On certain functional relations and a generalization of the $M_{k,m}$ function

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Introduction. Erdelyi has shown a method of finding recurrence relations and expansions of functions which are defined as Hankel transformations of any given function. He has used the method to obtain several new properties of the $M_{k,m}$ function and other known functions. He has also observed that a general kernel $K(s,t)$ would have similar properties. We find that the method yields several interesting relations when we consider the generalized Hankel transform defined by

$$A(t) = \int_0^\infty (st)^{(\lambda+\nu)/2} J^\lambda_{\nu+\lambda}(st^\lambda) B(s)\, ds$$

where $J^\lambda_\nu(x)$ is the Bessel-Maitland-Wright function defined by[1])

$$J^\lambda_\nu(x) = \sum_{r=0}^\infty \frac{(-1)^r x^r}{r!\,\Gamma(\nu+\lambda r+1)}.$$

This leads to a function which can be regarded as a generalization of Whittaker's $M_{k,m}$ function and has analogous properties.

Let us consider a function $f^\lambda_{\mu,\nu}(t)$ defined by

$$f^\lambda_{\mu,\nu}(t) = \int_a^b (st)^{\nu/2} J^\lambda_\nu(st^\lambda) s^\mu F(s)\, ds$$

where $F(s)$ is a suitable arbitrarily given function and λ, μ, ν belong to the range of values for which the integral converges. For $a=0$, $b=\infty$, we get the generalized Hankel transform.

1. Recurrence relations. We know that

$$\nu J^\lambda_\nu(z) = \lambda z J^\lambda_{\nu+\lambda}(z) + J^\lambda_{\nu-1}(z). \tag{1.1}$$

[1]) It is interesting to observe that for $\lambda=1$, $J^1_\nu(x) = x^{-\nu/2} J_\nu(2\sqrt{x})$ where $J_\nu(x)$ is the Bessel function.

Let us put st^λ for z in (1.1), multiply both sides by $(st)^{\nu/2} s^\mu F(s)$ and integrate from a to b. We then easily have

$$\nu f^\lambda_{\mu,\nu}(t) = \lambda t^{\lambda/2} f^\lambda_{\mu-\lambda/2+1,\nu+\lambda}(t) + t^{1/2} f^\lambda_{\mu+1/2,\nu-1}(t). \tag{1.2}$$

Similarly let us put st^λ for z in the relation

$$dJ^\lambda_\nu(z)/dz = -J^\lambda_{\nu+\lambda}(z).$$

Multiplying both sides by $\lambda (st)^{(\nu+\lambda)/2} F(s) s^{\mu-\lambda/2+1}$ and integrating with respect to s from a to b, we obtain

$$t^{(\nu-\lambda)/2+1} d\{t^{-\nu/2} f^\lambda_{\mu,\nu}(t)\}/dt = -\lambda f^\lambda_{\mu-\lambda/2+1,\nu+\lambda}(t). \tag{1.3}$$

Following the same procedure, we obtain from

$$\nu J^\lambda_\nu(z) = J^\lambda_{\nu-1}(z) - \lambda z\, dJ^\lambda_\nu(z)/dz$$

the relation

$$\nu f^\lambda_{\mu,\nu}(t) = t^{1/2} f^\lambda_{\mu+1/2,\nu-1}(t) - t^{\nu/2+1} d\{t^{-\nu/2} f^\lambda_{\mu,\nu}(t)\}/dt. \tag{1.4}$$

This can also be obtained at once from (1.2) and (1.3). Also it is easy to see that

$$d\{z^{\nu/\lambda} J^\lambda_\nu(z)\}/dz = z^{\nu/\lambda-1} \lambda^{-1} J^\lambda_{\nu-1}(z);$$

putting $z = st^\lambda$ and proceeding as before, we have

$$d\{t^{\nu/2} f^\lambda_{\mu,\nu}(t)\}/dt = t^{(\nu-1)/2} f^\lambda_{\mu+1/2,\nu-1}(t);$$

repeating the operation m times, we have

$$d^m\{t^{\nu/2} f^\lambda_{\mu,\nu}(t)\}/dt^m = t^{(\nu-m)/2} f^\lambda_{\mu+m/2,\nu-m}(t).$$

2. Certain integral representations. It can easily be verified that

$$\int_0^{\pi/2} J^\lambda_\nu(st^\lambda \sin^{2\lambda}\theta) \sin^{2\nu+1}\theta \cos^{2m-1}\theta\, d\theta = \tfrac{1}{2}\Gamma(m) J^\lambda_{\nu+m}(st^\lambda)$$

$$(\operatorname{re} m > 0, \quad \operatorname{re}\nu > -1).$$

This may be considered to be a generalization of Sonine's integral. Hence multiplying both sides by $s^\mu F(s)$ and integrating with respect to s from a to b, we have, on writing $\mu+\nu/2$ for μ,

$$\tfrac{1}{2}\Gamma(m) t^{-m/2} f^\lambda_{\mu-m/2,\nu+m}(t) = \int_0^{\pi/2} f^\lambda_{\mu,\nu}(t \sin^2\theta) \sin^{\nu+1}\theta \cos^{2m-1}\theta\, d\theta.$$

Making the substitution $t\sin^2\theta = x$, we have

$$\Gamma(m) t^{(m+\nu)/2} f^\lambda_{\mu-m/2,\nu+m}(t) = \int_0^t f^\lambda_{\mu,\nu}(x) x^{\nu/2} (t-x)^{m-1}\, dx,$$

$$(\operatorname{re} m > 0, \quad \operatorname{re}\nu > -1).$$

Again we have for $0<\lambda\leqslant 1$ (when $\lambda=0$, $J_\nu^\lambda(t)=[\Gamma(\nu+1)]^{-1}\exp(-t)$)

$$\int_0^\infty s^m J_\nu^\lambda(st^\lambda)\,ds = \frac{\Gamma(m+1)}{\Gamma(\nu-\lambda-\lambda m+1)}\, t^{-\lambda(1+m)},$$

when $0<\lambda<1$, $\operatorname{re} m>-1$ and when $\lambda=1$, $\nu-2m>1/2$ and $\operatorname{re} m>-1$. Instead of t^λ we write t, multiply by $(1/\lambda)t^{\mu+\nu/2}F(t)$ and integrate with respect to t from 0 to ∞. Then we have

$$\int_0^\infty s^{m\lambda+\lambda-\nu/2-1} f_{\mu,\nu}^\lambda(s)\,ds = \frac{\Gamma(m+1)}{\lambda\Gamma(\nu-\lambda-\lambda m+1)}\int_0^\infty t^{\mu-m+\nu/2-1}F(t)\,dt$$

since

$$\int_0^\infty J_\nu^\lambda(s^\lambda t)(st)^{\nu/2}t^\mu F(t)\,dt = f_{\mu,\nu}^\lambda(s),$$

provided the integral exists and a change in the order of integrations is permissible. Hence we have

$$\int_0^\infty s^{\lambda m+\lambda-\nu/2-1}f^\lambda_{\mu,\nu}(s)\,ds = \frac{\Gamma(m+1)}{\lambda\Gamma(\nu-\lambda-\lambda m+1)}\, G_{\mu+\nu/2-m-1}(0),$$

where we write

$$G_\mu(p) = \int_0^\infty \exp(-pt)t^\mu F(t)\,dt \qquad (p\geqslant 0)$$

and it is assumed that the integral is convergent. We have

(2.1) $$\int_0^\infty \exp(-pt)t^{\nu/2}f^\lambda_{\mu,\nu}(t)\,dt = \int_a^b s^{\mu+\nu/2}\exp(-sp^\lambda)p^{-(\nu+1)}F(s)\,ds$$

and, putting $a=0$, $b=\infty$, we have

$$\mathcal{L}_p[t^{\nu/2}f^\lambda_{\mu,\nu}(t)] = \frac{1}{p^\nu}\int_0^\infty \exp(-sp^{-\lambda})\,s^{\mu+\nu/2}F(s)\,ds = \frac{1}{p^{\nu-\lambda}}\mathcal{L}_{p^{-\lambda}}[s^{\mu+\nu/2}F(s)]$$

where

$$\mathcal{L}_p[F(t)] = p\int_0^\infty \exp(-pt)F(t)\,dt.$$

In other words, taking $\mu=0$ in (2.1) and putting

$$\mathfrak{h}_t^{\nu,\lambda}[F(s)] = \int_0^\infty J_\nu^\lambda(st^\lambda)(st)^{\nu/2}F(s)\,ds,$$

we obtain

(2.2) $$\mathcal{L}_p\left[t^{\nu/2}\mathfrak{h}_t^{\nu,\lambda}[F(s)]\right] = \frac{1}{p^{\nu-\lambda}}\mathcal{L}_{p^{-\lambda}}[t^{\nu/2}F(t)].$$

Similarly for another function $H(x)$, we have

$$\mathcal{L}_p\left[t^{\mu/2}\mathfrak{h}_t^{\mu,\lambda}[H(s)]\right] = \frac{1}{p^{\mu-\lambda}}\mathcal{L}_{p^{-\lambda}}[t^{\mu/2}H(t)].$$

Hence by the product theorem

$$\mathcal{L}_p\left[t^{\nu/2}\mathfrak{h}_t^{\nu,\lambda}[F(s)]\circ t^{\mu/2}\mathfrak{h}_t^{\mu,\lambda}[H(s)]\right] = \frac{1}{p^{\mu+\nu-2\lambda}}\mathcal{L}_{p^{-\lambda}}[t^{\nu/2}F(t)\circ t^{\mu/2}H(t)].$$

Interpreting this by (2.2), we have by Lerch's theorem

$$y^{(\mu+\nu+1)/2}\mathfrak{h}_y^{\mu+\nu+1,\lambda}\left[s^{-(\mu+\nu+1)/2}\{s^{\nu/2}F(s)\circ s^{\mu/2}H(s)\}\right] = y^{\nu/2}\mathfrak{h}_y^{\nu,\lambda}[F(s)]\circ y^{\mu/2}\mathfrak{h}_y^{\mu,\lambda}[H(s)]$$

where $A(x)\circ B(x)=\int_0^x A(\xi)B(x-\xi)\,d\xi$.

3. Infinite series expansions. We easily see that

$$\frac{1}{\Gamma(\nu+1)} = \sum_{r=0}^\infty \frac{1}{r!}J^\lambda_{\nu+\lambda r}(s)\cdot s^r.$$

Let us put st^λ for s, multiply both sides by $t^{\nu/2}s^{\mu+\nu/2}F(s)$ and integrate with respect to s from zero to infinity. We get

$$\frac{t^{\nu/2}G_{\mu+\nu/2}(0)}{\Gamma(\nu+1)} = \sum_{r=0}^\infty \frac{t^{\lambda r/2}}{r!}\, f^\lambda_{\mu+r-\lambda r/2,\nu+\lambda r}(t)$$

where we have assumed that the integrals exist and a change in the order of summation and integration is permissible.

Now we have the relation given by Delerue for the hyperbesselian function, viz.

$$J^{(n)}_{\lambda_1,\dots,\lambda_n}(x) = \frac{(x/(n+1))^{\lambda_1+\dots+\lambda_n}}{\Gamma(\lambda_1+1)\dots\Gamma(\lambda_n+1)}\, F_n\left[\lambda_1+1,\dots,\lambda_n+1; -\left(\frac{x}{n+1}\right)^{n+1}\right]$$

where n is a positive integer and $\lambda's>-1$. Also the Bessel-Maitland-Wright function is related to the hyperbesselian function by means of the following relation:

$$J^{(n)}_{\nu/n,(\nu-1)/n,\dots,(\nu-n+1)/n}(z) = \frac{(n+1)^{n/2}\left(\dfrac{n}{n+1}\right)^{\nu+1/2}}{(2\pi)^{(n-1)/2}z^{(n-2\nu-1)/2}}\, J_\nu^n\left\{\frac{1}{n}\left(\frac{nz}{n+1}\right)^{n+1}\right\}.$$

Then, using the following relation given by Delerue .

$$J^{(n)}_{\lambda_1,\dots,\lambda_n}(x)=\frac{\Gamma(\mu+1)}{\Gamma(\lambda_1-\mu)}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda_1-\mu+k)}{\Gamma(\lambda_1+k+1)\,k!}\left(\frac{x}{n+1}\right)^{\lambda_1-\mu+k}J^{(n)}_{\lambda_2+k,\dots,\lambda_n+k,\mu+k}(x)$$

and putting $\lambda_1=\nu/n$, $\lambda_2=(\nu-1)/n$, ..., $\lambda_n=(\nu-n+1)/n$ and $\mu=(\nu-n)/n$, we obtain

$$J^n_\nu(t)=\frac{\Gamma(\nu/n)}{n}\sum_{r=0}^{\infty}\frac{t^r}{\Gamma(\nu/n+r+1)}J^n_{\nu+nr-1}(t),$$

which is valid when $\nu/n\neq 0,-1,-2,\dots$ and when $n>0$. Put st^n for t, multiply both sides by $(st)^{\nu/2}s^\mu F(s)$ and integrate from a to b. We then have

$$\text{(3.1)}\qquad f^n_{\mu,\nu}(t)=\frac{\Gamma(\nu/n)}{n}\sum_{r=0}^{\infty}\frac{t^{(nr+1)/2}}{\Gamma(\nu/n+r+1)}f^n_{\mu+r-nr/2+1/2,\nu+nr-1}(t).$$

The following more general formula can readily be obtained:

$$\text{(3.2)}\qquad J^\lambda_\nu(t)=\sum_{r=0}^{\infty}c_r t^r J^\lambda_{l+\lambda r}(t).$$

The above is obtained by writing t^λ for t, multiplying by t^l and taking the image. We get

$$\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(l+\lambda r+1)}{r!\,\Gamma(\nu+\lambda r+1)}\cdot\frac{1}{p^{l+\lambda r}}=\exp(-p^\lambda)\sum_{r=0}^{\infty}\frac{c_r}{p^{l+\lambda r}}.$$

Hence we have

$$\text{(3.3)}\qquad c_r=\sum_{n=0}^{r}(-1)^{r-n}\frac{\Gamma(l+\lambda r-\lambda n+1)}{(r-n)!\,n!\,\Gamma(\nu+\lambda r-\lambda n+1)}.$$

Hence putting st^λ for t on both sides of (3.2) and multiplying by $(st)^{\nu/2}s^\mu F(s)$ we get the relation

$$\text{(3.4)}\qquad f^\lambda_{\mu,\nu}(t)=t^{(\nu-l)/2}\sum_{r=0}^{\infty}c_r t^{\lambda r/2}f^\lambda_{\mu+r+(\nu-l)/2-\lambda r/2,\,l+\lambda r}(t)$$

provided the infinite series converges.

We shall now obtain a multiplication theorem. We know that

$$\mathcal{L}_p[t^\nu J^\lambda_\nu(st^\lambda)]=\exp\left(-\frac{s}{p^\lambda}\right)p^{-\nu}=\exp\left(-\frac{1}{p^\lambda}\right)p^{-\nu}\exp\left(\frac{1-s}{p^\lambda}\right)$$

$$=\exp\left(-\frac{1}{p^\lambda}\right)p^{-\nu}\sum_{r=0}^{\infty}\frac{(1-s)^r}{r!}\cdot p^{-\lambda r}.$$

Hence we have

$$J^\lambda_\nu(mt)=\sum_{r=0}^{\infty}\frac{(1-m)^r}{r!}t^r J^\lambda_{\nu+\lambda r}(t)$$

or

$$J^\lambda_\nu(sm^\lambda t^\lambda)=\sum_{r=0}^{\infty}\frac{(1-m^\lambda)^r}{r!}s^r t^{\lambda r}J^\lambda_{\nu+\lambda r}(st^\lambda).$$

Multiplying both sides by $(mt)^{\nu/2}s^{\mu+\nu/2}F(s)$ and integrating from a to b, we have

$$m^{-\nu/2}f^\lambda_{\mu,\nu}(mt)=\sum_{r=0}^{\infty}\frac{(1-m^\lambda)^r}{r!}t^{\lambda r/2}f^\lambda_{\mu+r-\lambda r/2,\nu+\lambda r}(t).$$

It is assumed that the infinite series are convergent.

We shall now obtain certain applications of the above results.

4. Generalized Whittaker function. Let us put $F(t)=\exp(-t^{1/\lambda})$, $a=0$, $b=\infty$. Then, for $\mu=\nu/\lambda-\nu/2+1/\lambda-1$, we have

$$f^\lambda_{\mu,\nu}(t)=\lambda t^{\nu/2}\exp(-t^\lambda).$$

We are now in a position to define a function $M^\lambda_{k,m}(x)$ which has properties similar to those of the well-known $M_{k,m}$ function. We also obtain results generalizing Tricomi's expansion of $M_{k,m}$ functions in a series of Bessel functions.

Let $F(t)=\exp(-t^{1/\lambda})$, then[2])

$$f^\lambda_{\mu,\nu}(t)=\int_0^\infty(st)^{\nu/2}J^\lambda_\nu(st^\lambda)s^\mu\exp(-s^{1/\lambda})\,ds.$$

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[2]) As $x\to\infty$, $J^\mu_\lambda(x)=O\left[x^{-k(\lambda+1/2)}\exp\left\{(\mu x)^k\frac{\cos\pi k}{\mu k}\right\}\right]$, $k=1/(1+\mu)$.

As $x\to 0$, $J^\mu_\nu(x)=O(1)$ and $O(1/\Gamma(\nu+1))$ when ν is large.

We define $M^{\lambda}_{\mu,\nu}(t)$ by the relation

$$f^{\lambda}_{\mu,\nu}(t)=\frac{\Gamma(\mu+\nu/2+1)}{\Gamma(\nu+\lambda)}\exp\left(-\frac{t^{\lambda}}{2}\right)t^{-1/2}M^{\lambda}_{\mu+1/2,\nu/2}(t),$$

where $f^{\lambda}_{\mu,\nu}(t)$ is defined as above. For $\lambda=1$, $M^{\lambda}_{\mu+1/2,\nu/2}(t)$ reduces to the well-known $M_{\mu+1/2,\nu/2}(t)$ function. If we take further $\mu=\nu/2$, we get

$$f^{1}_{\nu/2,\nu}(t)=\exp(-t/2)t^{-1/2}M_{(\nu+1)/2,\nu/2}(t)=\exp(-t)t^{\nu/2}.$$

Also

$$M^{\lambda}_{(\nu+1)/\lambda-(\nu+1)/2,\nu/2}(t)=\frac{\lambda\Gamma(\nu+\lambda)}{\Gamma((\nu+1)/\lambda)}t^{(\nu+1)/2}\exp\left(-\frac{t^{\lambda}}{2}\right). \tag{4.1}$$

If λ is a positive integer ($=m$), then, using (3.1), we obtain

$$M^{m}_{\mu+1/2,\nu/2}(t)=\frac{\Gamma(\nu/m)\Gamma(\nu+m)}{m\Gamma(\mu+\nu/2+1)}\sum_{r=0}^{\infty}\frac{\Gamma(\mu+\nu/2+r+1)t^{(mr+1)/2}}{\Gamma(\nu/m+r+1)\Gamma(\nu+m+mr-1)}\times$$
$$\times M^{m}_{\mu+r-mr/2+1,(\nu+mr-1)/2}(t).$$

Using (4.1), we obtain

$$M^{m}_{\nu/m-(\nu+1)/2,\nu/2}(t)=t^{(\nu+1)/2}\exp\left(-\frac{t^{m}}{2}\right)\sum_{r=0}^{\infty}\frac{\Gamma(\nu+m)}{\Gamma(\nu/m+r+1)}t^{mr}.$$

Similarly starting with (3.4) we obtain the following more general relation:

$$\frac{\Gamma(\mu+\nu/2+1)}{\Gamma(\nu+\lambda)}M^{\lambda}_{\mu+1/2,\nu/2}(t)$$
$$=t^{(\nu-l)/2}\sum_{r=0}^{\infty}c_r\frac{\Gamma(\mu+r+\nu/2+1)}{\Gamma(l+\lambda r+\lambda)}t^{\lambda r/2}M^{\lambda}_{\mu+r+(\nu-l+1)/2-\lambda r/2,\,(l+\lambda r)/2}(t).$$

In particular when $l/\lambda=\mu+\nu/2-1/\lambda+1$, we have

$$\frac{\Gamma(\mu+\nu/2+1)}{\Gamma(\nu+\lambda)}M^{\lambda}_{\mu+1/2,\nu/2}(t)=\lambda t^{(\nu+1)/2}\exp\left(-\frac{t^{\lambda}}{2}\right)\sum_{r=0}^{\infty}c_r t^{\lambda r},$$

where c_r's have been defined before in (3.3). The infinite series are convergent as can easily be seen.

5. The polynomials $L^{\nu}_{m,\lambda}(t^{\lambda})$[3]. Again let $\mu=(\nu+m)/\lambda-\nu/2+1/\lambda-1$ where m is a positive integer and $F(s)=\exp(-s^{1/\lambda})$. We get

$$f^{\lambda}_{\mu,\nu}(t)=\lambda t^{\nu/2}\sum_{r=0}^{\infty}\frac{(-1)^r t^{\lambda r}}{r!}(\lambda r+\nu+1)\dots(\lambda r+\nu+m).$$

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[3]) We have found that the polynomials $L^{\nu}_{n,\lambda}(x)$ obtained by us have been treated by Angelescu as an interesting case of biorthogonal polynomials (1923).

Let

$$(\lambda r+\nu+1)\dots(\lambda r+\nu+m) \tag{5.1}$$
$$=a_0+\frac{a_1}{1!}r+\frac{a_2}{2!}r(r-1)+\dots+\frac{a_m}{m!}r(r-1)\dots(r-m+1).$$

Then

$$\sum_{r=0}^{\infty}\frac{(-1)^r t^{\lambda r}}{r!}(\lambda r+\nu+1)\dots(\lambda r+\nu+m)=\exp(-t^{\lambda})L^{\nu}_{m,\lambda}(t^{\lambda}),$$

where

$$L^{\nu}_{m,\lambda}(t)=a_0-\frac{a_1}{1!}t+\frac{a_2}{2!}t^2-\dots+(-1)^m\frac{a_m}{m!}t^m.$$

Now we have from (5.1)

$$\begin{aligned}
&a_0 &&=(\nu+1)\dots(\nu+m),\\
&a_0+a_1 &&=(\lambda+\nu+1)\dots(\lambda+\nu+m),\\
&a_0+2a_1+a_2 &&=(2\lambda+\nu+1)\dots(2\lambda+\nu+m),\\
&\dots\dots\dots &&\dots\dots\dots\\
&a_0+\binom{m}{1}a_1+\dots+\binom{m}{r}a_r+\dots+a_m &&=(m\lambda+\nu+1)\dots(m\lambda+\nu+m).
\end{aligned}$$

Hence

$$(-1)^n a_n=\sum_{r=0}^{n}(-1)^r\binom{n}{r}(\lambda r+\nu+1)\dots(\lambda r+\nu+m)\qquad(n=0,1,\dots,m).$$

We shall now show that in the expansion of

$$\exp t^{\lambda}\sum_{r=0}^{\infty}\frac{(-1)^r t^{\lambda r}}{r!}(\lambda r+\nu+1)\dots(\lambda r+\nu+m)$$

in powers of t^{λ} the coefficient of $t^{\lambda n}$ is zero when $n>m$. For the coefficient of $t^{\lambda n}$ is equal to

$$\frac{1}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(\lambda r+\nu+1)\dots(\lambda r+\nu+m)=\frac{1}{n!}\Delta_1^n\{(\lambda r+\nu+1)\dots(\lambda r+\nu+m)\}.$$

Let $n>m$. Since $(\lambda r+\nu+1)\dots(\lambda r+\nu+m)$ is a polynomial in r of degree m,

$$\Delta_1^n\{(\lambda r+\nu+1)\dots(\lambda r+\nu+m)\}=0$$

by the mean value theorem of Polya, where $\Delta_1^n f(x) = f^{(n)}(\xi)$ for some ξ for which $0<\xi<n-1$. In particular, the following are the explicit values of these polynomials:

$$L^{\nu}_{0,\lambda}(t) = 1,$$

$$L^{\nu}_{1,\lambda}(t) = \nu+1-\lambda t,$$

$$L^{\nu}_{2,\lambda}(t) = (\nu+1)(\nu+2)-(\lambda+2\nu+3)\lambda t+\lambda^2 t^2,$$

$$L^{\nu}_{3,\lambda}(t) = (\nu+1)(\nu+2)(\nu+3)-\{\lambda^2+\lambda(3\nu+6)+(3\nu^2+12\nu+11)\}\lambda t+ +(3\nu+6+3\lambda)\lambda^2 t^2-\lambda^3 t^3.$$

It is easy to see that using (1.1), (1.2), (1.3), (1.4) we obtain the following recurrence relations satisfied by $L^{\nu}_{m,\lambda}(t)$:

(5.2) $$\nu L^{\nu}_{m,\lambda}(t) = \lambda t L^{\nu+\lambda}_{m,\lambda}(t)+L^{\nu-1}_{m+1,\lambda}(t),$$

(5.3) $$\frac{d}{dt}\{\exp(-t)L^{\nu}_{m,\lambda}(t)\} = -\exp(-t)L^{\nu+m}_{m,\lambda}(t).$$

(5.4) $$\lambda\frac{d}{dt}\{t^{\nu/\lambda}\exp(-t)L^{\nu}_{m,\lambda}(t)\} = t^{(\nu-\lambda)/\lambda}\exp(-t)L^{\nu-1}_{m+1,\lambda}(t),$$

which follows easily from (5.3). From (5.4) we have

$$\lambda^n\left(t^{\lambda-1}\frac{d}{dt^\lambda}\right)^n\{t^\nu\exp(-t^\lambda)L^{\nu}_{m,\lambda}(t^\lambda)\} = t^{\nu-n}\exp(-t^\lambda)L^{\nu-n}_{m+n,\lambda}(t^\lambda).$$

6. Generating function of the polynomials. Since

$$\mathcal{L}_p[t^\nu J^\nu_p(t^\lambda)] = \exp(-p^{-\lambda})p^{-\nu},$$

we have

$$\mathcal{L}_p[t^{\nu+n}J^\lambda_\nu(t^\lambda)] = (-1)^n p\frac{d^n}{dp^n}\exp(-p^{-\lambda})p^{-(\nu+1)}.$$

On calculation, we find that the right hand side is the same as

$$p^{-\nu-n}\exp(-p^{-\lambda})L^{\nu}_{n,\lambda}(p^{-\lambda}).$$

We therefore have

$$L^{\nu}_{n,\lambda}(p^{-\lambda}) = (-1)^n p^{\nu+n+1}\exp p^{-\lambda}\frac{d^n}{dp^n}\exp(-p^{-\lambda})p^{-(\nu+1)}.$$

Putting $p^{-\lambda}=x$, we have

$$L^{\nu}_{n,\lambda}(x) = \lambda^n x^{-(\nu+n+1)/\lambda}\exp x\left(x^{1+1/\lambda}\frac{d}{dx}\right)^n\left(\exp(-x)x^{(\nu+1)/\lambda}\right)$$

or

$$L^{\nu}_{n,\lambda}(x^\lambda) = x^{-\nu-n-1}\exp x^\lambda\left(x^2\frac{d}{dx}\right)^n\left(x^{\nu+1}\exp(-x^\lambda)\right).$$

For $\lambda=1$, we get a new formula for the Laguerre polynomials, which is analogous to Rodrigues formula for the Legendre polynomials:

$$L^{\nu}_{n}(x) = x^{-\nu-n-1}\exp x\left(x^2\frac{d}{dx}\right)^n\left(x^{\nu+1}\exp(-x)\right).$$

Let us put

$$\exp(-t^\lambda)L^{\nu}_{m,\lambda}(t^\lambda) = F(t^\lambda,\nu).$$

We find that this satisfies the functional equation

$$\partial F/\partial t^\lambda = -F(t^\lambda,\nu+\lambda).$$

Writing t for t^λ, we get the relation

(6.1) $$\partial F/\partial t = -F(t,\nu+\lambda).$$

We can now prove the following theorem, which is a slight modification of Truesdell's theorem:

If $F(t,\nu)$ satisfies the equation (6.1), *then $F(t+y,\nu)$ possesses a Taylor series expansion and we have*

$$F(t+y,\nu) = \sum_{n=0}^{\infty}(-1)^n\frac{y^n}{n!}F(t,\nu+\lambda n).$$

In particular, we have

$$\exp(-y)L^{\nu}_{m,\lambda}(t+y) = \sum_{n=0}^{\infty}\frac{(-y)^n}{n!}L^{\nu+\lambda n}_{m,\lambda}(t).$$

Again we have

$$\frac{1}{p^{\nu+n+1}}\exp(-p^{-\lambda})L^{\nu}_{n,\lambda}\left(\frac{1}{p^\lambda}\right) = (-1)^n\frac{d^n}{dp^n}\left(\frac{\exp(-p^{-\lambda})}{p^{\nu+1}}\right)$$

so that

$$\frac{d}{dp}\left\{\frac{1}{p^{\nu+n+1}}\exp(-p^{-\lambda})L^{\nu}_{n,\lambda}\left(\frac{1}{p^\lambda}\right)\right\} = (-1)^n\frac{d^{n+1}}{dp^{n+1}}\left(\frac{\exp(-p^{-\lambda})}{p^{\nu+1}}\right) = -\frac{\exp(-p^{-\lambda})}{p^{\nu+n+2}}L^{\nu}_{n+1,\lambda}\left(\frac{1}{p^\lambda}\right).$$

Hence putting

$$F(z,n)=\frac{(-1)^n}{z^{\nu+n+1}}\exp(-z^{-\lambda})L_{n,\lambda}^{\nu}\left(\frac{1}{z^\lambda}\right)$$

and using Taylor's theorem, we obtain since $\partial F/\partial z=F(z,n+1)$,

$$\frac{1}{(z+y)^{\nu+n+1}}\exp\left(-\frac{1}{(z+y)^\lambda}\right)L_{n,\lambda}^{\nu}\left\{\frac{1}{(z+y)^\lambda}\right\}$$
$$=\sum_{m=0}^{\infty}\frac{(-y)^m}{m!}z^{-(\nu+n+m+1)}\exp\left(-\frac{1}{z^\lambda}\right)L_{n+m,\lambda}^{\nu}\left(\frac{1}{z^\lambda}\right).$$

After a slight transformation, we have

$$\frac{1}{(1-u)^{\nu+n+1}}\exp\left(\omega-\frac{\omega}{(1-u)^\lambda}\right)L_{n,\lambda}^{\nu}\left\{\frac{\omega}{(1-u)^\lambda}\right\}=\sum_{m=0}^{\infty}\frac{u^m}{m!}L_{m+n,\lambda}^{\nu}(\omega).$$

For $n=0$, we have in particular

$$\frac{1}{(1-u)^{\nu+1}}\exp\left(\omega-\frac{\omega}{(1-u)^\lambda}\right)=\sum_{m=0}^{\infty}\frac{u^m}{m!}L_{m,\lambda}^{\nu}(\omega).$$

References

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Sur le problème de Cauchy pour les systèmes d'équations aux dérivées partielles du premier ordre dans le cas hyperbolique de deux variables indépendantes

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Introduction. Nous nous occuperons du problème de Cauchy pour le système

(1) $$\partial z_i/\partial t=f_i(t,x,z_1,\ldots,z_n,\partial z_1/\partial x,\ldots,\partial z_n/\partial x)$$

du type hyperbolique (toutes les racines caractéristiques de la matrice $[\partial f_i/\partial q_j]$ sont réelles et différentes entre elles). M. Cinquini-Cibrario à démontré[1]) l'unicité et l'existence des solutions en supposant que les troisièmes dérivées des seconds membres satisfont à la condition de Lipschitz. Nous donnons ici un théorème d'existence et une méthode de construction des solutions. L'idée de la construction de la solution m'a été suggérée au cours d'une conférence prononcée par M. T. Ważewski à l'Institute Mathématique de l'Académie Polonaise des Sciences. M. Ważewski a introduit des surfaces polygonales dont les arrêtes sont des courbes intégrales de certaines équations différentielles ordinaires et tendent vers les caractéristiques, lorsque le nombre de ces équations tend vers l'infini. L'intégrale de l'équation aux dérivées partielles construite par M. Ważewski était la limite des surfaces polygonales. L'existence de cette limite était garantie par un théorème de Ważewski sur les inégalitées aux dérivées approximatives. La méthode de E. Baiada est aussi proche de la mienne. M. Baiada [1] a obtenu pour solution la limite des surfaces polygonales dont les arrêtes sont situées dans des plans $t=$const. Ces surfaces sont construites successivement dans les bandes $t_{i-1}\leqslant t\leqslant t_i$, $i=1,2,\ldots$, à partir des données de Cauchy (de même que dans la méthode de Cauchy-Lipschitz pour les équations différentielles ordinaires). Les deux méthodes sont également employées dans le cas d'une équation. S'il s'agit de systèmes, M. Conti [4] a généralisé la méthode de E. Baiada dans le cas où le second membre f_i ne dépend pas de $\partial z_j/\partial x$ $(j\neq i)$. Pour le théorème d'existence dans ce cas cf. aussi

[1]) [2] et [3]; on peut trouver en [3] la littérature de ce problème.