On the uniqueness of the non-negative solution of the homogeneous Cauchy problem for a system of partial differential equations

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In the present paper we shall deal with the problem of the (non-local) unicity of the non-negative solution of a system of partial differential equations of the form

(1) $$\partial u_i/\partial x = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{ijk}(x,y_1,\ldots,y_n)\,\partial u_j/\partial y_k + \sum_{j=1}^{m} b_{ij}(x,y_1,\ldots,y_n)\,u_j \qquad (i=1,2,\ldots,m)$$

satisfying the initial conditions

(2) $$u_i(0,y_1,\ldots,y_n)=0 \quad \text{for} \quad |y_k|\leqslant\beta \qquad (k=1,2,\ldots,n;\ \beta>0)$$

(Theorem 1). We shall also give a conclussion from theorem 1 for non-linear systems (Theorem 2).

THEOREM 1. *Let us assume that the coefficients* $a_{ijk}(x,y_1,\ldots,y_n)$, $b_{ij}(x,y_1,\ldots,y_n)$ *are measurable with respect to* $(y_1,\ldots,y_n)$[1]) *in the set* Z

$$0<x\leqslant\alpha \quad (\alpha>0), \qquad |y_k|\leqslant\beta \quad (k=1,2,\ldots,n)$$

and satisfy in this set for certain constants A, B, L *the following inequalities*:

(3) $$|a_{ijk}(x,y_1,\ldots,y_n)|\leqslant A \qquad (i,j=1,2,\ldots,m;\ k=1,2,\ldots,n),$$

(4) $$|a_{ijk}(\ldots,y_{k-1},y_k^{**},y_{k+1},\ldots)-a_{ijk}(\ldots,y_{k-1},y_k^{*},y_{k+1},\ldots)|\leqslant L|y_k^{**}-y_k^{*}| \qquad (i,j=1,2,\ldots,m;\ k=1,2,\ldots,n)$$[2]),

(5) $$|b_{ij}(x,y_1,\ldots,y_n)|\leqslant B \qquad (i,j=1,2,\ldots,m).$$

Under the above assumptions every solution $u_1(x,y_1,\ldots,y_n),\ldots,u_m(x,y_1,\ldots,y_n)$ *of system* (1) *of class* C^1 *in the pyramid*

1) The assumption of measurable has been introduced in order to avoid additional considerations.

2) Assumption (4) is satisfied in particular when a_{ijk} do not depend on y_k $(i,j=1,2,\ldots,m;\ k=1,2,\ldots,n)$.

$$R\,(0<x\leqslant\gamma=\min(\alpha,\beta/mA),\ |y_i|\leqslant\beta-mAx),$$

continuous in $\bar{R}$[3]) *and satisfying in pyramid* R *the inequalities*

(6) $$u_i(x,y_1,\ldots,y_n)\geqslant 0 \qquad (i=1,2,\ldots,m)$$

and conditions (2) *becomes identically zero in pyramid* R.

Remark 1. E. M. Landis in paper [2] has constructed an example of a system of form (1) (for $n=1$, $m=2$), with coefficients possessing a total differential, satisfying the assumption of our theorem with the exception of assumption (4) and admitting of a solution of class C^∞ satisfying conditions (2) and the inequalities

$$u_i(x,y_1)>0 \quad \text{for} \quad x\neq 0 \qquad (i=1,2).$$

It follows hence that condition (4) is essential.

Remark 2. In paper [3] we find an exemple of a system of form (1) with coefficients of class C^∞ (*i.e* satisfying also (4)), $(n=1,\ m=2)$, possessing a solution of class C^∞ satisfying conditions (2) and not becoming identically zero in any neighbourhood of the axis y. (This solution does not satisfy condition (6).) The example shows that the uniqueness of the solutions of system (1) satisfying condition (2) need not take place in the class of all functions of class C^1 (not necessarily satisfying inequalities (6)), even in the case of the coefficients of system (1) being of class C^∞. Under the assumption of the coefficients being analytic, the local uniqueness of the solutions of system (1) satisfying condition (2) in the class of all functions of class C^1 follows from the well-known theorem of Holmgren.

Proof. Let us introduce a section C_ξ of pyramid R by a plane $x=\xi$, given by the relations: $x=\xi$, $|y_i|\leqslant\beta-mA\xi$ $(i=1,2,\ldots,n)$ and sets P_ξ^j, Q_ξ^j bounding C_ξ, given by the relations:

$$P_\xi^j:\ x=\xi,\ |y_i|\leqslant\beta-mA\xi \quad (i=1,2,\ldots,j-1,j+1,\ldots,n) \quad y_j=-\beta+mA\xi,$$

$$Q_\xi^j:\ x=\xi,\ |y_i|\leqslant\beta-mA\xi \quad (i=1,2,\ldots,j-1,j+1,\ldots,n) \quad y_j=\beta-mA\xi.$$

Now let us introduce auxiliary (linear) operations, associating with the functions of the variables $x,y_1,\ldots,y_n$ the functions of the variable x given by the formulas

(7) $$H(f)=\overbrace{\int\limits_{C_x}\cdots\int}^{n} f(x,y_1,\ldots,y_n)\,dy_1\ldots dy_n,$$

3) By $\bar{R}$ we denote the closure of the set R.

$$V_j(f)=\overbrace{\int\dots\int}^{n-1}_{P_x^j} f(x,y_1,\dots,y_n)\,dy_1\dots dy_{j-1}\,dy_{j+1}\dots dy_n,$$

$$W_j(f)=\overbrace{\int\dots\int}^{n-1}_{Q_x^j} f(x,y_1,\dots,y_n)\,dy_1\dots dy_{j-1}\,dy_{j+1}\dots dy_n\ ^{4)}.$$

Let $u_1(x,y_1,\dots,y_n),\dots,u_m(x,y_1,\dots,y_n)$ be an arbitrary solution of system (1) of class C^1 in pyramid R, continuous in $\bar{R}$ and satisfying conditions (2) and inequalities (6).

Let us consider the function

(8) $$g(x)=\sum_{i=1}^{m} H(u_i).$$

It is easy to observe that

(9) $$dg(x)/dx=\sum_{i=1}^{m}\Big[H(\partial u_i/\partial x)-mA\sum_{k=1}^{n}\big(V_k(u_i)+W_k(u_i)\big)\Big]\quad \text{for}\quad 0<x<\gamma.$$

By (1) we have

(10) $$H(\partial u_i/\partial x)=H\Big(\sum_{j=1}^{m}\sum_{k=1}^{n}a_{ijk}\partial u_j/\partial y_k+\sum_{j=1}^{m}b_{ij}u_j\Big)$$
$$=\sum_{j=1}^{m}\sum_{k=1}^{n}H(a_{ijk}\partial u_j/\partial y_k)+\sum_{j=1}^{m}H(b_{ij}u_j).$$

On account of relation (4) the function $a_{ijk}(x,y_1,\dots,y_n)$ is absolutely continuous with respect to y_k. Therefore we can apply to the expression $H(a_{ijk}\partial u_j/\partial y_k)$ the theorem on integrating by parts on straight lines parallel to the axis y_k. We obtain

(11) $$H(a_{ijk}\partial u_j/\partial y_k)=W_k(a_{ijk}u_j)-V_k(a_{ijk}u_j)-H\big((\partial a_{ijk}/\partial y_k)u_j\big).$$

On the basis of relations (9), (10), (11) we have

(12) $$dg(x)/dx=\sum_{i=1}^{m}\sum_{j=1}^{m}H(b_{ij}u_j)-\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n}H\big((\partial a_{ijk}/\partial y_k)u_j\big)-$$
$$-\sum_{i=1}^{m}\sum_{k=1}^{n}mAV_k(u_i)-\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n}V_k(a_{ijk}u_j)-$$
$$-\sum_{i=1}^{m}\sum_{k=1}^{n}mAW_k(u_i)+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n}W_k(a_{ijk}u_j).$$

[4]) For $n=1$ we put $V_j(f)=f(x,-\beta+mAx)$, $W_j(f)=f(x,\beta-mAx)$.

By (5) and (6) the following inequality holds:

(13) $$|H(b_{ij}u_j)|\leqslant BH(u_j)\qquad (i,j=1,2,\dots,m)\ ^{5)}.$$

By (4) we have $|\partial a_{ijk}/dy_k|\leqslant L$, i.e. by (6)

(14) $$\big|H\big((\partial a_{ijk}/\partial y_k)u_j\big)\big|\leqslant LH(u_j)\qquad (i,j=1,2,\dots,m;\ k=1,2,\dots,n).$$

By (3) and (6) we obtain

$$|V_k(a_{ijk}u_j)|\leqslant AV_k(u_j),\qquad |W_k(a_{ijk}u_j)|\leqslant AW_k(u_j)$$

$(i,j=1,\dots,m;\ k=1,\dots,n)$ and hence

(15) $$\Big|\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n}V_k(a_{ijk}u_j)\Big|\leqslant mA\sum_{j=1}^{m}\sum_{k=1}^{n}V_k(u_j),$$

(16) $$\Big|\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n}W_k(a_{ijk}u_j)\Big|\leqslant mA\sum_{j=1}^{m}\sum_{k=1}^{n}W_k(u_j).$$

From relations (12), (13), (14), (15), (16) we obtain

$$dg(x)/dx\leqslant mB\sum_{j=1}^{m}H(u_j)+mnL\sum_{j=1}^{m}H(u_j)\qquad \text{for}\quad 0<x<\gamma,$$

therefore writing $C=mB+mnL$ and taking into account (8) we obtain the inequality

$$dg(x)/dx\leqslant Cg(x)\qquad \text{for}\quad 0<x<\gamma.$$

By (8), (7), (2) we have $g(0)=0$.

The function $G(x)=\exp(-Cx)g(x)$ is continuous for $0\leqslant x\leqslant\gamma$ and has the following properties:

$$G(0)=0,$$

$$dG(x)/dx=-C\exp(-Cx)g(x)+\exp(-Cx)dg(x)/dx$$
$$\leqslant -C\exp(-Cx)g(x)+\exp(-Cx)Cg(x)=0\qquad \text{for}\qquad 0<x<\gamma.$$

Therefore $G(x)\leqslant 0$ for $0\leqslant x\leqslant\gamma$ i.e. also $g(x)\leqslant 0$ for $0\leqslant x\leqslant\gamma$.

By (8), (7) we have hence

$$\sum_{i=1}^{m}\overbrace{\int\dots\int}^{n}_{C_x} u_i(x,y_1,\dots,y_n)\,dy_1\dots dy_n\leqslant 0\qquad \text{for}\qquad 0\leqslant x\leqslant\gamma$$

and hence by (6)

$$u_i(x,y_1,\dots,y_n)=0\qquad \text{for}\quad 0\leqslant x\leqslant\gamma,\quad |y_k|\leqslant\beta-mAx\qquad (i=1,2,\dots,m)$$

q. e. d.

[5]) In relation (13) and the succeeding relations symbols $|H(b_{ij}u_j)|$, $H(u_j)$, $V_k(u_j)$ etc. represent functions of one variable x.

Remark 3. Applying the transformation $\tilde{u}_j = e_j u_j$, $e_j = \pm 1$, we obtain an analogical theorem concerning the cases where some of the functions u_j are non-positive and the remaining functions u_j are non-negative.

THEOREM 2. *Let us assume that the functions*

$$F_i(x, y_1, \ldots, y_n, u_1, \ldots, u_m, q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{mn}) \qquad (i = 1, 2, \ldots, m)$$

are of class C^2 *in the set* S:

$$0 \leqslant x \leqslant a, \qquad |y_k| \leqslant \beta \qquad (k = 1, 2, \ldots, n;\ a > 0,\ \beta > 0)$$

and satisfy in this set the inequalities

$$|\partial F_i(x, \ldots, q_{mn}) / \partial q_{jk}| \leqslant A \qquad (i, j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n).$$

Let the systems of functions $u_1^*(x, y_1, \ldots, y_n), \ldots, u_m^*(x, y_1, \ldots, y_n)$ and $u_1^{**}(x, y_1, \ldots, y_n), \ldots u_m^{**}(x, y_1, \ldots, y_n)$ be of class C^2 in $\bar{R}$ and let them satisfy in pyramid R the system of partial differential equations

$$\frac{\partial u_i}{\partial x} = F_i\left(x, y_1, \ldots, y_n, u_1, \ldots, u_m, \frac{\partial u_1}{\partial y_1}, \ldots, \frac{\partial u_1}{\partial y_n}, \frac{\partial u_2}{\partial y_1}, \ldots, \frac{\partial u_m}{\partial y_n}\right)$$

$$(i = 1, 2, \ldots, m)$$

and the inequalities

$$u_i^{**}(x, y_1, \ldots, y_n) \geqslant u_i^*(x, y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, m).$$

Moreover, let

$$u_i^{**}(0, y_1, \ldots, y_n) = u_i^*(0, y_1, \ldots, y_n) \quad \text{for} \quad |y_k| \leqslant \beta$$

$$(k = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, m).$$

Under the above assumptions we have the following identities in pyramid R:

$$u_i^{**}(x, y_1, \ldots, y_n) = u_i^*(x, y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, m).$$

Theorem 2 can be reduced to theorem 1 by applying Hadamard's lemma (see [1], p. 352-354).

References

[1] J. Hadamard, *Leçons sur la propagation des ondes*, Paris 1903.

[2] Е. М. Ландис, *Пример неединственности решения задачи Коши для сыстемы вида* $\partial u_i/\partial x = \sum_j A_{ij}\partial u_j/\partial x + \sum_j B_{ij} u_j + f_i$ $(i, j = 1, 2)$, Математич. сборник 27 (69) 1950, p. 319-323.

[3] A. Pliś, *The problem of uniqueness for the solution of a system of partial differential equations*, Bull. Acad. Polon. Sci., Cl. III, 2 (1954), p. 55.

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