre à tout nombre positif ε un nombre positif η(ε) et un nombre naturel N(ε) tels que

\[ |F_k(x, y, y', y'', \ldots) - F_k(x, y', y'', \ldots)| < \varepsilon\]

si

\[ |y_v| < \eta, \quad |y_v'| < \eta, \quad |y_v''| < \eta \quad (v = 1, 2, 3, \ldots, N).\]

Cette définition de la continuité est équivalente à la définition de la continuité au sens de la métrique (6)."