$$(57) \qquad -\pi\mu(s) + \int_L \sin\varphi_{s\sigma} r_{s\sigma}^{-1} \mu(\sigma)\, d\sigma + \\ + a(s)\int_L \log r_{s\sigma}^{-1} \mu(\sigma)\, d\sigma + b(s)\int_L \cos\varphi_{s\sigma} r_{s\sigma}^{-1} \mu(\sigma)\, d\sigma = f(s)$$

bien étudiée, admet la solution *unique* μ pour *toute* la fonction $f(s)$ vérifiant la condition d'Hölder, ce qui a lieu si l'équation homogène correspondante n'a que la solution *nulle* et si l'*index* de l'équation (57) est égal à zéro. Sous cette hypothèse, les considérations qui suivront seront analogues aux précédentes et on concluera l'existence de la solution du problème (55), si la valeur absolue du paramètre λ est suffisamment petite.

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The problem of non-local existence for solutions of a linear partial differential equation of the first order

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E. Kamke has shown that the partial differential equation

$$(1) \qquad \partial z/\partial x + Q(x,y)\,\partial z/\partial y = 0$$

with a coefficient $Q(x,y)$ of class C^1 [1]) in a certain (open) region D admits in every closed and bounded subset of D a solution of class C^1, possessing a positive derivative with respect to y [2]). The problem of non-local existence of non-trivial [3]) solutions of class C^1 has been solved in the negative by T. Ważewski, who furnishes an example of a differential equation of form (1) such that each of its solutions of class C^1 in the whole region D is a constant function [4]). In this example the region D, constituting the domain of the function $Q(x,y)$, is simply connected, and $Q(x,y)$ is of high regularity in D.

In this paper we shall consider the problem of non-local existence of non-trivial solutions of equation (1), having a total differential (in the sense of Stolz) in region D. For the open simply connected region D we shall prove the existence of a solution having a total differential at every point of D and such that its derivative with respect to y is positive nearly everywhere in the set D (§ 1. Theorem 1). Consequently, such a solution

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[1]) A function continuous together with its derivatives of the first order is termed a function of class C^1.

[2]) An analogous theorem is also known for the equation

$$\partial z/\partial x + \sum_{i=1}^{n} Q_i(x, y_1, y_2, \ldots, y_n)\,\partial z/\partial y_i = 0$$

with a larger number of independent variables. The proof (in the case of two variables) is to be found in [1].

[3]) By a non-trivial solution of the equation (1) we mean a solution which is not identically equal to a constant.

[4]) See [6]. An example of such an equation defined over the whole plane is to be found in [5].

will be increasing with respect to y in every subset of D, normal with respect to the x-axis. By introducing small changes into the construction of the solution one may prove the existence of a solution satisfying the assertion of theorem 1 and such that its derivatives are bounded in every compact subset of region D.

Under the weaker assumption that D is a finitely connected set we shall prove the existence of a solution $z(x,y)$ which has a total differential and which is not a constant in any open subset of D (§ 2 Theorem 2). The components of the set $z(x,y)=z(x_0,y_0)$ are in this case integrals of the ordinary differential equation

$$\partial y/\partial x = Q(x,y). \tag{2}$$

The function $z(x,y)$ may, therefore, be used to find the integrals of (2). Moreover, we shall give an example of an ordinary differential equation of form (2) with the right side $Q(x,y)=Q^*(x,y)$ of class C^∞ in a region D, admitting an integral of ramification (see definition 4, § 3) dense everywhere in D (the set of components of the complement of D is in this example of the power of the continuum). Consequently, each function $z(x,y)$ continuous in region D and constant along the integrals of (2) (and thus constant on an arbitrary integral of ramification; see § 3) and, in particular, an arbitrary function $z(x,y)$ satisfying (1) (with $Q(x,y)=Q^*(x,y)$) and having a total differential at each point of D, is identically equal to a constant in the set D.

An analogous example of a system of two ordinary differential equations may be defined even in the whole space E_3, *i. e.* in a simply connected set [4]. The right sides of this system, $Q_1(x,y_1,y_2)$, $Q_2(x,y_1,y_2)$, may even be of class C^∞ in the entire E_3. Just, as in the above example, each solution of the equation

$$\partial z/\partial x + Q_1(x,y_1,y_2)\,\partial z/\partial y_1 + Q_2(x,y_1,y_2)\,\partial z/\partial y_2 = 0,$$

having a total differential at each point in E_3, is identically equal to a constant in E_3. This example, when compared with theorem 1, shows that the properties of non-local solutions of the partial differential equation

$$\partial z/\partial x + \sum_{i=1}^{k} Q_i(x,y_1,y_2,\ldots,y_k)\,\partial z/\partial y_i = 0$$

are different for $k=1$ and different for $k\geqslant 2$.

§ 1. THEOREM 1. *If the function $Q(x,y)$ is of class C^1 in the simply connected open region D, there exists a solution of equation (1) having a total differential in this region and such that its derivative with respect to y is positive nearly everywhere in the set D.*

Before constructing such a solution, let us introduce the following definition:

Definition 1. By *the emission zone of the set Z with respect to the ordinary differential equation R* we denote a set of points lying on the integrals of R which have at least one point in common with Z.

Now let us consider an arbitrary segment T contained in the region D, and parallel to the y-axis, and its emission zone S with respect to equation (2). Each integral of (2), contained in S, intersects T exactly at one point. Hence, we may define in S a constant function along the integrals by defining it arbitrarily on T. Since the function, constant along the integrals of (2) and having a total differential in a certain open set, is the solution of equation (1), the determining of the solution of (1) in the interior of the set S is not difficult. We shall construct the decomposition of the region D into the above kind of disjoint emission zones of segments T_i. For this purpose let us prove the following lemma.

LEMMA 1. *For the simply connected region D and equation* (2), *defined in it, with the right side of class C^1 there exists a sequence of segments $T_n(n=0,1,2,\ldots)$ belonging to D, $T_0\,(x=x_0, h_0<y<g_0)$, $T_n\big(x=x_n, y\in[h_n,g_n)\big)$*[5] *$(n=1,2,3,\ldots)$ such that*

1° $D=\sum_{n=0}^{\infty} S_n$ *where S_n denotes the emission zone of segment T_n with respect to equation* (2);

2° *the sets S_n are disjoint;*

3° *to the point (x_n,h_n) $(n=1,2,3,\ldots)$ corresponds a point (x_{i_n},v_n) belonging to $\overline{T}_{i_n}$*[6]*) $(i_n<n)$ such that if the integrals $y_m(x)$ $(m=1,2,\ldots)$ do not meet segment T_n, i.e. $\big(x_n,y_m(x_n)\big)\,\overline{\in}\,T_n$, and if, moreover $\lim_{m\to\infty} y_m(x_n)=h_n$, then, for sufficiently large m, $y_m(x)$ are defined for $x=x_{i_n}$ and $\lim_{m\to\infty} y_m(x_{i_n})=v_n$.*

(The geometrical significance of this assumption is illustrated in Fig. 1. It will be further clarified in the proof.)

The proof of lemma 1. Let T be a segment (not reduced to a point), parallel to the y-axis, and contained in region D. We shall prove that the emission zone S of T with respect to equation (2) has the following property:

Property W. A set Z has the property W if the following implication takes place. If $(x^*,y^*)\in\overline{Z}\cdot D$, there existe a positive number r

[5]) $y\in[h_n,g_n)$ means that in the case of $h_n<g_n$ we have $h_n\leqslant y<g_n$, and in the case of $h_n>g_n$ we have $g_n<y\leqslant h_n$.

[6]) $\overline{A}$ denotes the closure of the set A.

such that either the segment $x=x^*, y^*<y<y^*+r$ or the segment $x=x^*$, $y^*-r<y<y^*$ is contained in Z.

For the proof let us consider an arbitrary point $(x^{\sim}, y^{\sim})$ of the set $\bar{S}\cdot D$. Let K be a segment not reduce to a point, belonging to D, aod parallel to the y-axis with centre at $(x^{\sim}, y^{\sim})$. It is easily seen that on this segment there is a point $(x^{\sim}, y^{\approx})$ of the set S, differing from $(x, y^{\approx})$ [7]. Let us assume that $y^{\approx}<y^{\sim}$, because in the case of $y^{\approx}>y^{\sim}$ the proof is analogous. To prove property W, it suffices to show that the points $(x^{\sim}, y)$ belong to S for $y^{\approx}<y<y^{\sim}$. For indirect proof let us assume that the point $(x^{\sim}, q)$ $y^{\approx}<q<y^{\sim}$ does not belong to S. In this case the whole integral passing through this point does not belong to S. On account of the simple connectivity of D and of the assertion that the integral reaches with its ends to the boundary [8]), this integral divides region D into two parts. The set S, being connected, is entirely contained in the part to which the point $(x, y^{\approx})$ belongs, and therefore the point $(x^{\sim}, y^{\sim})$ does not belong to $\bar{S}$. Thus, it has been proved that S has the property W.

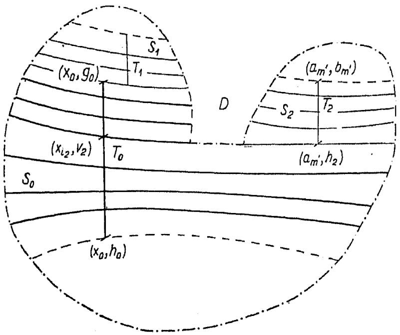

Fig. 1

Now let us consider then open subset G of D, composed of integrals [9]) (of equation (2)) and having the property W. Let the segment T $(x=a, y\in[h,g))$ be contiguous to the set G (*i. e.* let it belong to the difference $D-G$, and let the point (a,h) belong to $\bar{G}$). We assert that the set $G+S$, where S is the emission zone of the segment T, is an open set. Let us assume in addition that $h<g$. (In the case of $h>g$ the argumentation is analogous). On account of the property W and the inclusion $T\subset D-G$, there exists an r^* such that the segment $x=a$, $h-r^*<y<h$ belongs to the set G, and thus, denoting by $S^{\sim}$ the emission zone of the

[7]) In the case of $(x^{\sim}, y^{\sim})\in S$, it follows from the theorem on the continuous dependence of the integral on initial conditions, because a segment that is not reduced to a point is a set dense in itself.

[8]) A point in infinity is considered as belonging to the frontier of an unbounded region.

[9]) *i. e.*, if the point P belongs to G, the whole integral passing through P is contained in G.

segment $x=a$, $h-r^*<y<g$, we obtain $G+S=G+S^{\sim}$. Moreover, S, being the emission zone of an open segment, is an open set; therefore $G+S$ is indeed an open set. $G+S$ has, besides, the property W since as follows directly from the definition of W, the sum of a finite number of sets having the property W has this property. The above argumentation may also be applied to $G+S$ and the contiguous segment T^* to obtain an open set $G+S+S^*$ having the property W. Proceeding in this way we obtain a sequence of open sets, having the property W. If, moreover, the set G is connected, then all the sets obtained are connected. These remarks will now be utilized in the construction of the sequence T_i appearing in the formulation of lemma 1.

Let (a_1, b_1), $(a_2, b_2), \ldots$ be a sequence of points of the region D, everywhere — dense in this region (*e. g.* the sequence of points with rational coordinates of the set D) and $T_0(x=x_0, h_0<y<g_0)$ a segment contained together with its end-points in D. Let $(a_{\bar{m}}, b_{\bar{m}})$ be the first point of the sequence (a_m, b_m) that does not belong to $\bar{S}_0$ but can be connected with S_0 [10]) by a segment [11]) parallel to the y-axis and contained in D. Let us denote by $T_1(x=a_{\bar{m}}, y\in[h_1, b_{\bar{m}}))$ a segment having no points in common with S_0 and such that $(a_{\bar{m}}, h_1)\in.\bar{S}_0$ The segments $T_0, T_1, \ldots, T_k$ being analogously defined (Fig. 1), let us consider the first point of the sequence (a_m, b_m) not belonging to the set $\overline{S_0+S_1+\ldots+S_k}$ and connectible with the set $S_0+S_1+\ldots+S_k$ by a segment parallel to the y-axis and contained in D. Let this point be $(a_{m'}, b_{m'})$. By T_{k+1} $(x=a_{m'}, y\in[h_{k+1}, b_{m'}))$ we denote the segment belonging to D, having no poits in common with the set $S_0+S_1+\ldots+S_k$ and such that $(a_{m'}, h_{k+1})\in\overline{S_0+S_1+\ldots+S_k}$. It is possible because, as follows from the preceeding remarks, the set $S_0+S_1+\ldots+S_k$ is open.

It may easily be seen that property 2⁰ of this lemma is fulfilled. Relation 3⁰ of this lemma follows, as can easily be shown, from the property W of the sets S_i. For the indirect proof of relation 1⁰ let us assume that the set $D_0=\sum_{m=1}^{\infty} S_m$ is a proper part of the region D. Hence there exists a boundary point $P_0(x_0, y_0)$ of the set D_0, belonging to the region D. Let P' be a point of D_0 so close to P_0 that the integral passing through P' is defined in the neighbourhood x_0 and can be connected with P_0 by a segment parallel to the y-axis and contained in D. In view of the definition of the set D_0, there exists an N such that $P'\in S_N$. In view of

[10]) The emission zone S_0 of the segment T_0 parallel to the y-axis and not containing the end-points is an open set.

[11]) *i. e.* there exists a segment τ such that $(a_{\bar{m}}, b_{\bar{m}})\in\tau$, $\tau\in D$ and the sum of sets τ and S_0 is a connected set.

the simple connectivity of D and the theorem of the integral reaching the boundary with its ends, the integral passing through P_0 divides the region D into two open parts. That part to which P' belongs we denote by D_1, the other by D_2. The set D_0 has no points in common with the integral passing through P_0 (otherwise, since D_0 is composed of emission zones, the point P_0 would belong to the open set D_0), and, since it is a connected set, it belongs entirely to the set D_1. Let $(a_{\tilde{m}}, b_{\tilde{m}})$ be a point of the sequence (a_m, b_m) lying in D_2 so close to P_0 as to be connectible with the integral passing through P' by a segment contained in the region D and parallel to the y-axis. Then, as is seen from the definition of the sequence S_n; $(a_{\tilde{m}}, b_{\tilde{m}})$ belongs to the set

$$\overline{S_0+S_1+S_2+\ldots+S_{\tilde{m}+N}},$$

and thus also to the set $\bar{D}_0$, which is at variance with D_0 being contained in D_1, because $(a_{\tilde{m}}, b_{\tilde{m}})$ belongs to D_2 and the product $\bar{D}_1\cdot D_2$ is an empty set. Thus the proof of lemma 1 has been completed.

Using the decomposition of the region D into the sets S_i, we shall give in lemma 2 the definition of a continuous function in D with a derivatives with respect to y, positive and continuous in the set $G=\sum_{i=1}^{\infty} I(S_i)$, where $I(S_i)$ denotes the interior of S_i.

LEMMA 2. *For the simply connected region D and equation* (2), *defined in it, with the right side of class C^1 there exists a function $z(x,y)$ and an open set G, $G\subset D$, such that*

I) *the set $D-G$ is composed of a countable number of integrals,*
II) *the function $z(x,y)$ is of class C^1 in the set G,*
III) *the derivative $\partial z(x,y)/\partial y$ is positive in G,*
IV) *the function $z(x,y)$ is continuous in D,*
V) *the function $z(x,y)$ is constant along the integrals of equation* (2).

The proof of lemma 2. Let $f_0(y), f_1(y), f_2(y), \ldots$ be a sequence of bounded functions having continuous and positive derivatives, for $-\infty<y<+\infty$ and satisfying the conditions:

$$f_n(h_n)=f_{i_n}(v_n) \quad \text{for} \quad n=1,2,3,\ldots \tag{C}$$

We define the function $z(x,y)$ in the set D assuming that $z(x,y)=f_i(y)$ for $(x,y)\in T_i$ and that $z(x,y)$ constant along the integrals of equation (2). In view of property 2^0 of lemma 1 this definition is correct and in view of property 1^0 it is valid for the whole region D. The function $z(x,y)$ thus defined is expressed in every set S by the formula: $z(x,y)=f_i\big(g(x_i;x,y)\big)$

where $g(\xi;x,y)$ denotes the coordinate y for $x=\xi$ of the integral of (2) passing through the point (x,y). The function $g(x_i;x,y)$ is of class C^1 in the set S_i and has in this set a positive derivative with respect to y ([2], p. 155). Therefore, $z(x,y)$ is also of class C^1 in (the interior of) S_i and has in it a positive derivative with respect to y. Assuming that $G=S_0+I(S_1)+I(S_2)+\ldots$, we obtain properties II), III), property V) following directly from the definition, and IV) from the condition (C) and property 3^0 of lemma 1. Property I) results from the set $D-G$ being composed of integrals passing through the points (x_n, h_n) $(n=1,3,\ldots)$ The proof of lemma 2 is thus completed.

Remark. We shall match with the function $z(x,y)$ a suitable function $l(t)$ in such a manner that the compound function $z^*(x,y)=l\big(z(x,y)\big)$ will have a total differential at each point of the region D. The function $z^*(x,y)=l\big(z(x,y)\big)$ will satisfy the assertion of theorem 1. Such a function $l(t)$ will be given in lemma 3.

LEMMA 3. *For every positive function $g(s)$ weakly increasing*[12]) *for positive s belonging to a sufficiently small right-hand neighbourhood of zero, and for an arbitrary sequence of real numbers $t_1, t_2, t_3, \ldots$, there exists a sequence of positive numbres $c_i\leqslant 1$ and a function $l(t)$ differentiable at each point of the interval $(-\infty,+\infty)$, and such that its derivative is positive outside at most the set of measure zero*[13]), *for which the following relations hold:*

$$|l(t)-l(t_i)|\leqslant g(|t-t_i|) \quad \text{for} \quad 0<|t-t_i|\leqslant c_i. \tag{3}$$

Without limiting the generality of this lemma, we may, as is easily seen, additionally assume that the function $g(t)$ is continuous in the interval $[0,1]$, $g(t)\leqslant 1$ and that

$$g(0)=0, \tag{4}$$

because the relation (3) are inequalities of local character and with the function $g(t)$ we may match a function $h(t)$ so that in the right-hand neighbourhood of zero $h(t)\leqslant g(t)$, and $h(0)=0$, with $h(t)$ strongly increasing and continuous. Also without limiting generality, we may assume in the proof that the numbers t_i are distinct. Let us observe that in order that the function $k(t)$ be the derivative of the desired function it suffices that it have the following properties:

$$k(t)>0 \quad \text{nearly everywhere,} \tag{5}$$

[12]) *i. e.* $g(s_1)\geqslant g(s_2)$ for $s_1\geqslant s_2$.

[13]) The function constructed will be of class C^1 except the closure of the set of numbers t_i.

(6) $k(t)$ upper semi-continuous at the interval $(-\infty, +\infty)$,

(7) $$\underline{\lim}_{t\to t^*}\Big[\int_{t^*}^{t} k(s)\,ds\,(t-t^*)^{-1}\Big] \geqslant k(t^*) \quad \text{for every } t^*,$$

(8) there exists a sequence of positive numbers c_i such that

$$k(t)\leqslant g(|t-t_p|) \quad \text{for } |t-t_p|\leqslant c_p \quad (p=1,2,3,\ldots)^{14})$$

Indeed, let us suppose that $k(t)$ has the properties (5), (6), (7), (8). We shall prove that for the function $l(t)=\int_0^t k(s)\,ds$ the identity $l'(t)=k(t)$ holds and that this function satisfies the assertion of lemma 2.

At first we shall show that the derivative of the function $l(t)$ exists for every value of t and is equal to $k(t)$. For this purpose let us put arbitrarily $t=t^*$. It follows from relation (6) that

$$\overline{\lim}_{t\to t^*}\frac{l(t)-l(t^*)}{t-t^*} = \overline{\lim}_{t\to t^*}\frac{\int_{t^*}^{t} k(s)\,ds}{t-t^*} \leqslant k(t^*)$$

whereas relation (7) denotes that

$$\underline{\lim}_{t\to t^*}\frac{l(t)-l(t^*)}{t-t^*} = \underline{\lim}_{t\to t^*}\frac{\int_{t^*}^{t} k(s)\,ds}{t-t^*} \geqslant k(t^*),$$

hence indeed $l'(t^*)=k(t^*)$. By (5) it follows that $l'(t)$ is positive nearly everywhere. On the basis of (8) we obtain

$$|l(t)-l(t_i)| = |t-t_i|\,|l'(\tau)| \leqslant |t-t_i|\,g(|\tau-t_i|)$$

where τ lies between t, t_i, *i. e.* $|\tau-t_i|<|t-t_i|$; hence, on the basis of the assumption that $g(s)$ is increasing and since $c_i\leqslant 1$,

$$|l(t)-l(t_i)|\leqslant g(|t-t_i|) \quad \text{for } |t-t_i|\leqslant c_i.$$

The construction of the function $k(t)$ having the properties (5)-(8), we shall reduce to the conctruction of a sequence of functions $k_i(t)$, tending to $k(t)$, and of two sequences of positive numbers c_i, d_i. To obtain in the limit the relation (5)-(8) we choose the function $k_i(t)$ and the positive numbers c_i, d_i such that for $i=1,2,\ldots$ the following properties hold:

(9) $$k_i(t)>0 \quad \text{for } t\neq t_1,\ t\neq t_2,\ \ldots,\ t\neq t_i,$$

[14]) Additionally assuming (which also does not limit the generality of the lemma), that the function $g(s)$ is of class C^1 and $g(0)=g'(0)=0$, $g'(s)>0$ for $s>0$, one might replace the function $g(s)$ in (8) by its derivative.

(10) $$k_i(t)=k_{i-1}(t) \quad \text{for } |t-t_i|\geqslant 1/2^i \quad (i\geqslant 2),$$

(11) $$k_i(t)\leqslant k_{i-1}(t) \quad \text{for } i\geqslant 2,$$

(12) $k_i(t)$ uniformly continuous for $-\infty<t<+\infty$,

(13) $$\inf_{0<|t-t^*|<d_p}\frac{\int_{t^*}^{t} k_i(s)\,ds}{t-t^*} > k_i(t^*)-\frac{1}{p}, \quad p=1,2,\ldots,i \quad \text{for arbitrary } t^*,$$

(14) $$k_i(t)\leqslant g(|t-t_j|) \quad \text{for } |t-t_j|\leqslant c_j \quad (j=1,2,3,\ldots,i),$$

(14a) $$k_i(t)\leqslant 1.$$

It is easily seen that the function $k(t)=\lim k_i(t)$ satisfies relations (5)-(8), namely relation (5) in consequence of (9), (10), relation (6) in consequence of (11), (12)[15]), relation (7) in consequence of (13), and relation (8) in consequence of (14).

Before constructing the sequence of functions $k_i(t)$, we shall prove that for arbitrary positive numbers δ, ε and for an arbitrary real number q there exist a function $h(t)$ and a positive number r which have the following properties:

(15) $h(t)$ is continuous for $-\infty<t<+\infty$,

(16) $$h(t)\geqslant 1 \quad \text{for } |t-q|\geqslant\delta,$$

(17) $$h(t)=g(|t-q|) \quad \text{for } |t-q|\leqslant r,$$

(18) $$h(t)>0 \quad \text{for } t\neq q,$$

(19) the set of numbers s satisfying the relations

(Z) $$h(s)\geqslant h(t^*)-\varepsilon, \quad s\in[t,t^*],$$

where t,t^* are arbitrary numbers ($t<t^*$ or $t\geqslant t^*$) has a measure not smaller than $(1-\varepsilon)|t-t^*|$.

In the proof we assume that $q=0$. The general case is reducible to this one by changing the variable $t=\tilde{t}+q$. For the proof let us denote by $a(t)$ the function

$$a(t)=1-\varepsilon\left(\ln\frac{\varepsilon}{2+\varepsilon}\right)^{-1}\ln\frac{|t|}{\delta}$$

and by $g^*(s)$ the function $g^*(s)=\min\big(g(s),\varepsilon\big)$ for $0\leqslant s\leqslant 1$, $g^*(s)=\varepsilon$ for $s>1$.

[15]) On the basis of the theorem, on the decreasing sequence of continuous functions, see *e. g.* [3], p. 354, theorem 9.

We shall show that the function $h(t)=\max(a(t),g^*(|t|))$ for $t\neq 0$, $h(0)=0$ has properties (15)-(19). In view of the definition of $h(t)$ and equality (4), relation (17) follows from $g^*(s)=g(s)$ in a certain right-hand neighbourhood of zero and from $\lim_{t\to 0} a(t)=-\infty$. (As number r we may take $\min(r_1,r_2)$, where r_1 is the radius of the neighbourhood of zero in which $a(t)$ does not assume positive values, and r_2 is the radius of the neighbourhood in which $g^*(s)=g(s)$.) Relation (18) follows from the inequality $g^*(t)>0$ for $t>0$, and relation (15) from (17) and from the fact that the function $a(t)$ is continuous for $t\neq 0$. The validity of relation (16) follows from the inequality

$$a(t)\geqslant 1 \quad \text{for} \quad |t|\geqslant\delta \qquad \left(\text{because } \ln\frac{\varepsilon}{2+\varepsilon}<0\right).$$

In order to prove relation (19) let us observe that the function $a(t)$ fulfils functional equation

(F) $$a\left(\frac{\varepsilon}{\varepsilon+2}t\right)=a(t)-\varepsilon \qquad (\text{for} \quad t\neq 0).$$

From (F), from the evenness of the function $a(t)$, and from the fact that the function $a(t)$ increases for $t>0$ follows the validity of the inequalities

(N) $$a(s)\geqslant a(t^*)-\varepsilon \quad \text{for} \quad |s|\geqslant\frac{\varepsilon}{2+\varepsilon}|t^*|, \quad t^*\neq 0.$$

We shall prove now that $h(t)$ satisfies the inequality

(N′) $$h(s)\geqslant h(t^*)-\varepsilon \quad \text{for arbitrary } t^*,\ |s|\geqslant\frac{\varepsilon}{2+\varepsilon}|t^*|.$$

Therefore we shall distinguish two cases:

I. $h(t^*)=g^*(|t^*|)$.

In this case $h(t^*)-\varepsilon=g^*(|t^*|)-\varepsilon\leqslant 0$, since $g^*(s)\leqslant\varepsilon$; therefore (N′) is fulfilled in consequence of relation (18) (even for every s).

II. $h(t^*)\neq g^*(|t^*|)$, then $h(t^*)=a(t^*)$,

and hence on the basis of (N) we have

$$a(s)\geqslant h(t^*)-\varepsilon \quad \text{for} \quad s\geqslant\frac{\varepsilon}{2+\varepsilon}|t^*|.$$

Since $h(s)\geqslant a(s)$, we obtain the relation (N′).

To obtain relation (19) from (N′) we distinguish two cases:

I. $$|t-t^*|<\left(1-\frac{\varepsilon}{2+\varepsilon}\right)|t^*|.$$

In this case we have for $s\in[t,t^*]$ the inequality

$$|s-t^*|\leqslant|t-t^*|\leqslant\left(1-\frac{\varepsilon}{2+\varepsilon}\right)|t^*|,$$

and thus $|s|\geqslant|t^*|-|s-t^*|\geqslant|t^*|\varepsilon/(2+\varepsilon)$, *i. e.* (Z) holds in view of (N′) over the whole interval $[t,t^*]$.

II. $$|t-t^*|\geqslant 1-\frac{\varepsilon}{2+\varepsilon}|t^*|=\frac{2}{2+\varepsilon}|t^*|.$$

The set of points of the interval $[t,t^*]$, for which the inequality (Z) does not hold, has, in view of (N′), a measure not larger than $2|t^*|\varepsilon/(2+\varepsilon)$. Since is this case $|t-t^*|\geqslant 2|t^*|/(2+\varepsilon)$ the measure of points of the interval $[t,t^*]$ for which (Z) does not hold is not larger than $\varepsilon|t-t^*|$. In other words, the set of numbers of the interval $[t,t^*]$ for which (Z) holds has a measure not smaller than $(1-\varepsilon)[t-t^*]$. The existence of the function $h(t)$ (and the positive number r) satisfying the relations (15)-(19) is thus proved. The existence of the sequence $k_i(t)$ and the sequences of positive numbers c_i,d_j satisfying relations (9)-(14a) will be proved by induction.

Let $k_1(t)=1$ for $|t-t_1|\geqslant 1$, $k_1(t)=g(|t-t_1|)$ for $|t-t_1|\leqslant 1/2$. Now, in the interval $t_1-1\leqslant t\leqslant t_1-1/2$ and in the interval $t_1+1/2\leqslant t\leqslant t_1+1$ let $k_1(t)$ be a linear function.

Moreover, let us take $c_1=1/2$; obviously relations (9)-(12) and relations (14) and (14a) are thus fulfilled. Then we choose d_1 so that (13) is also fulfilled. For this purpose it suffices to choose d_1 so that $|k_1(t)-k_1(t^*)|\leqslant 1/2$ for t,t^* satisfying the inequality $|t-t^*|\leqslant d_1$, which is possible because of the uniform continuity of the function $k_1(t)$.

Suppose now that we have defined the functions $k_1(t),k_2(t),\ldots,k_{n-1}(t)$ and the positive numbers $c_1,c_2,\ldots,c_{n-1},d_1,d_2,\ldots,d_{n-1}$ in such a way that relations (9)-(14a) are fulfilled. We have to define $k_n(t)$ and c_n,d_n so that (9)-(14a) should be fulfilled for $i=n$. This end let us introduce the following notation:

(20) $$\varepsilon_0=\frac{1}{5}\min_{p=1,2,\ldots,n-1}\Big[\inf_{0<|t-t^*|\leqslant d_p}\Big[\int_{t^*}^{t}k_{n-1}(s)\,ds/(t-t^*)-k_{n-1}(t^*)+\frac{1}{p}\Big]\Big],$$

$$(21) \qquad q_0 = t_n.$$

By δ_0 let us denote a positive number smaller than $1/2^n$ chosen for ε_0 so that

$$(22) \qquad |k_{n-1}(t) - k_{n-1}(t_n)| \leqslant \varepsilon_0 \quad \text{for} \quad |t - t_n| \leqslant \delta_0.$$

(There exists such a number in view of the continuity of $k_{n-1}(t)$.) Let $h_0(t)$ be a function having properties (15)-(19) for $\delta = \delta_0$, $\varepsilon = \varepsilon_0$, $q = q_0$.

After these preparations we define (see Fig. 2)

$$(23) \qquad k_n(t) = \min\big(k_{n-1}(t), h_0(t)\big).$$

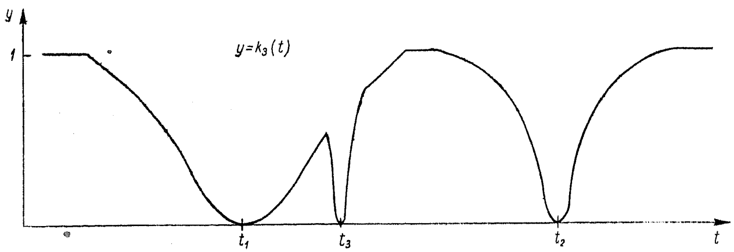

Fig. 2

For the function $k_n(t)$ relation (9) follows from the inductive assumptior and relation (18), relation (10) from (16) and the validity of (14a) fo $i = n-1$ in view of $\delta_0 \leqslant 1/2^n$, and relation (11) directly from the definition. Property (12) follows from the inductive assumption, the validity of (10) for $i = n$, and from (15). Relation (14a) is the result of the inductive assumption and of (11). Putting $c_n = r$, we obtain from (17) relation (14) for $j = n$, for the remaining j (14) is a consequence of the inductive assumption and of (11).

Now we shall prove the validity of (13) for $i = n$ and for $p = 1, 2, \ldots, n-1$ (the case $p = n$ will be considered later). For this purpose it suffices to show that the function $f(t) = k_n(t) - k_{n-1}(t)$ satisfies the inequality

$$(24) \qquad \int_{t^*}^{t} f(s)\,ds/(t-t^*) - f(t^*) \geqslant -4\varepsilon_0 \quad \text{for every} \quad t, t^*, \quad t \neq t^*.$$

Indeed, let us assume that this inequality is valid. We shall prove the validity of (13) for $i = n$, $p = 1, 2, \ldots, n-1$. We have, namely,

$$\inf_{0<|t-t^*|<d_p} \left(\frac{\int_{t^*}^{t} k_n(s)\,ds}{t-t^*} - k_n(t^*) + \frac{1}{p} \right)$$

$$= \inf \left[\left(\frac{\int_{t^*}^{t} k_{n-1}(s)\,ds}{t-t^*} - k_{n-1}(t^*) + \frac{1}{p} \right) + \left(\frac{\int_{t^*}^{t} f(s)\,ds}{t-t^*} - f(t^*) \right) \right]$$

$$\geqslant \inf \left(\frac{\int_{t^*}^{t} k_{n-1}(s)\,ds}{t-t^*} - k_{n-1}(t^*) + \frac{1}{p} \right) + \inf \left(\frac{\int_{t^*}^{t} f(s)\,ds}{t-t^*} - f(t^*) \right)$$

$$\geqslant 5\varepsilon_0 - 4\varepsilon_0 = \varepsilon_0 > 0,$$

therefore inequality (13) will be proved for $p = 1, 2, \ldots, n-1$, if we prove (24). Let us observe that by (23)

$$f(t) = \min\big(k_{n-1}(t), h_0(t)\big) - k_{n-1}(t) = \min\big(0, h_0(t) - k_{n-1}(t)\big).$$

By (16) and (14a) we have

$$(25) \qquad f(t) = 0 \quad \text{for} \quad |t - t_n| \geqslant \delta_0.$$

Let us introduce the following notation:

$$(25\text{a}) \qquad f_1(t) = \min\big(0, h_0(t) - k_{n-1}(t_n)\big), \qquad f_2(t) = f(t) - f_1(t).$$

Also for $f_1(t)$ in view of (16), (14a) the following relation is valid

$$(26) \qquad f_1(t) = 0 \quad \text{for} \quad |t - t_n| \geqslant \delta_0.$$

From (25) and (26) it follows that

$$(26\text{a}) \qquad f_2(t) = 0 \quad \text{for} \quad |t - t_n| \geqslant \delta_0.$$

Let us now estimate $|f_2(t)|$ for $|t - t_n| \leqslant \delta_0$

$$|f_2(t)| = \big|\min\big(0, h_0, (t) - k_{n-1}(t)\big) - \min\big((0, h_0(t) - k_{n-1}(t_n)\big)\big|$$

$$\leqslant \big|\big(h_0(t) - k_{n-1}(t)\big) - \big(h_0(t) - k_{n-1}(t_n)\big)\big| = |k_{n-1}(t) - k_{n-1}(t_n)|;$$

therefore, in consequence of (22), we have in this case $|f_2(t)| \leqslant \varepsilon_0$, which in view of (26a) gives the inequality $|g_2(t)| \leqslant \varepsilon_0$ for every t; thus the inequality

$$\Big(\int_{t^*}^{t} f_2(s)\,ds\Big)/(t-t^*) - f_2(t^*) \geqslant -2\varepsilon_0$$

is fulfilled for every $t, t^*, t \neq t^*$. For the proof of (24) it remains to show, (since $f(t) = f_1(t) + f_2(t)$ and the left side of (24) is a linear functional with respect to $f(t)$) that

$$\Big(\int_{t^*}^{t} f_1(s)\,ds\Big)/(t-t^*) - f_1(t^*) \geqslant -2\varepsilon_0 .$$

In consequence of (25a), (18), (15), (14a) the function $f_1(t)$ satisfies the inequalities

$$(27) \qquad -1 \leqslant f_1(t) \leqslant 0 \quad \text{for every} \quad t.$$

The function $h_0(t)$ has property (19) (for $\varepsilon = \varepsilon_0$); therefore, as follows directly from (Z), the function $j(t) = h_0(t) - k_{n-1}(t_n)$ also has this property. From the inequality $j(s) \geqslant j(t^*) - \varepsilon$, in view of the identity $f_1(t) = \min\big(0, j(t)\big)$, follows the inequality $f_1(s) \geqslant f_1(t^*) - \varepsilon$. Indeed, in the case of $j(t^*) - \varepsilon \geqslant 0$ we have $j(s) \geqslant 0$, hence $f_1(s) = 0 = f_1(t^*) \geqslant f_1(t^*) - \varepsilon$, whereas in the case of $j(t^*) - \varepsilon < 0$ we have $j(t^*) - \varepsilon \leqslant \min\big(0, j(s)\big) = f_1(s)$, and thus $f_1(t^*) - \varepsilon \leqslant f_1(s)$. Therefore, the function $f_1(t)$ also has property (19). Let us denote by B the set of points s belonging to the interval $[t, t^*]$ in which the following inequality does not hold

$$f_1(s) \geqslant f_1(t^*) - \varepsilon_0,$$

and by $|B|$ its measure. In view of relation (19) we obtain the inequality

$$(28) \qquad |B| \leqslant \varepsilon_0 |t - t^*|.$$

As a result of relations (27), (28) we have the following inequalities:

$$\begin{aligned} \int_{[t,t^*]-B} f_1(s)\,ds &\geqslant \big(|t-t^*| - |B|\big)\big(f_1(t^*) - \varepsilon_0\big) \\ &= |t-t^*|\big(f_1(t^*) - \varepsilon_0\big) + |B|\big(\varepsilon_0 - f_1(t^*)\big) \\ &\geqslant |t-t^*|\big(f_1(t^*) - \varepsilon_0\big), \end{aligned}$$

$$\int_B f_1(s)\,ds \geqslant -|B| \geqslant -|t-t^*|\,\varepsilon_0 .$$

Hence

$$\Big(\int_{t^*}^{t} f_1(s)\,ds\Big)/(t-t^*) - f_1(t^*) = \Big(\int_{[t,t^*]} f_1(s)\,ds\Big)/|t-t^*| - f_1(t^*) \geqslant -2\varepsilon_0;$$

thus the relations (13) are indeed valid for $i = n$, $p = 1, 2, \ldots, n-1$. We then choose d_n so that (13) be fulfilled also for $i = n$, $p = n$. For this purpose it suffices to chose d_n so that $|k_n(t) - k_n(t^*)| \leqslant 1/2n$ for $|t - t^*| \leqslant d_n$, which is possible because of the uniform continuity of $k_n(t)$. The proof of Lemma 3 is thus completed.

LEMMA 4. *If the function $z(x,y)$ is continuous in the region D and has a total differential outside the set in which it assumes a countable number of values, then there exists a function $l(t)$ everywhere differentiable with a positive derivative almost everywhere, and such that the composed function $z^*(x,y) = l\big(z(x,y)\big)$ has a total differential at each point of the set D*[16].

The proof of lemma 4. Let us denote by $D(r)$ the set of points which

I) are contained in a closed sphere of radius $1/r$ and centre at the point $(0,0)$;

II) belong with their open neighbourhoods of radius r to the set D. Evidently, the sets $D(r)$ are compact and not empty for sufficiently small r. By $d(r)$ let us denote the maximum number d such that, if

$$(29) \qquad P \in D(r), \qquad P^{\sim} \in D(r)$$

and the distance of the points $P, P^{\sim}$, $|P - P^{\sim}|$ is not larger than d, then $|z(P) - z(P^{\sim})| \leqslant r/2$. The set $D(r)$ being compact, the function $d(r)$ is positive for positive r (and assumes finite values for sufficiently small r provided $z(P)$ is not identically equal to a constant). Moreover, $d(r)$ is (weakly) increasing. We may, therefore, apply lemma 2 to the function $g(s) = sd(s)$ and to the sequence of values t_i assumed by the function $z(P)$ at points at which it has no total differential. Let $l(t)$ be the function appearing in the assertion of Lemma 2. Evidently, the function $z^*(x,y) = l\big(z(x,y)\big)$ has a total differential at those points of the region D at which $z(x,y)$ also has it. Therefore let the point P_0 be an arbitrary point of D at which $z(P)$ has no total differential. We shall prove that $z_x^*(P_0) = 0$, $z_y^*(P_0) = 0$ and that at point P_0 the function $z^*(P)$ has a total differential. For this purpose it suffices of course to consider the points P, for which

$$(30) \qquad z(P) - z(P_0) \neq 0.$$

For these points we shall prove the following inequality:

$$(31) \qquad d\big(|z(P) - z(P_0)|\big) \leqslant |P - P_0| \quad \text{for sufficiently small} \quad |P - P_0|.$$

For indirect proof, let us assume that in every neighbourhood of P_0 there exists a $P^{\sim}$ fulfilling (30) and such that

$$(32) \qquad |P^{\sim} - P_0| < d(r^{\sim}), \qquad \text{where} \quad r^{\sim} = |z(P^{\sim}) - z(P_0)|,$$

[16]) Lemma 4 remains valid under a more general assumption that $z(P)$ has a total differential outside the set in which it assumes values forming jointly a set composed of a countable number of sets of Jordan's measure zero.

hence for sufficiently small $|P^{\sim}-P_0|$ the assumptions of implication (29) $(P_0 \epsilon D(r), P^{\sim} \epsilon D(r))$ are satisfied, whence it follows that $|z(P^{\sim})-(P_0)| \leqslant r^{\sim}/2$, which is at variance with relations (30), (32). We have

$$|z^*(P)-z^*(P_0)| = |l(z(P)) - l(z(P_0))| \leqslant |z(P)-z(P_0)|\, d(|z(P)-z(P_0)|)$$

for sufficiently small $|P-P_0|$ as a result of (3) and the equality $g(s)=s\,d(s)$, and hence, on the basis of (31) for sufficiently small $|P-P_0|$ we have

$$|z^*(P)-z^*(P_0)| \leqslant |P-P_0|\, |z(P)-z(P_0)|.$$

Since $|z(P)-z(P_0)| \to 0$ when $P \to P_0$, the proof of the existence of a total differential in the set D, and hence of lemma 4, has been completed.

The proof of theorem 1. We may apply lemma 4 to the function $z(x,y)$ of lemma 2, because this function is not differentiable at most in the set $D-G$ composed of a countable number of integrals, *i. e.*, in a set in which it assumes a countable number of values. The function $z(x,y)$ being constant along the integrals of (2), the function $z^*(x,y) = l(z(x,y))$ is also constant along the integrals of (2), and since it has a total differential, it is the solution of (1). To complete the proof it remains to show that the set of points at which the derivative $\partial z^*(x,y)/\partial y$ is zero, has the measure zero. For this purpose it suffices to prove that for each point P of the open set G there is a neighbourhood Ω of the point P such that the set of points belonging to Ω at which the derivative $\partial z^*(x,y)/\partial y$ becomes zero has the measure zero, because as a result of property I) of lemma 2, $D-G$ has the measure zero. Therefore let P be an arbitrary point of G. Let us denote by K a closed circle with the centre at point P, contained in G, and by Z a set of points of K, at which the derivative $\partial z^*(x,y)/\partial y$ is zero.

The transformation T $u=x$, $v=z(x,y)$ transforms K into the set A of variables u,v, and Z into the set B (subset of the set A). The function $z(x,y)$ satisfies properties II), III) of lemma 2, hence the inverse transformation T^{-1} is of class C^1 in A. This transformation transforms the set B into the set Z. The set B has the field measure zero because it is contained in the Cartesian product of the u-axis and the set of the linear measure zero, of value v, for which the derivative $l'(v)$ is zero, as a result of property III) of lemma 2 and the equality $\partial z^*(x,y)/\partial y = l'(z(x,y))\,\partial z(x,y)/\partial y$. The set Z has, therefore, a field measure zero as an image of a set of field measure zero through the transformation T^{-1} of class C^1. Thus theorem 1 has been proved.

§ 2. THEOREM 2. *If the function $Q(x,y)$ is of class C^1 in a finitely-connected open region D, then there exists a solution of equation* (1) *having a total differential in this region and identically equal to a constant in no open (non-empty) subset of the region D*[17]*.*

The proof of theorem 2. We shall reduce theorem 2 to theorem 1. Let C^1 be an integral reaching with one end to a certain point of the unbounded component of the supplement of $\mathring{D}$[18]), and by the other end at a certain point of bounded components. We shall prove that such an integral exists, provided the supplement of the set D has a bounded component. Indeed, let us denote by b the largest coordinate x of the points belonging to the bounded components of the supplement of the set D, and by T the segment of maximum lenght, belonging to the bounded components of the supplement of D and lying on the straight line $x=b$ (this segment may be reduced to a point). Let K be an open rectangle containing T and having no points in common with the unbounded component of the set D. Let us denote by P_1 a point lying on the straight line $x=b$ above the segment T and belonging to the product DK, and by P_2 an analogous point below T. The integrals passing through these points belong to DK for $b-d \leqslant x \leqslant b+d$, where d is a certain positive number; hence the straight line $x=b+d$ intersects these integrals at points belonging to the rectangle K. The segment of the straight line between the above integrals belongs to the region D (because K contains no points of the unbounded component of the supplement of D, and for points of the bounded components of the supplement of D the inequality $x \leqslant b$ is valid). One of the integrals starting from this segment must reach with its left end to a certain point of bounded components, since otherwise all these integrals would exists in the interval $[b-d, b+d]$, which contradicts the fact that T does not belong to D. This integral reaches with its right end to a point of the unbounded component (since to the right of $x=b$ there are no points of bounded components); hence it is the desired integral C_1. The supplement of the set $D_1 = D - C_1$ has one bounded component less than the supplement of D. An analogous argumentation may be applied to the set D_1 (if its supplement has bounded

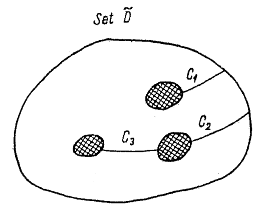

Fig. 3

[17]) A. Bielecki has furnished an example of a double connected region and of an equation defined in it for which the assertion of theorem 1 is not fulfilled.

[18]) We say that the integral $y=y(x)$ reaches the point P with its left end if there exists on the integral a sequence of points (x_m, y_m) such that (x_m, y_m) tends to the point P, and $x_m \to a$, where a is the left end-point of the interval in which the considered integral is defined. Analogously we define reaching a point by an integral with its right end.

components) and obtain the region $D_2=D_1-C_2$, whose supplement has two bounded components less than that of D. Proceding in this way we shall finally obtain the region $D^{\sim}=D-C_1-C_2-\ldots-C_k$ (Fig. 3), whose supplement has no bounded components, *i. e* a simply connected region, to which theorem 1 may be applied. Let us denote by $u(x,y)$ the bounded solution of equation (1), fulfilling the assertion of theorem 1. (If the solution $z(x,y)$ satisfies the assertion of theorem 1, then the function $w(x,y)=\operatorname{arc\,tg}[z(x,y)]$ is a bounded solution of equation (1) satisfying the assertion of theorem 1.) When tending to integrals C_i, the function has one-side limits (as a result of boundedness and monotony [19])). Let us denote them by $g_1,\ldots,g_n$ $(n=2k)$. The function $w\big(u(x,y)\big)$, where $w(u)=(u-g_1)(u-g_2)\ldots(u-g_n)$, tends to zero when the point tends to the integrals C_i. Defining it on these integrals as equal to zero, we obtain a continuous function in D, having a total differential except at most at those points where its value is zero. One may, therefore, apply to it lemma 4 and obtain the desired solution $s(x,y)$. The function $s(x,y)$ is not identically equal to a constant in any open subset of its domain, because the continuous functions $u(x,y)$, $w(u)$, $l(w)$ have the above property. The proof of theorem 2 has been completed.

§ 3. Before furnishing an example, we introduce the following definitions:

Definition 1. The points A,B are called *directly conjugate with respect to the system of ordinary differential equations* U if there exist sequences of points A_m,B_m,A_m tending to A and B_m to B, such that for each index m the points A_m and B_m lie on the same integral of U.

Definition 2. The points A^* and A^{**} are called *conjugate with respect to* U if there exists a finite sequence of points $A_1,A_2,\ldots,A_k$ such that $A_1=A^*,A_k=A^{**}$ and for the index $i=1,2,\ldots,k-1$ the points A_1,A_{i+1} are directly conjugate with respect to U.

Definition 3. We give the term *an integral of ramification* to the totality of points conjugate to a given one with respect to U [20]).

The relation of conjugation is reflexive, transitive, and symmetrical. Points lying on the same integral are conjugate, but not necessarily vice versa, *e. g.* for the equation $y'=0$ considered on a plane with the origin of coordinates removed, the points (1,0), (−1,0) are conjugate (directly) though they do not lie on the same integral. (They belong to the same integral of ramification.)

[19]) This function tends to the same limit, when tending to an arbitrary point of a given integral C_1 through points below (above) the integral in view of the theorem on the continuous dependence of an integral on initial conditions, and of the fact that it is constant along the integrals.

[20]) The definition of the integral of ramification has been introduced by T. Ważewski in connection with the first integrals of an ordinary differential equation. The above definition concerns particularly one equation.

The principal property of integrals of ramification is the following: Every function continuous in the domain of the right-hand members of the system U and constant along the integrals of the system U, is constant along the integrals of ramification of the system U. To prove this property it suffices to show that, in view of definitions 2 and 3, the function $f(x,y)$ constant along the integrals and continuous, assumes the same values at the directly conjugated points A and B. To this end let us observe that, in view of definition 1, the equality $f(A_m)=f(B_m)$ holds; therefore, in virtue of the continuity of $f(x,y)$, we have $f(A)=f(B)$. From this property of integrals of ramification it follows also that if there exists an integral of ramification of a certain system of ordinary differential equations U, everywhere-dense in the domain of the right-hand members of the system U, then each function continuous in this region and constant along the integrals is in U identically equal to a constant. It is known that to the functions which are constant along the integrals of the system of ordinary differential equations

$$y_i'=Q_i(x,y_1,\ldots,y_n)\qquad (i=1,2,\ldots,n)$$

belong the solutions of the partial differential equation

$$\partial z/\partial x+\sum_{i=1}^{n}Q_i(x,y_1,\ldots,y_n)\,\partial z/\partial y_i=0,$$

having a total differential everywhere.

Example. We shall now construct a differential equation

$$\partial y/\partial x=Q(x,y) \tag{R}$$

with the right side of class C^∞ [21]) in the open region D, having an integral of ramification everywhere-dense in D.

Let $\{w_i\}$ be a sequence of rational numbers in the interval (0,1), in which each rational number belonging to this interval appears exactly once. It is easily seen that with each number w_i we may associate an interval $T_i=(a_i,b_i)$, contained in the interval [1/2,1], such that following implcations are valid:

I) $w_i<w_j\supset a_i<b_i<a_j<b_j$,

[21]) A function is said to be of class C^∞ if its derivatives of all orders are continuous. Through a proper approximation of the function $Q(x,y)$ by an analytical function, analogous example with the function $Q(x,y)$ analytical in D may be obtained.

II) if for a chosen sequence $w_{j_m} \to 1$, then $a_{j_m} \to 1$ [22]).

We shall construct the equation (R) in a certain subset D of the square $K(|x| \leqslant 1, y \leqslant 1)$ so that the following relation will be valid:

$$Q(x,y) = Q(-x,-y) \quad \text{for} \quad (x,y) \in D.$$

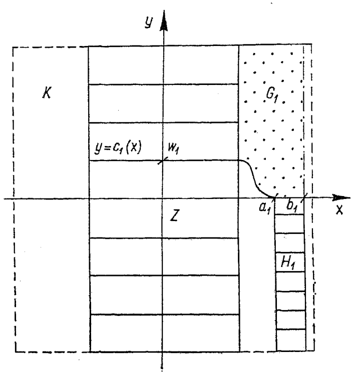

Fig. 4

It suffices, therefore, to define the set D and the function $Q(x,y)$ in D for $x \geqslant 0$.

Remark. The function $Q(x,y)$ will at first be constructed also at some boundary points of the open region D, in order that some auxiliary sets (whose interiors will serve to construct the region D) be closed.

By Z let us denote the set composed of the rectangle $-1/2 < x < 1/2$, $|y| \leqslant 1$ and of the segment $y=1$; $1/2 < x \leqslant 1$. We shall define $Q(x,y)$ at first in the set Z putting

$$Q(x,y) = 0 \quad \text{for} \quad (x,y) \in Z. \tag{33}$$

We shall extend the definition of the function $Q(x,y)$ to certain subsets of the square K. This extension will be done in a countable number of steps.

Step one. Let $y = c_1(x)$ be an integral of the equation (R) considered in Z, satisfying the initial condition $c_1(0) = w_1$ [23]) (of course $c_1(x) = w_1$ and the function is defined for $-1/2 < x < 1/2$). Let us further elongate the curve $y = c_1(x)$ on the remaining part of interval $[-1/2, b_1]$ (*i. e.* on the interval $[1/2, b_1]$) so that

1) the function $c_1(x)$ will be of class C^∞ in the interval $[-1/2, b_1]$,
2) $0 < c_1(x)$ for $1/2 < x < a_1$,
3) $c_1(x) = 0$ for $a_1 \leqslant x \leqslant b_1$,
4) the part of the curve $y = c_1(x)$, defined in determining the elongation (*i. e.* $y = c_1(x)$ for $1/2 < x \leqslant b_1$), will run in the set $K - Z$ (*i. e.* that

[22]) The sequence of segments T may be *e.g.* of the same kind as the segments contiguous to Cantor's set.

[23]) The term integral of the equation (R) considered in Z is given to the integral contained in Z and reaching to the boundary of Z by its two "ends".

$c_1(x) < 1$ for $1/2 < x \leqslant b_1$ [24])). Let us denote this part of the curve $y = c_1(x)$ by $\tilde{C_1}$.

Now let us assume that $Q(x,y) = dc_1(x)/dx$ on $\tilde{C_1}$. In consequence of such a definition the whole curve $y = c_1(x)$ defined in the interval $-1/2 \leqslant x \leqslant b_1$ is an integral. Let us denote by G_1 the set of points $1/2 < x \leqslant b_1$, $c_1(x) < y < 1$ and by H_1 the rectangle $a_1 \leqslant x \leqslant b_1$, $-1 \leqslant y < 0$, and let us assume $Q(x,y) = 0$ for $(x,y) \in H_1$.

We extend the function $Q(x,y)$ from the set $Z + \tilde{C_1} + H_1$ to the set G_1, retaining class C^∞ [25]). We shall prove that for the equation (R), considered in the interior of the set $Z_1 = Z + \tilde{G_1} + H_1$, the points $(0, w_1)$ and $\big((a_1+b_1)/2, 0\big)$ are directly conjugated. For the proof let us consider the sequence of integrals $y = y(x; 0, w_1 + 1/n)$ [26]). Because of the regularity of the function $Q(x,y)$ in the set Z_1 and the theorem of continuous dependence of the integral on initial conditions, the above sequence of integrals tends at $n \to \infty$ to the integral $y = c_1(x)$ [27]). The integrals $y = y(x; 0, w_1 + 1/n)$ belong (starting from a certain n) to the interior of Z_1 and are defined in the interval $-1/2 < x < b_1$; and, since

$$y\big((a_1+b_1)/2;\ 0, w_1 + 1/n\big) \to 0, \qquad y(0, 0, w_1 + 1/n) \to w_1 \quad \text{for} \quad n \to \infty,$$

therefore the points $(0; w_1)$, $\big((a_1+b_1)/2, 0\big)$ are indeed conjugated. The first step is thus completed.

Remark. The curve $y = c_1(x)$ does not constitute one intergal in the interior of Z_1, because the points $y = c_1(x)$, $1/2 \leqslant x \leqslant a_1$ do not belong to the interior of Z_1. It will not constitute one integral also in the set D, since the point $(a_1, 0)$ will not belong to this set.

Let us assume that the function $Q(x,y)$ has already been defined in the set Z_k. We shall construct the set Z_{k+1} and extend the definition of $Q(x,y)$ to this set.

[24]) *E.g.* we may assume that

$$c_1(x) = w_1 \left[1 - \frac{\int_{1/2}^{x} \exp[(s-1/2)(s-a_1)]^{-1} ds}{\int_{1/2}^{a_1} \exp[(s-1/2)(s-a_1)]^{-1} ds} \right] \quad \text{for} \quad 1/2 < x < a_1$$

and $c_1(x) = 0$ for $a_1 \leqslant x \leqslant b_1$.

[25]) Such an extension is possible on the basis of the theorem on extending the function from a closed set with the retention of regularity, see [7].

[26]) $y = y(x; u, v)$ denotes an integral passing through the point (u, v).

[27]) The integrals of this sequence exist starting from a certain n.

Let $y = c_{k+1}(x)$ be an integral of the equation (R) in Z_k, fulfilling the initial condition: $c_{k+1}(0) = w_{k+1}$. Let us elongate the curve $y = c_{k+1}(x)$ (see Fig. 5) over the remaining part of the interval $[-1/2, b_{k+1}]$ so that

1') the function $c_{k+1}(x)$ will be of class C^∞,

2') $0 < c_{k+1}(x)$ for $x < a_{k+1}$,

3') $c_{k+1}(x) = 0$ for $a_{k+1} \leqslant x \leqslant b_{k+1}$,

4') the part of the curve $y = c_{k+1}(x)$, defined in the elongation, will run in the set $K - Z_k$[28]).

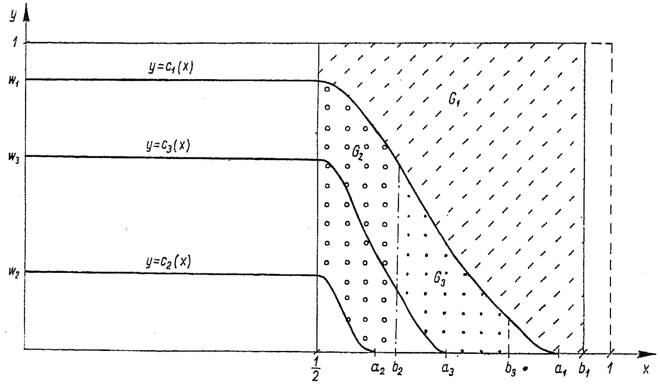

Fig. 5

Let us denote this part of the curve by $C_{k+1}^{\sim}$. Let us define the function $Q(x,y)$ along the curve $C_{k+1}^{\sim}$ by the following formula: $Q(x,y) = dc_{k+1}(x)/dx$. In consequence of this definition the whole curve $y = c_{k+1}(x)$ is an integral of the equation (R). By $G^{\sim}$ let us denote the set of points

$$-1/2 \leqslant x \leqslant b_{k+1}, \qquad c_{k+1}(x) < y < 1,$$

and by G_{k+1} the following set $G_{k+1} = G^{\sim} - Z_k$. Let H_{k+1} be a rectangle $a_{k+1} \leqslant x \leqslant b_{k+1}, -1 \leqslant y < 0$. Let us assume that $Q(x,y) = 0$ for $(x,y) \in H_{k+1}$. Let us extend the function $Q(x,y)$ from the set $Z_k + C_{k+1}^{\sim} + H_{k+1}$ to the set G_{k+1}, retaining class C^∞. Analogously as at the first step, we may see, that the point $(0, w_{k+1})$ and $\big((a_{k+1}+b_{k+1})/2, 0\big)$ are conjugated with respect to (R) considered in the interior of the set $Z_{k+1} = Z_k + G_{k+1} + H_{k+1}$. Similarly to the curve $y = c_1(x)$, also the curve $y = c_{k+1}(x)$ will not constitute one integral in the region D, because the point $(a_{k+1}, 0)$ will not belong to D. Applying mathematical induction, we have defined in this way the function $Q(x,y)$ in the set $\sum_{i=1}^{n} Z_i$, this function being of class C^∞ in the set $\sum_{i=1}^{n} I(Z_i)$, where $I(Z_i)$ denotes the interior of the set Z_i. Moreover, we have $Q(x,y) = 0$ in the set $\sum_{i=1}^{n} H_i$. Let us assume that $Q(x,y) = 0$ also at the remaining points of the rectangle H: $1/2 \leqslant x < 1$, $-1 < y < 0$. Thus the points $\big((a_i+b_i)/2, 0\big)$ will be directly conjugated with the point $(0,0)$ with respect to the equation (R) defined in the open region D

$$D = \sum_{i=1}^{\infty} I(Z_i) + H + \sum_{i=1}^{\infty} I(Z_i^*) + H^*$$

where H^*, Z_i^* denote sets symmetrical to the sets H, Z_i with respect to the origin of a system of coordinates. Indeed, the sequences of points $\big((a_i+b_i)/2, -1/(n+1)\big)$, $\big(0, -1/(n+1)\big)$, lying on the same integrals $-1/2 < x < 1$, $y = -1/(n+1)$, tend at $n \to \infty$ to the points $\big((a_i+b_i)/2, 0\big)$, $(0,0)$; hence these points are conjugated. The points $(0, w_i)$ are conjugate to the points $\big((a_i+b_i)/2, 0\big)$; therefore, in view of the above, they are conjugate to the point $(0,0)$. As a result of symmetry, also the points $(0, -w_i)$ are conjugate to $(0,0)$; hence the points belonging to the integral of ramification L passing through $(0,0)$ are lying everywhere-densely on the segment $x = 0$, $-1 < y < 1$. One may easily observe that through this segment pass all integrals of the equation (R) (defined in D) not lying on the x-axis and thus, in view of the theorem on the continuous dependence of the integral on initial conditions, the integral of ramification L is everywhere-dense in the region D.

[28]) The existence of a curve satisfying the above conditions (particularly also property 4')) follows from implication I) $w_i < w_j \supset a_i < b_i < a_j < b_j$.

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