Elle est holomorphe dans le domaine D_∞, le point $z=\infty$ y compris, continue dans $D_\infty+E$ et son module sur le continu F_k est égal à $e^{l_k/\lambda}$, car

$$|\varphi(z, D_\infty, 1, g)| = e^{l_k}, \qquad |\varphi(z, D_\infty, 1, 0)| = 1 \quad \text{si} \quad z \in F_k.$$

D'après ce qui précède, lorsque z parcourt une fois un contour $C_k \subset D_\infty$ entourant le continu F_k et n'entourant aucun des continus F_i, $i \neq k$, l'argument de $\varphi(z, D_\infty, 1, g)$ croit de $2\pi\alpha_k(g)$ et celui de $\varphi(z, D_\infty, 1, 0)$ augmente de $2\pi\alpha_k(0)$. Par suite, lorsque z parcout C_k, l'argument de $F(z)$

1° augmente de 2π si $k=1$,
2° diminue de 2π si $k=2$,
3° ne change pas si $k=3, 4, \ldots, p$.

Il s'ensuit que la fonction $w=F(z)$ est uniforme et univalente dans le domaine $D_\infty(E)$ et représente ce domaine sur une couronne circulaire $r<|w|<R$ pourvue de $p-2$ coupures situées sur des circonférences concentriques (une telle couronne est dite domaine canonique de Koebe).

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Remarks on the stability problem for parabolic equations

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The problem of the stability of solutions of parabolic equations has been investigated by Bellman [1], Prodi [4] and Narasimhan [3].

In the first part of this paper our considerations are based on the generalized Westphal-Prodi theorem given in [2]. In the second part we discuss the stability problem for systems of purely non-linear equations of parabolic type. We apply a theorem concerning the evaluation of solutions of parabolic equations given by J. Szarski in [5].

Part I. 1. Suppose G is an open and bounded region lying in the space E^m of points $(x_1, \ldots, x_m)$. Denote by B the Cartesian product of G and the interval $(0, \infty)$, $B = G\times(0,\infty)$. We denote the boundary of G by Γ. $\bar{B}$ denotes the closure of B.

Suppose the sequence of functions $u_1(x,t), \ldots, u_n(x,t)$ is a solution of the parabolic system[1])

$$\frac{\partial z_s}{\partial t} = F_s\left(x, t, z_1, \ldots, z_n, \frac{\partial z_s}{\partial x_i}, \frac{\partial^2 z_s}{\partial x_i \partial x_k}\right) \qquad (s=1,2,\ldots,n). \tag{1}$$

We say that $u=(u_1,\ldots,u_n)$ is a *stable solution* of (1) if for every $\varepsilon>0$ there exists such $\delta>0$ that for every solution $v=(v_1,\ldots,v_n)$ of (1) such that $u_i(x,t)=v_i(x,t)$ for $(x,t)\in\Gamma\times\langle 0,\infty)$ $(i=1,\ldots,n)$ and $|u_i(x,0)-v_i(x,0)|<\delta$ $(i=1,\ldots,n)$ we have the inequalities $|u_i(x,t)-v_i(x,t)|<\varepsilon$, $(x,t)\in B$ $(i=1,\ldots,n)$.

Now we investigate systems of the form

$$\frac{\partial z_s}{\partial t} = L_s[z_s] + f_s(x, t, z_1, \ldots, z_n) \qquad (s=1,2,\ldots,n), \tag{2}$$

where L_s is the elliptic differential operator of the form

$$L_s[v] = \sum_{i,k=1}^{m} a_{ik}^s(x)\frac{\partial^2 v}{\partial x_i \partial x_k},$$

* I wish to express here my thanks to J. Szarski for reading the manuscript of this paper and for his valuable remarks.

[1]) On the definition of the parabolic system see [2] and [5]. Our systems are normal parabolic systems.

i. e. the form

$$\sum_{i,k=1}^{m} a_{ik}^{s}(x)\,\xi_i\,\xi_k \geqslant 0 \quad \text{for} \quad x \in G$$

and arbitrary $\xi_1, \ldots, \xi_m$. The functions a_{ik}^{s} are defined for $x \in G$.

Suppose that $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$, $\bar{\bar{u}} = (\bar{\bar{u}}_1, \ldots, \bar{\bar{u}}_n)$ are two solutions of (2) such that $\bar{u}_i(x,t) = \bar{\bar{u}}_i(x,t)$ for $(x,t) \in \Gamma \times \langle 0, \infty)$. Let the functions $\sigma_s(x, t, u_1, \ldots, u_n) \geqslant 0$ be defined for $x \in G$, $t > 0$ and $u_i \geqslant 0$. We have the following theorem implied by theorem 1 of [2]:

THEOREM 1. *Let the functions* $\sigma_s(x, t, u_1, \ldots, u_n)$ *satisfy the* (W) *condition with respect to* $u_1, \ldots, u_n$ (see [2]). *Suppose that*

$$|f_s(x, t, \bar{v}_1, \ldots, \bar{v}_n) - f_s(x, t, \bar{\bar{v}}_1, \ldots, \bar{\bar{v}}_n)| \leqslant \sigma_s(x, t, |\bar{v}_1 - \bar{\bar{v}}_1|, \ldots, |\bar{v}_n - \bar{\bar{v}}_n|)$$

$$(s = 1, 2, \ldots, n).$$

We assume that the functions $z_s(x,t)$ *are continuous in* $\bar{B}$ *and posses continuous derivatives* $\partial^2 z_s/\partial x_i \partial x_k$ *for* $(x,t) \in B$. *Suppose that* $z_s > 0$ *for* $(x,t) \in \bar{B}$. *Let the following inequalities be satisfied:*

$$L_s[z_s] + \sigma_s(x, t, z_1, \ldots, z_n) < \frac{\partial z_s}{\partial t} \quad (s = 1, 2, \ldots, n),$$

$$|\bar{u}_s(x, 0) - \bar{\bar{u}}_s(x, 0)| < z_s(x, 0) \quad (s = 1, 2, \ldots, n).$$

Under our assumptions the following inequalities hold:

$$|\bar{u}_s(x,t) - \bar{\bar{u}}_s(x,t)| < z_s(x,t) \quad (s = 1, 2, \ldots, n), \quad (x,t) \in \bar{B}.$$

2. Suppose L_s is the Laplace operator, *i. e.* the system (2) has the form

$$\frac{\partial z_s}{\partial t} = \Delta z_s + f_s(x, t, z_1, \ldots, z_n) \quad (s = 1, 2, \ldots, n). \tag{3}$$

The stability problem of solutions of such equations has been considered by Prodi [4] in the case $n = 1$. Put $\sigma_s = \lambda(v_1 + \ldots + v_n)$, λ being a constant. We assume that $n\lambda < \mu_1$ where μ_1 is the smallest, positive characteristic value of the equation $\Delta\varphi + \mu\varphi = 0$ considered in G, of the eigenproblem with homogenous boundary conditions $\varphi(x) = 0$ for $x \in \Gamma$. Applying the arguments of Prodi one can construct a function $z(x)$ which satisfies the inequality $\Delta z + (\mu_1 - \varepsilon) z < 0$ in G and $z(x) > 0$ for $x \in \bar{G}$. ε is a suitable positive number such that $\mu_1 - n\lambda - \varepsilon > 0$. We put $a = \mu_1 - n\lambda - \varepsilon > 0$. Then the function

$$\bar{z}(x,t) = e^{-at} z(x)$$

fulfils the inequality $\Delta\bar{z} + n\lambda\bar{z} < \partial\bar{z}/\partial t$. Let us write $z_s(x,t) = \bar{z}(x,t)/n$; then for z_s the following inequalities hold:

$$\Delta z_s + \lambda(z_1 + z_2 + \ldots + z_n) < \frac{\partial z_s}{\partial t} \quad (s = 1, 2, \ldots, n).$$

Suppose now that

$$|f_s(x, t, \bar{v}_1, \ldots, \bar{v}_n) - f_s(x, t, \bar{\bar{v}}_1, \ldots, \bar{\bar{v}}_n)| \leqslant \lambda \sum_{i=1}^{n} |\bar{v}_i - \bar{\bar{v}}_i| \tag{4}$$

$$(s = 1, 2, \ldots, n).$$

If $(\bar{u}_1, \ldots, \bar{u}_n)$, $(\bar{\bar{u}}_1, \ldots, \bar{\bar{u}}_n)$ are two solutions of (3) such that

$$\bar{u}_i(x,t) = \bar{\bar{u}}_i(x,t) \quad \text{for} \quad (x,t) \in \Gamma \times \langle 0, \infty), \quad |\bar{u}_s(x,0) - \bar{\bar{u}}_s(x,0)| < \eta z_s(x,0),$$

then by theorem 1

$$|\bar{u}_s(x,t) - \bar{\bar{u}}_s(x,t)| < \eta z_s(x,t) \quad (x \in G,\ t > 0).$$

Let $\varepsilon > 0$. We define

$$\delta = \varepsilon \frac{\inf z(x)}{\sup z(x)}.$$

Therefore if $|\bar{u}_s(x,0) - \bar{\bar{u}}_s(x,0)| < \delta$, then $|\bar{u}_s(x,t) - \bar{\bar{u}}_s(x,t)| < \varepsilon$. Hence we have the following conclusion: *if* $n\lambda < \mu_1$ *and* (4) *are satisfied, then every solution of* (3) *is stable.*

3. In the previous example we have applied the method of Prodi. Now we shall show how the method of Narasimhan may be applied to the systems of the form (2). We put $\sigma_s = \lambda(v_1 + \ldots + v_n)$, λ being a positive constant. In may be assumed that $x_i \geqslant 0$ for $(x_1, \ldots, x_m) \in \bar{G}$. Let us denote

$$X_k = \max_{x \in G} x_k.$$

We write $A = \delta + e^{D(X_1 + \ldots + X_m)}$ where $0 < \delta < 1$, D being a positive constant to be specified. Let us define

$$z_s(x,t) = [A - e^{D(x_1 \ldots + x_m)}]\, e^{-at}.$$

Hence the inequality $L_s[z_s] + \lambda \sum_{i=1}^{n} z_i < \partial z_s/\partial t$ means that

$$(n\lambda + a)(A - e^{D(x_1 + \ldots + x_m)}) < D^2 e^{D(x_1 + \ldots + x_m)} \sum_{i,k=1}^{m} a_{ik}^{s}(x). \tag{5}$$

Let

$$a = \min_{s} [\inf_{x \in G} \sum_{i,k=1}^{m} a_{ik}^{s}(x)] \tag{6}$$

and $a > 0$. Write

$$\gamma = \max_{x \varepsilon G} \sum_{i=1}^{m} (X_i - x_i).$$

Suppose that for certain $a > 0$ there exists $D > 0$ such that

(7) $$e^{D\gamma} \leqslant \frac{aD^2}{n\lambda + a}.$$

On the other hand since $\delta < 1 \leqslant e^{D(x_1+\ldots+x_m)}$, we have

$$A - e^{D(x_1+\ldots+x_m)} < A - \delta = e^{D(X_1+\ldots+X_m)}.$$

Therefore

(8) $$(n\lambda + a)(A - e^{D(x_1+\ldots+x_m)}) < (n\lambda + a) e^{D(X_1+\ldots+X_m)}.$$

But (7) implies

(9) $$(n\lambda + a) e^{D(X_1+\ldots+X_m)} \leqslant aD^2 e^{D(x_1+\ldots+x_m)}.$$

According to (6) we have

(10) $$aD^2 e^{D(x_1+\ldots+x_m)} \leqslant D^2 e^{D(X_1+\ldots+X_m)} \sum_{i,k=1}^{m} a_{ik}^{s}(x).$$

The inequalities (8), (9) and (10) imply the inequality (5).

We see now that the construction of z_s in the prescribed form depends on the existence of positive solutions on D of the inequality (7), where a is a positve constant. The discussion of this inequality may be conducted by elementary methods.

In the same way as it has been done in section 2, if (7) is satisfied, the suitable stability condition for system (2) may be formulated, the form of z_s being used.

Part II. 1. Let us consider the system (1) and the system of ordinary differential equations

(11) $$y_i' = \sigma_i(t, y, \ldots, y_n),$$

σ_i being continuous for $t \geqslant 0, y_i \geqslant 0$ and satisfying the (W)-condition. Suppose that $\sigma_i \geqslant 0$ and denote by $\omega_i(t, \varepsilon_1, \ldots, \varepsilon_n)$ the right maximal integral of (11) valid in $\langle 0, \infty)$ such that $\omega_i(0, \varepsilon_1, \ldots, \varepsilon_n) = \varepsilon_i \geqslant 0$. We formulate a theorem which is an immediate consequence of theorem 2.1 [5].

THEOREM 2. *Suppose that $(\bar{u}_1, \ldots, \bar{u}_n)$, $(\bar{\bar{u}}_1, \ldots, \bar{\bar{u}}_n)$ are two solutions of* (1). *Let*

(12) $$|F_s(x, t, \bar{v}_1, \ldots, \bar{v}_n, q_i, p_{ik}) - F_s(x, t, v_1, \ldots, v_n, q_i, p_{ik})| \leqslant \sigma_s(t, |\bar{v}_1 - v_1|, \ldots, |\bar{v}_n - v_n|).$$

We assume that

$$|\bar{u}_i(x, 0) - u_i(x, 0)| \leqslant \varepsilon_i \quad (i = 1, 2, \ldots, n), \quad x \epsilon \bar{G},$$

$$|\bar{u}_i(x, t) - u_i(x, t)| \leqslant \varepsilon_i \quad (i = 1, 2, \ldots, n), \quad (x, t) \epsilon \Gamma \times (0, \infty).$$

As our assumptions are satisfied, the inequalities

$$|\bar{u}_i(x, t) - u_i(x, t)| \leqslant \omega_i(t, \varepsilon_1, \ldots, \varepsilon_n) \quad (i = 1, \ldots, n)$$

hold for $(x, t) \epsilon \bar{B}$.

Suppose now that the right maximal integral ω_i of (11) such that $\omega_i(0) = 0$ is identically equal to zero, *i. e.* $\omega_i(t) \equiv 0$. In that case we say that $\omega_i(t) \equiv 0$ is *stable in the sense of Liapunoff — shortly (L)-stable —* if for every $\varepsilon > 0$ there is such a $\delta > 0$ that for every solution $\omega_i(t)$ $(i = 1, \ldots, n)$ of (11) such that $|\omega_i(0)| < \delta$ we have $|\omega_i(t)| < \varepsilon$ for $t > 0$.

Let us introduce the following definition: the solution $(\bar{u}_1, \ldots, \bar{u}_n)$ of (1) is *stable in the wider sense* if for every $\varepsilon > 0$ there exists such a $\delta > 0$ that for every solution $(u_1, \ldots, u_n)$ of (1) such that

$$|\bar{u}_i(x, 0) - u_i(x, 0)| < \delta \quad (i = 1, 2, \ldots, n), \quad x \epsilon \bar{G},$$

$$|\bar{u}_i(x, t) - u_i(x, t)| < \delta \quad (i = 1, 2, \ldots, n), \quad (x, t) \epsilon \Gamma \times (0, \infty)$$

we have $|\bar{u}_i(x, t) - u_i(x, t)| < \varepsilon$ $(i = 1, 2, \ldots, n)$, $(x, t) \epsilon \bar{B}$.

We have the following

THEOREM 3. *Suppose that $\omega_i(t) \equiv 0$ is an (L)-stable solution of* (11). *We assume that* (12) *holds. Then every solution of* (1) *is stable in the wider sense.*

2. Example. Let (11) be a linear system of the form

(13) $$y_i' = \sum_{k=1}^{n} a_{ik}(t) y_k, \quad a_{ik}(t) \geqslant 0,$$

with continuous coefficients $a_{ik}(t)$ for $t \geqslant 0$. Write

$$\varphi(t) = \max_{i,k=1,\ldots,n} |a_{ik}(t)|$$

and suppose that

$$\int_0^\infty \varphi(t)\, dt < +\infty.$$

It is well known that $y_i \equiv 0$ is an (L)-stable solution of (13). Hence if

$$|F_s(x, t, \bar{v}_1, \ldots, \bar{v}_n, q_i, p_{ik}) - F_s(x, t, \bar{\bar{v}}_1, \ldots, \bar{\bar{v}}_n, q_i, p_{ik})| \leqslant \sum_{n=1}^{n} a_{sk}(t) |\bar{v}_k - \bar{\bar{v}}_k|$$

then every solution of (1) is stable in the wider sense.

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Differential inequalities of parabolic type

by W. Mlak (Kraków)*

In connection with the stability problem of solutions of parabolic equations some theorems concerning certain differential inequalities have been discussed (see [1] and [2]).

In this paper we discuss some generalizations of the theorems about differential inequalities of the form

$$\frac{\partial z_s}{\partial t} < F_s\left(x, t, z_1, \ldots, z_n, \frac{\partial z_s}{\partial x_i}, \frac{\partial^2 z_s}{\partial x_i \partial x_k}\right) \quad (s = 1, 2, \ldots, n).$$

1. Notation and definitions. We investigate a hypercylinder of the form $G \times (0, T\rangle$ $(T > 0)$ where G is an open bounded region lying in the space E^m of points $(x_1, \ldots, x_m)$. We write $B = G \times (0, T\rangle$; Γ being the boundary of G we write $C = \Gamma \times (0, T\rangle$; $\bar{B}$ denotes the closure of B, $\bar{C} = \Gamma \times \langle 0, T\rangle$.

Suppose that $F(x_1, \ldots, x_m, t, u_1, \ldots, u_n, q_1, \ldots, q_m, p_{11}, \ldots, p_{mm})$, written shortly as $F(x, t, u, q_i, p_{ik})$, satisfies the following condition: For every system of numbers $\bar{r}_{ik}$ $(i, k = 1, \ldots, m)$, r_{ik} $(i, k = 1, \ldots, m)$ such that the quadratic form

$$\sum_{i,k=1}^{m} (\bar{r}_{ik} - r_{ik}) \xi_i \xi_k$$

is non-negative for arbitrary $\xi_1, \ldots, \xi_m$, the following inequality holds:

$$F(x, t, u, q_i, \bar{r}_{ik}) \geqslant F(x, t, u, q_i, r_{ik}).$$

$F(x, t, u, q_i, p_{ik})$ is then called the *elliptic function with regard to* p_{ik}. A system of equations

$$\frac{\partial u_s}{\partial t} = F_s\left(x, t, u_1, \ldots, u_n, \frac{\partial u_s}{\partial x_i}, \frac{\partial^2 u_s}{\partial x_i \partial x_k}\right) \quad (s = 1, 2, \ldots, n)$$

is called *parabolic* if every F_s is elliptic.

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