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A method of determining the existence domain for solutions of partial differential equations of the first order

by A. Pliś (Kraków)

1. In the present paper we shall deal with the Cauchy problem for the differential equation

$$\frac{\partial z}{\partial x} = f\left(x, y_1, \dots, y_n, z, \frac{\partial z}{\partial y_1}, \dots, \frac{\partial z}{\partial y_n}\right). \tag{1}$$

The domain in which a solution of class C^2 [1]) exists for equation (1) with the initial condition

$$z(a, y_1, \dots, y_n) = \omega(y_1, \dots, y_n), \tag{2}$$

will be determined "along a given characteristic".

For this purpose we shall introduce a system of supplementary characteristic equations for equation (1) (system (4)) [2]). It is satisfied by the derivatives $z_{y_i y_j}$ $(i, j = 1, 2, \dots, n)$ of any solution z of equation (1) along the characteristic (Theorem 1).

Theorem 2 together with its application to the estimation of the existence domain for solutions of the equation

$$\frac{\partial z}{\partial x} = F\left(x, y_1, \dots, y_n, \frac{\partial z}{\partial y_1}, \dots, \frac{\partial z}{\partial y_n}\right)$$

with initial condition (2) was stated in [3].

2. Let us denote by $C^{k,m}$ $(0 \leqslant k \leqslant m)$ the class of functions which are of class C^k and possess continuous derivatives to the m-th order with respect to all variables with the possible exception of the variable x.

Assumption $A(a_1, a_2)$. Suppose that the function $f(x, P)$, where $P = (p_1, \dots, p_{2n+1}) = (Y, z, Q) = (y_1, \dots, y_n, z, q_1, \dots, q_n)$, defined in an

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[1]) A function continuous together with its derivatives to the m-th order is termed a *function of class* C^m.

[2]) Analogous systems have been used for the quasilinear partial differential equation of the first order in papers [2], [4].

open set G is of class $C^{1,2}$ and the system of classical characteristic equations

$$(3)\qquad \begin{cases} y_i' = -f_{q_i}(x, Y, z, Q) \quad (i = 1, 2, \ldots, n), \\ z' = f(x, Y, z, Q) - \sum_{k=1}^{n} q_k f_{q_k}(x, Y, z, Q), \\ q_i' = f_{y_i}(x, Y, z, Q) + q_i f_z(x, Y, z, Q) \quad (i = 1, 2, \ldots, n) \end{cases}$$

has a solution $Y = \Phi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$, $z = \zeta(x)$, $Q = \Gamma(x) = (\gamma_1(x), \ldots, \gamma_n(x))$ defined in the interval $a_1 < x < a_2$ $(a_1 < a < a_2)$[3].

We shall also use the following system of supplementary characteristic equations:

$$(4)\qquad t_{ij}' = \sum_{k=1}^{n}\sum_{m=1}^{n} f_{q_k q_m}(x, P) t_{ki} t_{mj} + \sum_{k=1}^{n} \big(f_{q_k y_i}(x, P) + f_{q_k z}(x, P) q_i\big) t_{kj} + $$
$$+ \sum_{k=1}^{n} \big(f_{q_k y_j}(x, P) + f_{q_k z}(x, P) q_j\big) t_{ki} + f_z(x, P) t_{ij} +$$
$$+ f_{y_i y_j}(x, P) + f_{y_i z}(x, P) q_j + f_{y_j z}(x, P) q_i + f_{zz}(x, P) q_i q_j \qquad (i, j = 1, 2, \ldots, n),$$

where $P = (\Phi(x), \zeta(x), \Gamma(x))$.

We shall denote by $Y(x, V) = (y_1(x, V), \ldots, y_n(x, V))$, $s(x, V)$, $Q(x, V) = (q_1(x, V), \ldots, q_n(x, V))$, where $V = (v_1, \ldots, v_n)$, the solution of the system of ordinary differential equations (3) satisfying the initial conditions

$$(5)\qquad Y(a, V) = V, \quad s(a, V) = \omega(V), \quad q_i(a, V) = \omega_{v_i}(V) \qquad (i = 1, 2, \ldots, n)$$

for V from a certain neighbourhood of the point $H = \Phi(a)$, where $\omega(H) = \zeta(a)$, $\omega_{v_i}(H) = \gamma_i(a)$ $(i = 1, 2, \ldots, n)$. It is evident that

$$(5')\qquad Y(x, H) = \Phi(x), \quad s(x, H) = \zeta(x), \quad Q(x, H) = \Gamma(x) \quad \text{for } a_1 < x < a_2.$$

Moreover we shall use the following notation:

$$u_i^j(x) = \frac{\partial y_i(x, V)}{\partial v_j} \quad \text{for} \quad V = H.$$

3. THEOREM 1. *Let us make assumption* $A(a_1, a_2)$. *Suppose that a solution* $z(x, Y)$ *of equation* (1) *with condition* (2) *is of class* C^2 *in a neighbourhood of the arc* $Y = \Phi(x)$, $a_1 < x < a_2$[4].

[3]) It follows that the points $x = \xi$, $Y = \Phi(\xi)$, $z = \zeta(\xi)$, $Q = \Gamma(\xi)$ belong to G for $a_1 < \xi < a_2$.

[4]) We give the term *neighbourhood of a set* S to an open set containing the set S.

Then

(6) *the functions* $t_{ij} = z_{y_i y_j}(x, \Phi(x))$ $(i, j = 1, 2, \ldots, n)$ *fulfil system* (4) *in the interval* $J: a_1 < x < a_2$,

(7) *the functions* $u_i^j(x)$ *satisfy the inequality* $\det(u_i^j(x)) \neq 0$ *for* $x \in J$.

Proof of theorem 1. At first we shall prove relation (7). From (3) and from the identities

$$(8)\qquad s(x, V) = z(x, Y(x, V)), \quad q_i(x, V) = z_{y_i}(x, Y(x, V)) \qquad (i = 1, 2, \ldots, n)$$

it follows that the functions $y_i(x, V)$ $(i = 1, 2, \ldots, n)$ satisfy the following differential equations:

$$(9)\qquad y_i'(x, V) = -f_{q_i}\Big(x, Y(x, V), z\big(x, Y(x, V)\big), z_{y_1}\big(x, Y(x, V)\big), \ldots, z_{y_n}\big(x, Y(x, V)\big)\Big)$$

in a certain neighbourhood of the segment $a_1 < x < a_2$, $V = H$. Differentiating the relations (9) on both sides with respect to v_j $(1 \leqslant j \leqslant n)$ we obtain the following equations fulfilled indentically by the functions $u_i = u_i^j(x)$ $(i = 1, 2, \ldots, n)$:

$$(10)\qquad u_i' = \sum_{k=1}^{n} \Big[-f_{q_i y_k}(Z) - f_{q_i z}(Z) z_{y_k}(X) - \sum_{m=1}^{n} f_{q_i q_m}(Z) z_{y_m y_k}(X)\Big] u_k \qquad (i = 1, 2, \ldots, n),$$

where $X = (x, Y(x, H))$ and $Z = (X, z(X), z_{y_1}(X), \ldots, z_{y_n}(X))$.

By virtue of (5) we have $\det(u_i^j(a)) = 1$; then inequality (7) follows from (10).

Now we shall prove property (6). We begin with a particular case: we suppose that the solution $z(x, y)$ is of class C^3. Under this stronger assumption we have the relation

$$(11)\qquad d\big(z_{y_i y_j}(x, \Phi(x))\big)/dx = z_{y_i y_j x}(x, \Phi(x)) + \sum_{k=1}^{n} \varphi_k'(x) z_{y_i y_j y_k}(x, \Phi(x)).$$

By virtue of the identity

$$z_x = f(x, y_1, \ldots, y_n, z, z_{y_1}, \ldots, z_{y_n})$$

we have

$$(12)\qquad z_{y_i y_j x}(x, Y) = f_{y_i y_j}(U) + f_{y_i z}(U) z_{y_j} + f_{y_j z}(U) z_{y_i} + f_{zz}(U) z_{y_i} z_{y_j} +$$
$$+ \sum_{k=1}^{n} \big(f_{q_k y_i}(U) + f_{q_k z}(U) z_{y_i}\big) z_{y_k y_j} + f_z(U) z_{y_i y_j} +$$
$$+ \sum_{k=1}^{n} \big(f_{q_k y_j}(U) + f_{q_k z}(U) z_{y_j}\big) z_{y_k y_i} +$$
$$+ \sum_{k=1}^{n}\sum_{m=1}^{n} f_{q_k q_m}(U) z_{y_k y_i} z_{y_m y_j} + \sum_{k=1}^{n} f_{q_k}(U) z_{y_i y_j y_k},$$

where $U = (x, y_1, \ldots, y_n, z, z_{y_1}, \ldots, z_{y_n})$. From (11), (12), (8), (5′) and from identities $\varphi_k'(x) = -f_{q_k}(x, \Phi(x), \zeta(x), \Gamma(x))$ it follows that relation (6) is satisfied in J.

Now we shall consider the general case. Let $[b_1, b_2]$ be arbitrary closed subinterval of the interval (a_1, a_2) containing the point $x = a$ $(a_1 < b_1 \leqslant a \leqslant b_2 < a_2)$. We shall approximate the function $f(x, P)$ and the function $\omega(Y) = z(a, Y)$ defined in a neighbourhood N of the point $Y = H$ by functions of class C^3. It is known that for any positive number r there exist functions $f^r(x, P)$, $\omega^r(Y)$ of class C^3 possessing the following properties:

$$(13)\quad |f-f^r| < r, \quad |f_{p_i}-f^r_{p_i}| < r, \quad |f_{p_i p_j}-f^r_{p_i p_j}| < r \quad (i, j = 1, 2, \ldots, 2n+1)$$

in G, and

$$(14)\quad |\omega-\omega^r| < r, \quad |\omega_{y_i}-\omega^r_{y_i}| < r, \quad |\omega_{y_i y_j}-\omega^r_{y_i y_j}| < r \quad (i, j = 1, 2, \ldots, n)$$

in N.

Let us denote by $Y^r(x, V)$, $s^r(x, V)$, $Q^r(x, V)$ the solution of system (3^r) with initial condition (5^r) where (3^r), (5^r) are obtained from (3), (5) by substituting $f = f^r$ and $\omega = \omega^r$. The righthand members of system (3) are of class $C^{0,1}$ and the initial values (5) are of class C^1. Hence the functions $Y^r(x, V)$ together with their derivatives of the first order are uniformly convergent for $r \to 0$ in a neighbourhood of the segment $V = H$, $b_1 \leqslant x \leqslant b_2$. The limit function $Y(x, V)$ fulfils inequality (7) for $b_1 \leqslant x \leqslant b_2$. Then it is easy to observe that for sufficiently small r the transformations $Y = Y^r(x, V)$ possess inverse transformations $V = V^r(x, Y)$ defined in a neighbourhood M of the arc $b_1 \leqslant x \leqslant b_2$, $Y = \Phi(x)$. Each function $z^r(x, Y) = s^r(x, V^r(x, Y))$ is of class C^3 and satisfies equation (1^r) with condition (2^r), where (1^r), (2^r) are obtained from (1), (2) by substituting $f = f^r$, $\omega = \omega^r$. Hence the functions $t^r_{ij} = z^r_{y_i y_j}(x, Y^r(x, H))$ satisfy system (4^r) obtained from (4) by substituting $f = f^r$. It is evident that the functions t^r_{ij} uniformly tend to the functions $t_{ij} = z_{y_i y_j}(x, \Phi(x))$ in the interval $b_1 \leqslant x \leqslant b_2$. Therefore, the functions $t_{ij} = z_{y_i y_j}(x, \Phi(x))$ satisfy system (4) in the interval $b_1 < x < b_2$. Thus the proof of theorem 1 is completed.

4. THEOREM 2. *Let us make assumption* $A(d_1, d_2)$. *Suppose that a solution* $t_{ij} = \tau_{ij}(x)$ $(i, j = 1, 2, \ldots, n)$ *of system* (4) *is defined in the interval* (d_1, d_2). *Let a function* $\omega(Y)$ *be of class* C^2 *in a neighbourhood of* H *and satisfy the conditions*

$$(C)\quad \begin{aligned} \omega(H) &= \zeta(a), \\ \omega_{y_i}(H) &= \gamma_i(a) \quad (i = 1, 2, \ldots, n), \\ \omega_{y_i y_j}(H) &= \tau_{ij}(a) \quad (i, j = 1, 2, \ldots, n). \end{aligned}$$

Then there exists a function $z(x, y)$ *of class* C^2 *in a certain neighbourhood of the arc* $Y = \Phi(x)$, $d_1 < x < d_2$, *satisfying equation* (1) *with condition* (2).

Moreover, the following identities are satisfied:

$$(15)\quad z_{y_i y_j}(x, \Phi(x)) = \tau_{ij}(x) \quad (i, j = 1, 2, \ldots, n)$$

for $d_1 < x < d_2$, *and for* $1 \leqslant j \leqslant n$ *the functions* $u_i = u_i^j(x)$ *satisfy the following system of ordinary differential equations:*

$$(16)\quad u_i' = \sum_{k=1}^{n}\Big(-f_{q_i y_k}(W) - f_{q_i z}(W)\gamma_k(x) - \sum_{m=1}^{n} f_{q_i q_m}(W)\tau_{mk}(x)\Big)u_k$$

$(i = 1, 2, \ldots, n)$, *where* $W = (x, \Phi(x), \zeta(x), \Gamma(x))$, *for* $d_1 < x < d_2$; *besides property* (7) *is fulfilled for* $d_1 < x < d_2$.

Proof. Property (7) is satisfied in a neighbourhood of the point $x = a$. Let (c_1, c_2) $(d_1 \leqslant c_1 < a < c_2 \leqslant d_2)$ be the largest subinterval of the interval (d_1, d_2) containing the point $x = a$ at which (7) is satisfied. It may easily be verified that the transformation $Y = Y(x, V)$ has an inverse transformation $V = V(x, Y)$ defined in a neighbourhood K of the arc $Y = \Phi(x)$ consisting of the sets

$$S_\nu:\quad c_1 + \frac{1}{\nu} < x < c_2 - \frac{1}{\nu}, \quad |Y - \Phi(x)| < \varepsilon_\nu \quad (\nu = N, N+1, \ldots),$$

where N is a certain natural number and ε_ν are sufficiently small positive numbers. The function $z(x, Y) = s(x, V(x, Y))$ is of class C^2 in K and satisfies equation (1) with condition (2). In virtue of theorem 1 applied to interval $J = (c_1, c_2)$ the functions $z_{y_i y_j}(x, \Phi(x))$ $(i, j = 1, 2, \ldots, n)$ satisfy system (4) in the interval (c_1, c_2). Hence from the uniqueness of solutions of system (4) it follows that identities (15) are fulfilled in (c_1, c_2). Relation (16) results from (10), (8), (5′) and (15) for (c_1, c_2).

Now we shall prove that the intervals (d_1, d_2), (c_1, c_2) are identical. Suppose the contrary. Then $c_1 > d_1$ or $c_2 < d_2$. We shall consider only the case of $c_2 < d_2$, because the other case is analogous. System (16) is fulfilled in the interval (a, c_2) and its coefficients are bounded in (a, c_2). Hence (see [1], p. 235, Satz 4)

$$\inf_{a < x < c_2}\big(\det(u_i^j(x))\big) > 0.$$

Then for a certain positive number l property (7) is satisfied in the interval (c_1, c_2+l). Hence the interval (c_1, c_2) is not the largest subinterval of (d_1, d_2) in which property (7) is fulfilled. Thus our supposition has led to a contradiction. This completes the proof.

5. Theorems 1 and 2 imply the following one.

COROLLARY. *Let us make assumption* $A(a_1, a_2)$. *The existence of a system of functions* $t_{ij}(x)$ $(i, j = 1, 2, \ldots, n)$ *defined in the interval* (a_1, a_2) *and satisfying the system* (4) *is necessary and sufficient for the existence of a function* $z(x, Y)$ *of class* C^2 *defined in a neighbourhood of the arc* $Y = \Phi(x)$, $a_1 < x < a_2$, *and satisfying equation* (1) *with condition* (2), *where* $\omega(Y)$ *is a given function of class* C^2 *in a neighbourhood of* $Y = H$ *and fulfils the relations* (C).

Remark 1. A proof of relation (16) independent of theorem 1 is included in [5].

Remark 2. Theorem 1 is formulated with the assumptions $f \in C^{1,2}$, $z \in C^2$. It holds true, however, for $f \in C^{0,2}$, $z \in C^{1,2}$.

Theorem 2 holds true for $f \in C^{l,m}$ $(2 \leqslant m \leqslant \infty,\ 0 \leqslant l \leqslant m)$, $\omega \in C^m$, $z \in C^{k,m}$, where $k = \min(l+1, m)$. It is also true for analytic f, ω, z.

Remark 3. Theorem similar to theorem 2 is also true for the equation $F(x_1, \ldots, x_n, z, z_{y_1}, \ldots, z_{y_n}) = 0$, where the function $F(x_1, \ldots, x_n, z, q_1, \ldots, q_n)$ is of class C^2 and satisfies the inequality

$$\sum_{i=1}^{n} (F_{q_i})^2 > 0,$$

and for the initial condition given on a certain hypersurface of class C^2, which is not tangent to the characteristics. The precise formulation of this theorem is not difficult. The same applies to theorem 1.

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Sur certaines propriétés des intégrales de l'équation $y' = f(x, y)$, dont le second membre est doublement périodique

par C. OLECH (Kraków)

Le but de la présente note est de démontrer quelques propriétés des intégrales de l'équation différentielle

$$(1) \qquad y' = g(x, y),$$

où $g(x, y)$ est une fonction définie et continue sur le plan tout entier et doublement périodique, c'est-à-dire il existe deux nombres positifs u, w tels que l'identité

$$g(x, y) = g(x+pu, y+qw)$$

ait lieu pour tout couple de nombres entiers p, q. Dans la suite je supposerai que $u = w = 1$.

H. Poincaré [1] a démontré le théorème suivant:

THÉORÈME P. *Si par chaque point du plan il ne passe qu'une seule intégrale* $\varphi(x)$ *de l'équation* (1), *les limites*

$$\lim_{x\to+\infty} [\varphi(x)/x], \qquad \lim_{x\to-\infty} [\varphi(x)/x]$$

existent, sont finies et égales.

En relation avec ce théorème M. T. Ważewski a considéré à son séminaire la fonction $f(x)$, définie et continue pour $x \in (-\infty, +\infty)$, jouissant de la

Propriété A. *Pour tout couple de nombres entiers* p, q *la fonction*

$$f_{p,q}(x) \overset{\text{df}}{=} f(x+p)+q$$

satisfait, pour tout x, *à l'une des relations suivantes*

$$f_{p,q}(x) > f(x) \quad \text{ou} \quad f_{p,q}(x) < f(x)$$

ou bien $f_{p,q}(x) = f(x)$.

M. T. Ważewski a aussi formulé le

THÉORÈME W. *Si la fonction* $f(x)$, *définie et continue pour* $x \in (-\infty, +\infty)$, *a la propriété* A, *les limites*

$$\lim_{x\to-\infty} [f(x)/x], \qquad \lim_{x\to+\infty} [f(x)/x]$$

existent, sont finies et égales.