Remark 2. The assertion of our theorem remains true if $\varphi_i(t)$ are ACG in Δ. In that case the condition N may be replaced by the following one. For almost all points of the set

$$Z_k = \underset{t}{E}\{\tau_k(t) < \varphi_k(t) < \tau_k(t)+\varepsilon_k(t),\ t \epsilon \Delta\}$$

the following inequality holds:

$$\varphi_k^{\cdot}(t) \leqslant f_k(t, \varphi_1(t), \ldots, \varphi_n(t)). \tag{8}$$

$\varphi_k^{\cdot}(t)$ denotes here the approximative derivative of the function $\varphi_k(s)$ at the point t. It is easy to see that the condition given above implies the condition N. In fact, if $\varphi_k(t)$ is ACG in Δ, so is

$$\psi_k(t) = \varphi_k(t) - \int_{t_0}^{t} f_k(z, \varphi_1(z), \ldots, \varphi_n(z))\, dz.$$

The inequality (8) states that almost everywhere in Z_k the approximative derivative of $\psi_k(t)$ is non-positive. Therefore one can conclude ([1], p. 225) that the function $\psi_k(t)$ is non-increasing in every component of Z_k. But it is sufficient for the inequality

$$\overline{D}_{-}\varphi_k(t) \leqslant f_k(t, \varphi_1(t), \ldots, \varphi_n(t))$$

to be satisfied in Z_k. Thus the condition N holds.

Remark 3. What has been proved for the right-hand maximal integral can also be proved in a similar way for the remaining extreme integrals.

In the case of the left-hand integrals one must introduce a suitable condition instead of (M) for the right-hand members of differential equations ([2], p. 137).

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Equations satisfied by the extremal schlicht functions with a pole

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In the present paper I give the differential functional equations which must be satisfied by extremal schlicht functions with a pole. These equations are analogical to the Schaeffer-Spencer equations for the regular schlicht functions.

Consider the class Σ of functions regular and schlicht for $0<|z|<1$, which have the expansion of the form $F(z) = z^{-1}+b_1z+b_2z^2+\ldots$

The first p coefficients of each function of this form determine a point of the real $2p$-dimensional Euclidean space. To the function $F(z) = z^{-1}+ +b_1z+b_2z^2+\ldots$ corresponds the point with the coordinates $(x_1, y_1, x_2, y_2, \ldots, x_p, y_p)$ where $b_j = x_j+iy_j$. To the schlicht functions corresponds a certain set D in this space. This set contains the origin of the coordinate system, because the function $F(z) = z^{-1}$ belongs to the class Σ, it is connected, since together with the function $F(z) = z^{-1}+b_1z+b_2z^2+\ldots$ the function $\varrho F(z\varrho) = z^{-1}+b_1\varrho^2z+\ldots$ also belongs to the class Σ, which means that every point of the set D may be joined to the origin of the system by a curve lying in D. From the surface-theorem ([1], p. 72-76) it follows that the set D is bounded and from the normality [2] of the family Σ it follows that it is closed.

Now consider the region G in this space, which contains the set D and define in it an arbitrary real-valued function $E(x_1, y_1, x_2, y_2, \ldots, x_p, y_p)$ of $2p$ arguments, continuous together with its first partial derivatives and satisfying the condition

$$\sum_{k=1}^{p}\left((\partial E/\partial x_k)^2+(\partial E/\partial y_k)^2\right) > 0$$

for every point of the set D.

The function E may be treated as a functional defined for the functions belonging to the class Σ. Let us introduce additional symbols

$$E_k = \tfrac{1}{2}\{\partial E/\partial x_k - i\partial E/\partial y_k\}, \qquad \overline{E}_k = \tfrac{1}{2}\{\partial E/\partial x_k + i\partial E/\partial y_k\}.$$

Since the derivatives do not vanish in the interior of the set D, therefore the function E has the extreme value only on its boundary.

THEOREM. *The function $F(z)$ regular and schlicht for $0<|z|<1$, whose the coefficients give the extreme value of the function E, satisfies the equation*

$$\big(zF'(z)/F(z)\big)^2 \sum_{k=2}^{p+1} A_k F^k(z) = -\sum_{k=-p-1}^{p+1} (B_k/z_k)$$

where

$$A_k = \sum_{n=k-1}^{p} b_n^{[k-1]} E_n,$$

$$B_0 = \sum_{n=1}^{p} (n+1)\,\mathrm{re}\,\{b_n E_n\},$$

$$B_1 = \hat{B}_1,$$

$$B_k = \hat{B}_k - E_{k-1}, \qquad k = 2,3,\ldots,p-1,$$

$$B_p = -E_{p-1},$$

$$B_{p+1} = -E_p,$$

$$\hat{B}_k = \sum_{n=1}^{p-k} nb_n E_{k+n}, \qquad k = 1,2,\ldots,p-1,$$

$$B_{-k} = \overline{B}_k, \qquad k = 1,2,\ldots,p+1,$$

$$[F(z)]^{-k} = z^k + b_{k+2}^{[k]} z^{k+2} + \ldots + b_p^{[k]} z^p + \ldots$$

Let us observe that the obtained equation is not a differential equation but a differential-functional one.

The proof of this theorem is based on the following

LEMMA. *Let Γ be an analytic Jordan arc with the end points α and β, which does not pass through the zero point on function $F(z)=z^{-1}+b_1z+\ldots$ and is contained in the region $0<|z|<1$. Let $p_\varepsilon(z)$ be a regular function in the neighbourhood of the arc Γ satisfying the conditions*

$$|p_\varepsilon(z)| \leqslant M, \qquad |p_{\varepsilon'}(z)-p_{\varepsilon''}(z)| \leqslant M|\varepsilon'-\varepsilon''|, \qquad p_\varepsilon(\alpha) = p_\varepsilon(\beta) = 0$$

for $|\varepsilon|<\varepsilon_0$. Then for every sufficiently small ε there exists a function, regular and schlicht for $0<|z|<1$, of the following form:

$$F^*(z) = F(z) + \frac{\varepsilon}{2\pi i}\int_\alpha^\beta \frac{p(u)}{2u^2}\left\{\left(\frac{uF'(u)}{F(u)}\right)^2 \frac{2F(z)F(u)}{F(u)-F(z)} - zF'(z)\frac{u+z}{u-z} - F(z)\right\}du + \frac{\bar{\varepsilon}}{2\pi i}\int_\alpha^\beta \frac{\overline{p}(u)}{2\bar{u}^2}\left\{zF'(z)\frac{1+\bar{u}z}{1-\bar{u}z} + F(z)\right\}d\bar{u} + o(\varepsilon).$$

$p_0(u)$ is denoted by $p(u)$.

The proof of this lemma is identical with the proof of an analogical lemma of Spencer ([3], lemma VI, p. 32) for regular functions. It suffices to replace the class of regular functions by the class of functions with a pole, and the condition that the arc Γ does not pass through the point $z=0$ by the condition that it does not pass by the possible zero of the function $F(z)$.

Making use of the formula of the lemma we shall calculate the coefficients of the expansion of the function $F^*(z)$. Namely

$$b_k^* = b_k + \frac{\varepsilon}{2\pi i}\int_\alpha^\beta \frac{p(u)}{2u^2}\left\{-\left(u\frac{F'(u)}{F(u)}\right)^2 2\sum_{n=1}^{k} b_k^{[n]} F^{n+1}(u) - (k+1)b_k + \frac{2}{u^{k+1}} - 2\sum_{n=1}^{k-1}\frac{nb_n}{u^{k-n}}\right\}du + \frac{\bar{\varepsilon}}{2\pi i}\int_\alpha^\beta \frac{\overline{p}(u)}{2\bar{u}^2}\left\{(k+1)b_k - 2\bar{u}^{k+1} + 2\sum_{n=1}^{k-1} nb_n \bar{u}^{k-1}\right\}d\bar{u} + o(\varepsilon).$$

From $E\big(F^*(z)\big) - E\big(F(z)\big) = 2\,\mathrm{re}\,\{\sum_{k=1}^{p} E_k(b_k^*-b_k)\} + o(\varepsilon)$ follows

$$E(F^*) - E(F) = 2\,\mathrm{re}\left\{\frac{\varepsilon}{2\pi i}\int_\alpha^\beta \frac{p(u)}{2u^2}\left[-\left(u\frac{F'(u)}{F(u)}\right)^2 \cdot 2\sum_{k=2}^{p+1} F^k(u) \sum_{n=k-1}^{p} b_n^{[k-1]} E_n - \sum_{k=1}^{p}(k+1)b_k E_k + 2\sum_{k=1}^{p}\frac{E_k}{u^{k+1}} - 2\sum_{k=2}^{p}\frac{1}{u^k}\sum_{n=1}^{p-k} nb_n E_{k+n}\right]du + \frac{\bar{\varepsilon}}{2\pi i}\int_\alpha^\beta \frac{\overline{p}(u)}{2\bar{u}^2}\left[\sum_{k=1}^{p}(k+1)b_k E_k - 2\sum_{k=1}^{p}\bar{u}^{k+1}E_k + 2\sum_{k=2}^{p}\bar{u}_k \sum_{n=1}^{p-k} nb_n E_{k+n}\right]d\bar{u}\right\} + o(\varepsilon).$$

Using the notation of the theorem we obtain

$$E\big(F^*(z)\big) - E\big(F(z)\big) = -2\,\mathrm{re}\left\{\frac{\varepsilon}{2\pi i}\int_\alpha^\beta \frac{p(u)}{2u^2}\left[\left(\frac{uF'(u)}{F(u)}\right)^2 \sum_{k=2}^{p+1} A_k F^k(u) + \sum_{k=-p-1}^{p+1}\frac{B_k}{u^k}\right]du\right\} + o(\varepsilon).$$

Now suppose that the point corresponding to the function $F(z)$ gives the maximum of the functional E. We know that this point lies on the boundary of the set D. Then the increment at this point must be non-positive. Since ε may be arbitrary, therefore its coefficient must vanish. The integral must vanish for all the functions $p(u)$, and therefore the expression in brackets must vanish, which proves the assertion of the theorem.

The deduction of these equations is analogical to the deduction of the equations for the regular functions contained in the book cited above.

This result may be used to verify a certain hypothesis of W. Wolibner [4] namely that the module of the pth coefficient of a schlicht function with a pole assumes the greatest value for the function

$$\varphi_p = \{z^{(p+1)/2} + z^{-(p+1)/2}\}^{2/(p+1)} = z^{-1} + \frac{2}{p+1} z^p + \ldots$$

Let us observe that if a certain function $F(z) = z^{-1} + b_1 z + \ldots$ gives the maximum of the module of the coefficient b_p then the function

$$\exp[-i\vartheta/(p+1)]F(\exp[-i\vartheta/(p+1)]z) = z^{-1} + b_1 z + \ldots, \qquad \vartheta = \arg\{b_p\}$$

gives the maximum of the real part of b_p. Therefore we may assume that $E = \operatorname{re}\{b_p\} = x_p$. Obviously this functional satisfies the desired conditions and

$$E_k = \begin{cases} 0 & \text{for} \quad k \neq p, \\ 1/2 & \text{for} \quad k = p. \end{cases}$$

The equation for this functional has the form

$$(zF'(z)/F(z))^2 \sum_{k=2}^{p+1} b_p^{[k-1]} F^k(z)$$
$$= -\{(p+1)\operatorname{re}\{b_p\} + \sum_{k=1}^{p-1} (b_{p-k} z^{-k} + \bar{b}_{p-k} z^k) - z^{-p-1} - z^{p+1}\}.$$

Since

$$\varphi_p(z) = z^{-1} + \frac{2}{p+1} z^p + \ldots, \qquad [\varphi_p]^{-(k-1)} = z^{k-1} - \frac{2(k-1)}{p+1} z^{k+p} + \ldots$$

therefore substituting the function $\varphi_p(z)$ into the equation we obtain

$$[z\varphi_p'(z)/\varphi_p(z)]^2 \varphi_p^{p+1}(z) = z^{-p-1} - 2 + z^{p+1}$$

which, as can be shown by easy calculation, holds. It follows that the function $\varphi_p(z)$ may give the maximum of the module of the coefficient b_p.

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