où $l=0,\pm 1,\ldots$ [3]). Nous sommes arrivés au système d'équations analogues, à un système que j'ai obtenu dans un autre travail (voir [3], p. 191) en remplaçant en ce dernier le système r par r^K et φ par φ/K dans les expressions $A(\varrho,\varphi)$ et $C(\varrho,\varphi)$ qui s'y trouvent. Analogiquement, on peut, de même que dans le travail cité ci-dessus, remplacer le système d'équations (44) par le système suivant:

$$(45)\qquad \frac{A(\varrho,\varphi/K)}{C(\varrho,\varphi/K)}U(\varrho,r)-V(\varrho,r)=0,\qquad \varphi=\left[U^2(\varrho,r)-V^2(\varrho,r)\right]^{1/2}+2l\pi,$$

où nous avons posé, pour abréger,

$$U(\varrho,r)=\log\left(\frac{1-\varrho}{1+\varrho}:\frac{1-r^K}{1+r^K}\right),\qquad V(\varrho,r)=\log\frac{\varrho(1-r^{2K})}{Tr^K(1-\varrho^2)}.$$

De cette façon nous avons démontré, selon (45) et (38), la première partie du théorème (I) pour la famille F_T

En appliquant la méthode tout à fait analogue a celle dans le travail cité (comparer [3], p. 192-199), on obtient ce qui suit: pour chaque solution ϱ,φ des équations (45) pour chaque l satisfaisant aux conditions $0<\varrho<r$, $\sin\left[U^2(\varrho,r)-V^2(\varrho,r)\right]^{1/2}>0$, il existe une fonction univalente $f^*(z)$ dans $|z|<1$ de la famille F_T telle que $f^*(r^K)=\varrho e^{i\varphi}$, ou bien

$$I\left\{\left[f^*(r^K)/a_1\right]^{1/K}\right\}=\varrho^{1/K}\sin(\varphi/K)a^{-1/K}.$$

En choisissant celle pour laquelle le produit $\varrho^{1/K}\sin(\varphi/K)a_1^{-1/K}$ est le plus grand, nous démontrons la seconde partie de la thèse I pour les fonctions de la famille F_T. En passant de la famille F_T par φ_T à Φ_M, où $T=M^{-K}$, nous arrivons à la thèse demandée. Enfin, on démontre le théorème (II), comme dans le travail cité ci-dessus (voir [3], p. 199-200).

Travaux cités

[1] Z. Charzyński et W. Janowski, *Sur l'équation générale des fonctions extrémales dans la famille des fonctions univalentes bornées*, Ann. Univ. M. Curie-Skłodowska 4, Lublin 1950, p. 41-56.

[2] W. Janowski, *Le maximum d'argument des fonctions univalentes bornées*, Ann. Univ. M. Curie-Skłodowska 4, Lublin 1950, p. 57-72.

[3] — *Le maximum de la partie imaginaire des fonctions univalentes bornées*, ce volume, p. 182-200.

[4] — *Sur le maximum du module des fonctions univalentes bornées*, Bul. Soc. Sci. et L. de Łódź, Cl. III, Vol. VIII, 6(1955).

[3]) On peut se borner évidemment aux valeurs $l=0,1,\ldots,K-1$.

On the geometrical significance of curvatures of higher orders for curves lying in n-dimensional spaces

by S. Gołąb (Kraków)

Burali-Forti is responsible for the following geometrical interpretation connected with the first and the second curvature of curves lying in the three-dimensional Euclidean space.

Let $\varkappa_1$ and $\varkappa_2$ denote, respectively, the first and the second curvature of a curve C at a regular point M. Fixing the point M_0 let us denote by M a neighbouring variable point on the curve C, by M' the projection of the point M on the tangent to C at the point M_0, and by M'' the projection of the point M on the plane exactly tangent at the point M_0.

Then we have the following formulas:

$$(1)\qquad \lim_{M\to M_0}\left(2\overline{M_0M'}/\overline{M_0M}^2\right)=\varkappa_1 \text{ [1]},$$

$$(2)\qquad \lim_{M\to M_0}\left(3\overline{M_0M''}/(\overline{M_0M}\cdot\overline{M_0M'})\right)=|\varkappa_2|.$$

These formulas may also serve to define curvatures $\varkappa_1$ and $\varkappa_2$ at point M_0. In this way we obtain a fairly general definition of those notions, which, for instance, does not imply at all the rectifiability of the curve in the neighbourhood of point M_0.

The object of this note is to generalize formulas (1) and (2). Before formulating this generalization we shall introduce certain symbols.

We shall assume that a curve C lying in an n-dimensional Euclidean space satisfies the assumptions under which Frenet's equations are valid. Denoting by σ an arc of curve C we shall mark by commas the differentiation with respect to the arc. Further, let us denote by $\boldsymbol{t}_i$ $(i=1,2,\ldots,n)$ the orthonormal system of vectors of Frenet's n-hedron, and by $\varkappa_j$ $(j=1,2,\ldots,n-1)$ the successive curvatures of curve C, it being assumed that

$$\varkappa_p>0\qquad (p=1,2,\ldots,n-2).$$

[1]) $\overline{AB}$ denotes the distance of points A and B.

Then Frenet's equations assume the form

$$
\begin{aligned}
&\boldsymbol{t}_1' = \varkappa_1 \boldsymbol{t}_2,\\
(3)\qquad &\boldsymbol{t}_j' = -\varkappa_{j-1}\boldsymbol{t}_{j-1} + \varkappa_j \boldsymbol{t}_{j+1} \qquad (j=2,3,\ldots,n-1),\\
&\boldsymbol{t}_n' = -\varkappa_{n-1}\boldsymbol{t}_{n-1}.
\end{aligned}
$$

Let us establish a point M_0 on C (corresponding to the value $\sigma=0$) and let M denote a neighbouring point (corresponding to the value $\sigma\neq 0$). Let us denote by E_j $(j=1,2,\ldots,n-1)$ a j-dimensional linear space osculating at point M_0, *i.e.* spanned on the vectors of the sequence $\boldsymbol{t}_1^0,\ldots,\boldsymbol{t}_j^0$.

Let us denote by M_j the orthogonal projection of point M on the space E_j.

Finally let us introduce the notation

$$(4)\qquad \delta_j = \overline{MM_j} \qquad (j=0,1,\ldots,n-1).$$

With the above notation we assert that the following relations hold:

$$(5)\qquad \lim_{M\to M_0} \big(\delta_j/(\delta_0\,\delta_{j-1})\big) = \varkappa_j/(j+1) \qquad (j=1,2,\ldots,n-2);$$

but if we define the distance δ^*_{n-1} (of the point M from the osculating hyperplane E_{n-1}) algebraically, assuming that

$$(6)\qquad \delta^*_{n-1} = \pm\overline{MM_{n-1}},$$

the sign $+$ being taken for those points for which the component of the vector $\overline{M_0M}$ in the direction of the vector $\boldsymbol{t}_n^0$ is positive[2]), then we shall have

$$(7)\qquad \lim_{M\to M_0}\big(\delta^*_{n-1}/(\delta_0\,\delta_{n-2})\big) = \varkappa_{n-1}/n.$$

In order to prove relations (5) and (7) we shall first prove the following

LEMMA. *Denoting by $\boldsymbol{r}(\sigma)$ the radius-vector of curve C, we have relations of the form*

$$(8)\qquad d^j\boldsymbol{r}/d\sigma^j = \sum_{m=1}^{j-1} a_m^j(\sigma)\boldsymbol{t}_m + \prod_{m=1}^{j-1}\varkappa_m \boldsymbol{t}_j \qquad (j=2,3,\ldots,n-1),$$

where the coefficients a_m^j are certain algebraical functions of the curvatures $\varkappa_m$ and their derivatives.

Indeed, for $j=2$ we have $d^2\boldsymbol{r}/d\sigma^2 = d\boldsymbol{t}_1/d\sigma = \varkappa_1\boldsymbol{t}_2$, *i.e.* taking $a_1^2=0$ we have an agreement with formula (8).

[2]) *i.e.* if $\overline{M_0M} = \lambda\boldsymbol{t}_n^0 + \boldsymbol{v}$ where the vector $\boldsymbol{v}$ is linearly independent of $\boldsymbol{t}_n^0$, then $\lambda>0$.

Let us assume that formula (8) holds for $j=p\geqslant 2$ $(p\leqslant n-2)$

$$(9)\qquad d^p\boldsymbol{r}/d\sigma^p = \sum_{m=1}^{p-1} a_m^p(\sigma)\boldsymbol{t}_m + \prod_{m=1}^{p-1}\varkappa_m \boldsymbol{t}_p.$$

Let us differentiate both sides of the above relation. We shall obtain

$$(10)\qquad d^{p+1}\boldsymbol{r}/d\sigma^{p+1} = \sum_{m=1}^{p-1}(da_m^p/d\sigma)\boldsymbol{t}_m + (\varkappa_1\cdot\ldots\cdot\varkappa_{p-1})'\boldsymbol{t}_p + \\ + \sum_{m=1}^{p-1} a_m^p\, d\boldsymbol{t}_m/d\sigma + \prod_{m=1}^{p-1}\varkappa_m\, d\boldsymbol{t}_p/d\sigma.$$

Applying Frenet's formulas we have

$$
\begin{aligned}
d\boldsymbol{t}_m/d\sigma &= -\varkappa_{m-1}\boldsymbol{t}_{m-1} + \varkappa_m\boldsymbol{t}_{m+1} \qquad (\varkappa_0=0),\\
d\boldsymbol{t}_p/d\sigma &= -\varkappa_{p-1}\boldsymbol{t}_{p-1} + \varkappa_p\boldsymbol{t}_{p+1},
\end{aligned}
$$

which on being substituted into (10) gives

$$
\begin{aligned}
d^{p+1}\boldsymbol{r}/d\sigma^{p+1} &= \sum_{m=1}^{p-1}(da_m^p/d\sigma)\boldsymbol{t}_m - \sum_{m=1}^{p-1} a_m^p\varkappa_{m-1}\boldsymbol{t}_{m-1} + \sum_{m=1}^{p-1} a_m^p\varkappa_m\boldsymbol{t}_{m+1} + \\
&\quad + (\varkappa_1\cdot\ldots\cdot\varkappa_{p-1})'\boldsymbol{t}_p - \varkappa_1\cdot\ldots\cdot\varkappa_{p-2}\varkappa_{p-1}^2\boldsymbol{t}_{p-1} + \varkappa_1\cdot\ldots\cdot\varkappa_{p-1}\varkappa_p\boldsymbol{t}_{p+1}\\
&= \sum_{m=1}^{p-1}(da_m^p/d\sigma)\boldsymbol{t}_m - \sum_{m=0}^{p-2} a_{m+1}^p\varkappa_m\boldsymbol{t}_m + \sum_{m=2}^{p} a_{m-1}^p\varkappa_{m-1}\boldsymbol{t}_m - \\
&\quad - \varkappa_1\cdot\ldots\cdot\varkappa_{p-2}\varkappa_{p-1}^2\boldsymbol{t}_{p-1} + (\varkappa_1\cdot\ldots\cdot\varkappa_{p-1})'\boldsymbol{t}_p + \prod_{m=1}^{p}\varkappa_m\boldsymbol{t}_{p+1}\\
&= \sum_{m=1}^{p} a_m^{p+1}\boldsymbol{t}_m + \prod_{m=1}^{p}\varkappa_m\boldsymbol{t}_{p+1},
\end{aligned}
$$

where

$$
\begin{aligned}
a_1^{p+1} &= da_1^p/d\sigma - a_2^p\varkappa_1,\\
a_j^{p+1} &= da_j^p/d\sigma - a_{j+1}^p\varkappa_j + a_{j-1}^p\varkappa_{j-1} \qquad (j=2,3,\ldots,p-2),\\
a_{p-1}^{p+1} &= da_{p-1}^p/d\sigma + a_{p-1}^p\varkappa_{p-1} - \varkappa_1\cdot\ldots\cdot\varkappa_{p-2}\varkappa_{p-1}^2,\\
a_p^{p+1} &= a_{p-1}^p\varkappa_{p-1} + (\varkappa_1\cdot\ldots\cdot\varkappa_{p-1})',
\end{aligned}
$$

and thus the lemma is proved.

We shall now prove relation (5) and (7).

Let us fix the origin of the system at point M_0 and let us give to the axes of the system the direction of the vectors $\boldsymbol{t}_1^0,\ldots,\boldsymbol{t}_n^0$ respectively.

Let us denote by $\boldsymbol{r}_j$ $(j=1,2,\ldots,n-1)$ the radius-vector of point M_j.

On the basis of simple formulas of the n-dimensional analytic geometry we obtain

$$\boldsymbol{r}_j = \sum_{m=1}^{j} \omega_m \boldsymbol{t}_m^0 \qquad \text{where} \quad \omega_m = \boldsymbol{r}\boldsymbol{t}_m^0\ ^{3}).$$

Hence we obtain $\delta_j = |\boldsymbol{r}-\boldsymbol{r}_j| = |\boldsymbol{R}_j|$.

It is obvious that the expansions for the quantities $\boldsymbol{R}_j$ must be found. Therefore let us briefly write $\boldsymbol{u}_i \overset{\text{df}}{=} (d^i\boldsymbol{r}/d\sigma^i)_0$. It will be observed that we have

$$(11) \qquad \boldsymbol{u}_1 = \boldsymbol{t}_1^0,$$

and on the basis of the lemma

$$(12) \qquad \boldsymbol{u}_j = \sum_{m=1}^{j-1} a_m^j(0)\, \boldsymbol{t}_m^0 + \varrho_j \boldsymbol{t}_j^0 \qquad (j = 2, 3, \ldots, n),$$

where $\varrho_j = \varkappa_1(0)\cdot\ldots\cdot\varkappa_{j-1}(0)$ $(j=2,3,\ldots,n)$ we have

$$(13) \qquad \boldsymbol{r} = \sum_{i=1}^{n} \boldsymbol{u}_i \sigma^i/i! + \boldsymbol{e},$$

where the vector $\boldsymbol{e}$ (dependent on σ) contains terms of higher degree with respect to σ than n.

In order to calculate $\boldsymbol{r}_j$ we must find the expansions of the scalar coefficients ω_k. Now

$$(14) \qquad \omega_k = \boldsymbol{t}_k^0 \boldsymbol{r} = \boldsymbol{t}_k^0\Big(\sum_{i=1}^{n} \boldsymbol{u}_i \sigma^i/i! + \boldsymbol{e}\Big) = \sum_{i=1}^{n} (\boldsymbol{t}_k^0 \boldsymbol{u}_i)\, \sigma^i/i! + \boldsymbol{e}\boldsymbol{t}_k^0.$$

On the basis of (11) and (12) we write out the table of the values of the scalar coefficients $\boldsymbol{t}_k^0 \boldsymbol{u}_i$:

	$\boldsymbol{u}_1$	$\boldsymbol{u}_2$	$\boldsymbol{u}_3$	$\boldsymbol{u}_4$ ...
$\boldsymbol{t}_1^0$	1	a_1^2	a_1^3	a_1^4 ...
$\boldsymbol{t}_2^0$	0	ϱ_2	a_2^3	a_2^4 ...
$\boldsymbol{t}_3^0$	0	0	ϱ_3	a_3^4 ...
$\boldsymbol{t}_4^0$	0	0	0	ϱ_4 ...
...	...	...	...	...

and thus we have the following general formulas:

$$\boldsymbol{t}_k^0 \boldsymbol{u}_i = \begin{cases} 0 & \text{for} \quad k > i, \\ \varrho_i & \text{for} \quad k = i \qquad (\varrho_1 \overset{\text{df}}{=} 1), \\ a_k^i(0) & \text{for} \quad k < i. \end{cases}$$

Hence it follows that for the values of ω_k we have the formula

$$\omega_k = \varrho_k \sigma^k/k! + \sum_{i=k+1}^{n} a_k^i(0)\, \sigma^i/i! + \boldsymbol{e}\boldsymbol{t}_k^0 \qquad (k = 1, 2, \ldots, n-1).$$

[3]) $\boldsymbol{uv}$ denotes the scalar product of the vectors $\boldsymbol{u}, \boldsymbol{v}$.

Now let us calculate

$$\begin{aligned}\boldsymbol{r}_j = \sum_{m=1}^{j} \omega_j \boldsymbol{t}_j^0 = \boldsymbol{t}_1^0\{\sigma + a_1^2\sigma^2/2! + a_1^3\sigma^3/3! + a_1^4\sigma^4/4! + a_1^5\sigma^5/5! + \ldots\} + \\ + \boldsymbol{t}_2^0\{\varrho_2\sigma^2/2! + a_2^3\sigma^3/3! + a_2^4\sigma^4/4! + a_2^5\sigma^5/5! + \ldots\} + \\ + \boldsymbol{t}_3^0\{\varrho^3\sigma^3/3! + a_3^4\sigma^4/4! + a_3^5\sigma^5/5! + \ldots\} + \\ + \boldsymbol{t}_4^0\{\varrho_4\sigma^4/4! + a_4^5\sigma^5/5! + \ldots\} + \\ + \ldots\ldots\ldots\ldots\ldots\ldots + \\ + \boldsymbol{t}_j^0\{\varrho_j\sigma^j/j! + a_j^{j+1}\sigma^{j+1}/(j+1)! + \ldots\} + \\ + \boldsymbol{E},\end{aligned}$$

where $\boldsymbol{E}$ is a certain vector containing in the expansion higher powers than σ^n.

Obviously we can write

$$\begin{aligned}\boldsymbol{r}_j &= \sigma\boldsymbol{t}_1^0 + \{a_1^2\boldsymbol{t}_1^0 + \varrho_2\boldsymbol{t}_2^0\}\sigma^2/2! + \{a_1^3\boldsymbol{t}_1^0 + a_2^3\boldsymbol{t}_2^0 + \varrho_3\boldsymbol{t}_3^0\}\sigma^3/3! + \ldots + \\ &\quad + \Big\{\sum_{m=1}^{j-1} a_m^j\boldsymbol{t}_m^0 + \varrho_j\boldsymbol{t}_j^0\Big\}\sigma^j/j! + \sum_{i=j+1}^{n}\Big(\sum_{m=1}^{j} a_m^i\boldsymbol{t}_m^0\Big)\sigma^i/i! + \boldsymbol{E} \\ &= \sum_{m=1}^{j}\boldsymbol{u}_m\sigma^m/m! + \sum_{i=j+1}^{n}\Big(\sum_{m=1}^{j} a_m^i\boldsymbol{t}_m^0\Big)\sigma^i/i! + \boldsymbol{E}.\end{aligned}$$

Hence on the basis of (13) we have

$$\begin{aligned}\boldsymbol{R}_j = \boldsymbol{r} - \boldsymbol{r}_j &= \sum_{i=j+1}^{n}\Big[\boldsymbol{u}_i - \sum_{m=1}^{j} a_m^i\boldsymbol{t}_m^0\Big]\sigma^i/i! + \boldsymbol{e} - \boldsymbol{E} \\ &= \Big[\boldsymbol{u}_{j+1} - \sum_{m=1}^{j} a_m^{j+1}\boldsymbol{t}_m^0\Big]\sigma^{j+1}/(j+1)! + \boldsymbol{E}_j^*,\end{aligned}$$

where $\boldsymbol{E}_j^*$ contains terms of higher degree than σ^{j+1}. But on the basis of (12) we can write

$$\boldsymbol{u}_{j+1} - \sum_{m=1}^{j} a_m^{j+1}\boldsymbol{t}_m^0 = \sum_{m=1}^{j} a_m^{j+1}\boldsymbol{t}_m^0 + \varrho_{j+1}\boldsymbol{t}_{j+1}^0 - \sum_{m=1}^{j} a_m^{j+1}\boldsymbol{t}_m^0 = \varrho_{j+1}\boldsymbol{t}_{j+1}^0.$$

Thus we finally have

$$\boldsymbol{R}_j = \varrho_{j+1}\boldsymbol{t}_{j+1}^0\sigma^{j+1}/(j+1)! + \boldsymbol{E}_j^*.$$

Let us assume that $j \leqslant n-2$. Then we have

$$\begin{aligned}\delta_j/(\delta_0\,\delta_{j-1}) &= |\boldsymbol{R}_j|/(\overline{MM}_0|\boldsymbol{R}_{j-1}|) = |\boldsymbol{R}_j|/(|\boldsymbol{r}|\cdot|\boldsymbol{R}_{j-1}|) \\ &= \frac{|\sigma|^{j+1}\{\varkappa_1\varkappa_2\cdot\ldots\cdot\varkappa_j + \varepsilon_j\}\,j!}{(j+1)!\,|\sigma|^j\{\varkappa_1\varkappa_2\cdot\ldots\cdot\varkappa_{j-1} + \varepsilon_{j-1}\}\,|\sigma|\cdot[1+\varepsilon_0]} \\ &= \frac{\varkappa_1\varkappa_2\cdot\ldots\cdot\varkappa_j + \varepsilon_j}{(j+1)(1+\varepsilon_0)(\varkappa_1\varkappa_2\cdot\ldots\cdot\varkappa_{j-1} + \varepsilon_{j-1})},\end{aligned}$$

where ε_0, ε_{j-1}, ε_j are infinitely small with respect to σ. The last formula directly implies formula (5).

Let us now take $j=n-1$ and suppose that $\varkappa_{n-1}\neq 0$. In any case we have

$$\lim_{M\to M_0} \delta_{n-1}/(\delta_0\,\delta_{n-2}) = |\varkappa_{n-1}|/n.$$

This it only remains to show that δ^*_{n-1} and $\varkappa_{n-1}$ are of the same sign. From the expansion $\boldsymbol{r}=\sum_{i=1}^{n}\boldsymbol{u}_i\sigma^i/i!+\boldsymbol{e}$ and from formulas (12) for $\boldsymbol{u}_i$ it follows that the component of vector $\boldsymbol{r}$ in the direction of $\boldsymbol{t}_n^0$ is of the same sign as ϱ_n. But $\varrho_n=\varkappa_1\varkappa_2\cdot\ldots\cdot\varkappa_{n-1}$, and since $\varkappa_i>0$ for $i=1,2,\ldots,n-2$, therefore ϱ_n is of the same sign as $\varkappa_{n-1}$.

Thus formula (7) is proved.

It will be observed that formulas (1) and (2) are particular cases of formulas (5) and (7).

On the concept of the centre of the second curvature and on a generalization of a certain geometrical meaning of v. Lilienthal

by S. Gołąb (Kraków)

The concept of the first curvature of a skew curve has — as we know — many geometrical meanings. Among them there are two such that by means of certain geometrical constructions we directly obtain the centre of the first curvature as a certain point S_1 and only in the second place the curvature $\varkappa_1$ itself. In other interpretations (which may also serve as definitions) we first obtain the curvature $\varkappa_1$ and only then, after a suitable definition of the principal normal, the centre of the curvature S_1.

The above-mentioned two interpretations which directly lead to the obtaining of point S_1 are the following. In the first we consider the circle drawn through three neighbouring points of the curve C — the centre of the limiting position of the variable circle is the centre of the first curvature S_1. In the second interpretation we consider the polar straight line as the limiting position of the intersection of two neighbouring planes normal to the curve, and then the intersection point of the polar with the principal normal gives us point S_1.

Both interpretations may be generalized to the case of curves lying in an n-dimensional Euclidean space. The first needs no essential change. For the second we must first define the polar figure connected with a variable point M of curve C. We define that figure as the limiting position of the intersection of two hyperplanes normal to curve C at neighbouring points. The polar figure (belonging to point M) will be a certain $(n-2)$-dimensional linear space B_{n-2}. Projecting point M upon B_{n-2} (the concept of projecting being well-defined), we obtain the *centre of the first curvature* S_1.

The second curvature $\varkappa_2$ of skew curves in the three-dimensional space does not possess so many geometrical interpretations as the first curvature $\varkappa_1$, and in the second place, as far as *I* know, there exists only one interpretation which gives a certain uniquely determined point S_2 (connected with point M of the curve) called the *centre of the second curvature*. Other geometrical interpretations directly lead to $\varkappa_2$ and, moreo-