where ε_0, ε_{j-1}, ε_j are infinitely small with respect to σ. The last formula directly implies formula (5).

Let us now take $j=n-1$ and suppose that $\varkappa_{n-1}\neq 0$. In any case we have

$$\lim_{M\to M_0} \delta_{n-1}/(\delta_0\,\delta_{n-2}) = |\varkappa_{n-1}|/n.$$

This it only remains to show that δ^*_{n-1} and $\varkappa_{n-1}$ are of the same sign. From the expansion $\boldsymbol{r}=\sum_{i=1}^{n}\boldsymbol{u}_i\sigma^i/i!+\boldsymbol{e}$ and from formulas (12) for $\boldsymbol{u}_i$ it follows that the component of vector $\boldsymbol{r}$ in the direction of $\boldsymbol{t}^0_n$ is of the same sign as ϱ_n. But $\varrho_n=\varkappa_1\varkappa_2\cdot\ldots\cdot\varkappa_{n-1}$, and since $\varkappa_i>0$ for $i=1,2,\ldots,n-2$, therefore ϱ_n is of the same sign as $\varkappa_{n-1}$.

Thus formula (7) is proved.

It will be observed that formulas (1) and (2) are particular cases of formulas (5) and (7).

On the concept of the centre of the second curvature and on a generalization of a certain geometrical meaning of v. Lilienthal

by S. Gołąb (Kraków)

The concept of the first curvature of a skew curve has — as we know — many geometrical meanings. Among them there are two such that by means of certain geometrical constructions we directly obtain the centre of the first curvature as a certain point S_1 and only in the second place the curvature $\varkappa_1$ itself. In other interpretations (which may also serve as definitions) we first obtain the curvature $\varkappa_1$ and only then, after a suitable definition of the principal normal, the centre of the curvature S_1.

The above-mentioned two interpretations which directly lead to the obtaining of point S_1 are the following. In the first we consider the circle drawn through three neighbouring points of the curve C — the centre of the limiting position of the variable circle is the centre of the first curvature S_1. In the second interpretation we consider the polar straight line as the limiting position of the intersection of two neighbouring planes normal to the curve, and then the intersection point of the polar with the principal normal gives us point S_1.

Both interpretations may be generalized to the case of curves lying in an n-dimensional Euclidean space. The first needs no essential change. For the second we must first define the polar figure connected with a variable point M of curve C. We define that figure as the limiting position of the intersection of two hyperplanes normal to curve C at neighbouring points. The polar figure (belonging to point M) will be a certain $(n-2)$-dimensional linear space B_{n-2}. Projecting point M upon B_{n-2} (the concept of projecting being well-defined), we obtain the *centre of the first curvature* S_1.

The second curvature $\varkappa_2$ of skew curves in the three-dimensional space does not possess so many geometrical interpretations as the first curvature $\varkappa_1$, and in the second place, as far as I know, there exists only one interpretation which gives a certain uniquely determined point S_2 (connected with point M of the curve) called the *centre of the second curvature*. Other geometrical interpretations directly lead to $\varkappa_2$ and, moreover, give no indication as to the construction of point S_2 when the curvature $\varkappa_2$ is known. The above-mentioned only interpretation is due to v. Lilienthal.

The object of this note is a generalization of v. Lilienthal's construction to curves lying in a n-dimensional space ($n \geqslant 3$). That construction makes it possible to define points $S_1, S_2, \ldots, S_{n-1}$ as the centres of the first, the second, ..., the $(n-1)$th curvature. However, in my opinion the construction in question has two disadvantages. First, we make use in it of Frenet's n-hedron in whose definition automatically appear the curvatures $\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1}$, so that it can hardly be said that the definition of points S_j precedes the definition of curvatures $\varkappa_j$. The second disadvantage is rather of an aesthetic nature. Namely, as we shall see, there is a certain asymetry in the distribution of points S_j on the axes of Frenet's n-hedron.

We do not quote the original construction of Lilienthal because it will become a particular case of our general definition.

Let us consider a regular curve C in the Euclidean-space R_n and let

$$\boldsymbol{t}_i \qquad (i=1,2,\ldots,n) \tag{1}$$

denote a sequence of unit vectors of Frenet's n-hedron connected with the variable point M of curve C. Denoting by σ the arc of curve C we obtain the generalized formulas of Frenet

$$d\boldsymbol{t}_i/d\sigma = -\varkappa_{i-1}\boldsymbol{t}_{i-1} + \varkappa_i \boldsymbol{t}_{i+1} \qquad (i=1,2,\ldots,n), \tag{2}$$

where we assume for symmetry

$$\varkappa_0 = \varkappa_n = 0. \tag{3}$$

Let us denote by l_i the axis passing through M and directed according to vector $\boldsymbol{t}_i$.

Let us fix on curve C a point M_0 and let us denote the neighbouring point of curve C by M. The length of the arc from M_0 to M let us denote by σ:

$$\overset{\frown}{M_0 M} = \sigma. \tag{4}$$

The vectors of Frenet's n-hedron at point M will be denoted by $\boldsymbol{t}_i$, and the corresponding vectors at point M_0 by $\boldsymbol{t}_i^0$. Similarly by ascribing a circle we shall distinguish the elements connected with point M_0 from the corresponding elements connected with the neighbouring point M.

Let us establish number j as one of the numbers of the sequence $2, 3, \ldots, n-1$.

On the axis l_{j+1}^0 we take into account point A_j with the abscisse $(-\sigma)$; thus with respect to the local system of reference with the beginning

at M_0 the radius-vector of point A_j will be $-\sigma \boldsymbol{t}_{j+1}^0$. Through point A_j we draw a straight line w_j parallel to the axis l_j. Let us denote by w_j' the projection of the line w_j upon the plane π_j spanned on the straight lines l_j^0 and l_{j+1}^0. Finally, let us denote by B_j the intersection point of the straight line w_j' with the straight line l_j^0. The limiting position of point B_j when $M \to M_0$ will be denoted by S_j.

The point S_j obtained in this way will be called the *centre of the jth curvature of curve C at point M_0*.

We assert that with the assumptions under which Frenet's equations hold, point S_j exists and that the measure of the vector $M_0 S_j$ (counted on the axis l_j^0) is $1/\varkappa_j^0$.

Proof. The projection $\boldsymbol{t}_j'$ of the vector $\boldsymbol{t}_j$ on the plane π_j will be expressed by the formula

$$\boldsymbol{t}_j' = \alpha \boldsymbol{t}_j^0 + \beta \boldsymbol{t}_{j+1}^0, \tag{5}$$

where the scalar coefficients α, β are the following scalar products:

$$\alpha = \boldsymbol{t}_j \boldsymbol{t}_j^0, \qquad \beta = \boldsymbol{t}_j \boldsymbol{t}_{j+1}^0. \tag{6}$$

Consequently, the equation of the straight line w_j' will be $\boldsymbol{x} = -\sigma \boldsymbol{t}_{j+1}^0 + \varrho \boldsymbol{t}_j'$, where $\boldsymbol{x}$ denotes the radius-vector of the variable point of the straight line w_j', ϱ is a parameter variable along w_j'.

In order to find the intersection point B_j of the straight line w_j' and the straight line l_j^0 we must solve the system of equations

$$\boldsymbol{x} = -\sigma \boldsymbol{t}_{j+1}^0 + \varrho \boldsymbol{t}_j', \qquad \boldsymbol{x} = \tau \boldsymbol{t}_j^0, \tag{7}$$

where τ is a parameter variable along the straight line l_j^0.

From equations (7), taking into account (5), we obtain

$$\tau \boldsymbol{t}_j^0 = -\sigma \boldsymbol{t}_{j+1}^0 + \varrho [\alpha \boldsymbol{t}_j^0 + \beta \boldsymbol{t}_{j+1}^0],$$

which, on account of the linear independence of vectors $\boldsymbol{t}_j^0$ and $\boldsymbol{t}_{j+1}^0$, leads to the following conclusions:

$$\tau = \varrho \alpha, \qquad 0 = -\sigma + \varrho \beta. \tag{8}$$

In system (8) α, β, σ are given quantities; ϱ, τ are the sought ones.

The second equation of (8) gives

$$\varrho = \sigma / \beta. \tag{9}$$

In order to calculate β let us write

$$\boldsymbol{t}_j = \boldsymbol{t}_j^0 + \sigma \{(d\boldsymbol{t}_j/d\sigma)_0 + \boldsymbol{e}\},$$

where $\boldsymbol{e}$ is a vector infinitely small with respect to σ.

Hence we have

$$\beta = \boldsymbol{t}_j \boldsymbol{t}_{j+1}^0 = \boldsymbol{t}_j^0 \boldsymbol{t}_{j+1}^0 + \sigma \{ \boldsymbol{t}_{j+1}^0 (d\boldsymbol{t}_j/d\sigma)_0 + \varepsilon \}, \tag{10}$$

where ε is a scalar quantity infinitely small with respect to σ. But on the basis of Frenet's equation we have

$$(d\boldsymbol{t}_j/d\sigma)_0 = -\varkappa_{j-1}^0 \boldsymbol{t}_{j-1}^0 + \varkappa_j^0 \boldsymbol{t}_{j+1}^0.$$

Substituting this into (10) and taking into account the fact that

$$\boldsymbol{t}_{j+1}^0 \boldsymbol{t}_{j+1}^0 = 1, \qquad \boldsymbol{t}_j^0 \boldsymbol{t}_{j+1}^0 = 0, \qquad \boldsymbol{t}_{j+1}^0 \boldsymbol{t}_{j-1}^0 = 0$$

we obtain $\beta = \sigma\{\varkappa_j^0 + \varepsilon\}$, whence (by (9)) $\varrho = 1/(\varkappa_j^0 + \varepsilon)$. Substituting this into the first equation of (8) we have

$$\tau = \alpha/(\varkappa_j^0 + \varepsilon). \tag{11}$$

But we have

$$\alpha = \boldsymbol{t}_j \boldsymbol{t}_j^0 = (\boldsymbol{t}_j^0 + \boldsymbol{e})\boldsymbol{t}_j^0 = 1 + \bar{\varepsilon}, \tag{12}$$

where $\bar{\varepsilon}$ is infinitely small with respect to σ.

Formulas (11) and (12) give

$$\tau = (1 + \bar{\varepsilon})/(\varkappa_j^0 + \varepsilon). \tag{13}$$

Assuming that $\varkappa_j^0 \neq 0$ we obtain for the coordinate of point B_j on the axis l_j^0 the value

$$(1 + \bar{\varepsilon})/(\varkappa_j^0 + \varepsilon).$$

When $\sigma \to 0$, then $\varepsilon \to 0$, $\bar{\varepsilon} \to 0$ and $B_j \to S_j$, where point S_j is defined as a radius-vector $(1/\varkappa_j^0)\boldsymbol{t}_j^0$ and thus the theorem is proved.

Point S_j, called the centre of the jth curvature, lies on the straight line l_j^0.

The above reasoning may also be applied to the case of $j=1$. Then, however, we should obtain the centre of the first curvature S_1 lying on a tangent straight line, and not on the principal normal as in the classical definition of the centre of the first curvature. If we want to preserve the classical definition of the centre of the first curvature and retain the above construction for $j=2, \ldots, n-1$ we obtain an asymmetry consisting in the fact that both the centre of the first curvature and the centre of the second curvature lie on the axis l_2^0, the centres of the succeeding curvatures lie on the axes l_j^0 where $j=3, \ldots, n-1$, while none of the curvature centres lie on the axes l_1^0 and l_n^0.

For $n=3$ and $j=2$ we obtain exactly the construction of v. Lilienthal.

Sur certaines inégalités aux dérivées partielles relatives aux fonctions possédant la différentielle approximative

par T. Ważewski (Kraków)

Les intégrales des systèmes d'équations différentielles ordinaires peuvent être approchées par des *fonctions brisées* ayant pour diagrammes des lignes brisées (méthode *polygonale* de Euler, Cauchy et Peano). On sait qu'il est possible d'évaluer l'exactitude d'une telle approximation au moyen de théorèmes convenables sur les inégalités differentielles ordinaires.

Dans le présent article nous traitons un problème analogue relatif à l'approximation des surfaces intégrales de certains systèmes d'équations aux dérivées partielles par des surfaces brisées. Le théorème 2 du § 5 sert à évaluer l'exactitude d'une telle approximation.

Les surfaces brisées sont définies comme diagrammes des fonctions possédant *la différentielle ε-approximative* qui au cas $\varepsilon=0$ se réduit à la différentielle classique au sens de Stolz. Cette notion se rattache de près à la notion de contingent de M. Bouligand. Nous nous servons aussi des notions de *dérivées partielles ε-approximatives* et de *gradients ε-approximatifs* constituant une généralisation des dérivées et des gradients au sens classique (§ 2).

Le théorème 1 du § 4 fournit une inégalité permettant d'évaluer le nombre dérivé supérieur d'une fonction auxiliaire $M(r)$ qui est égale au maximum que prend une fonction de plusieurs variables sur une surface mobile dépendant du paramètre réel r. Ce théorème permet de ramener l'examen des inégalités aux dérivées partielles à celui des inégalités différentielles ordinaires.

Le théorème 3 du § 6 fournit une limitation de l'exactitude avec laquelle une surface brisée approche l'intégrale d'un système d'équations aux dérivées partielles.

Les théorèmes 1, 2, 3 constituent une généralisation de nos résultats antérieurs [3] et [4] qui correspondent au cas $\varepsilon=0$, c'est-à-dire au cas des inégalités aux dérivées partielles au sens classique.

La méthode de démonstration de ces théorèmes est, à quelques détails près, la même que celles des articles [3] et [4]. Cependant, grâce au lemme 1 du § 3, les résultats du présent article s'appliquent aussi au