On the greatest prime factors of certain products

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P. L. Chebyshev has proved a theorem about the greatest prime factors of the products $\prod_{n=1}^{x}(1+n^2)$ (see [2], p. 559). This theorem has been generalized by Ivanov ([2], p. 562). The consequence of Chebyshev's theorem is Stormer's theorem about the numbers $i(i-1)(i-2)\dots(i-x)$ ([2], p. 561).

The purpose of this paper is to prove some theorems about the greatest prime divisors of certain products. Theorem 1 is a generalization of the theorem of Ivanov, theorem 2 is a generalization of that of Stormer.

Note that another generalization of Ivanov's theorem is given by P. Erdös [1].

THEOREM 1 (Ivanov's case: $a_n=n$, Chebyshev's case: $a_n=n$, $A=1$). *Let $\{a_n\}$ be a sequence of integers $0<a_1<a_2<\dots<a_n<\dots$ and A a positive integer. Let P_x be the greatest prime factor of*

$$(1)\qquad \prod_{n=1}^{x}(A+a_n^2) \qquad (x\text{-integer and } \geqslant 1),$$

$$(2)\qquad \text{if } \lim_{n\to\infty}\frac{\log(a_1a_2\dots a_n)}{a_n\log a_n}>\frac{1}{2}, \text{ then } \lim_{x\to\infty}\frac{P_x}{a_x}=\infty,$$

$$(3)\qquad \text{if } \lim_{n\to\infty}\frac{\log(a_1a_2\dots a_n)}{a_n\log n}>\frac{1}{2}, \text{ then } \lim_{x\to\infty}\frac{P_x}{x}=\infty$$

$$\left(\text{in particular, for } a_n=n \quad \lim_{n\to\infty}\frac{\log n!}{n\log n}=1\right).$$

Proof of (2). The numbers $k<4A$, $(k,4A)=1$ can be divided into two classes U, V with equal numbers of elements, such that if $p\equiv l_1 \pmod{4A}$ for some $l_1\in U$, then the congruence $y^2\equiv -A \pmod{p^m}$ is solvable (p denotes a prime, m positive integer), if $p\equiv l_2 \pmod{4A}$ for some $l_2\in V$, then the congruence is unsolvable. In the first case the congruence ha exactly two roots mod p^m (see e. g. [3], p. 121-127).

If $p|4A$, then the number of roots of the congruence $y^2\equiv -A \pmod{p^m}$ is $\leqslant C_p$, where C_p does not depend on m (see e. g. [2], p. 563).

If, for instance $A=1$, then the congruence $y^2\equiv -1 \pmod p$ is solvable for $p\equiv 1 \pmod 4$ and unsolvable for $p\equiv 3 \pmod 4$ (see e. g. [5], p. 82). In the case $p=2$ this congruence has one root for $m=1$ and no roots for $m>1$ (see e. g. [5], p. 84). Denoting by D_{pmx} the number of those factors $(A+a_i^2)$ $(i=1,2,\dots,x)$ which are divisible by p^m, we easily find that

$$D_{pmx}\begin{cases}\leqslant C_p\left(\dfrac{a_x}{p^m}+1\right) & \text{for } p|4A,\\[2ex] \leqslant 2\left(\dfrac{a_x}{p^m}+1\right) & \text{for } p\equiv l_1 \pmod{4A},\quad l_1\in U,\\[2ex] =0 \quad \text{for } p\equiv l_2 \pmod{4A}, & l_2\in V.\end{cases}$$

Suppose that the theorem is false. Thus there exists such $g>0$ and such a sequence $x\to\infty$ that

$$(4)\qquad \frac{P_x}{a_x}\leqslant g.$$

For these x we have

$$\prod_{n=1}^{x}(A+a_n^2)\leqslant \prod_{p|4A}p^{c_pu}\prod_{\substack{p\leqslant ga_x,\, l\in U\\ p\equiv l \pmod{4A}}}p^{2u},$$

where

$$u=\{(a_x/p+1)+(a_x/p^2+1)+\dots\}.$$

The sums in the exponents have

$$m_0=\left[\frac{\log(A+a_x^2)}{\log p}\right]$$

terms, because if $p^m|A+a_k^2$ then $p^m\leqslant A+a_k^2\leqslant A+a_x^2$. We try to estimate the logarithm of (1)

$$\sum_{n=1}^{x}\log(A+a_n^2)\leqslant \sum_{p|4A}C_p\log p\left\{\sum_{m=1}^{m_0}\left(\frac{a_x}{p^m}+1\right)\right\}+$$

$$+\sum_{\substack{p\leqslant ga_x,\, l\in U\\ p\equiv l \pmod{4A}}}2\log p\left\{\sum_{m=1}^{m_0}\left(\frac{a_x}{p^m}+1\right)\right\},$$

$$\sum_{p|4A}C_p\log p\left\{\sum_{m=1}^{m_0}\left(\frac{a_x}{p^m}+1\right)\right\}=O\left\{\sum_{p|4A}\left(\sum_{m=1}^{\infty}\frac{a_x}{p^m}\right)\right\}+O(\log a_x)=O(a_x),$$

$$\sum_{\substack{p\leqslant ga_x, l\in U\\ p\equiv l \pmod{4A}}} 2\log p\left\{\sum_{m=1}^{m_0}\left(\frac{a_x}{p^m}+1\right)\right\}\leqslant 2a_x\sum_{\substack{p\leqslant ga_x, l\in U\\ p\equiv l \pmod{4A}}}\log p\left\{\sum_{m=1}^{\infty}\frac{1}{p^m}\right\}+$$

$$+O\left(\sum_{p\leqslant ga_x}\log p\cdot\frac{\log a_x}{\log p}\right).$$

But $\pi(ga_x)=O\left(\frac{a_x}{\log a_x}\right)$, hence $O\left(\sum_{p\leqslant ga_x}\log a_x\right)=O\left(\log a_x\cdot\pi(ga_x)\right)=O(a_x)$.

We know also that

$$\sum_{\substack{p\leqslant ga_x\\ p\equiv l \pmod{4A}}}\frac{\log p}{p}=\frac{1}{\varphi(4A)}\log a_x+O(1)$$

([2], p. 450). Hence

$$\sum_{\substack{p\leqslant ga_x, l\in U\\ p\equiv l \pmod{4A}}} 2\log p\left\{\sum_{m=1}^{m_0}\left(\frac{a_x}{p^m}+1\right)\right\}\leqslant 2a_x\left(\frac{\varphi(4A)}{2}\cdot\frac{1}{\varphi(4A)}\cdot\log a_x+O(1)\right)+O(a_x)$$

$$=a_x\log a_x+O(a_x)$$

and finally

$$\sum_{n=1}^{x}\log(A+a_n^2)=a_x\log a_x+O(a_x). \qquad (5)$$

Besides $\sum_{n=1}^{x}\log(A+a_n^2)\geqslant 2\sum_{n=1}^{x}\log a_n$. But

$$\lim_{n\to\infty}\frac{\log(a_1a_2\ldots a_n)}{a_n\log a_n}>\frac{1}{2}.$$

Let this limit be $1/2+\varrho$ ($\varrho>0$) (if it is ∞, then the reasoning is analogous). Hence, for any $\varepsilon>0$, we have

$$\frac{\sum_{n=1}^{x}\log a_n}{a_x\log a_x}>\frac{1}{2}+\varrho-\varepsilon$$

(for x sufficiently large). For $\varepsilon=\varrho/4$, we have

$$\frac{\sum_{n=1}^{x}\log(A+a_n^2)}{2a_x\log a_x}=\frac{\sum_{n=1}^{x}\log(A+a_n^2)}{2\sum_{n=1}^{x}\log a_n}\cdot\frac{2\sum_{n=1}^{x}\log a_n}{2a_x\log a_x}>\frac{1}{2}+\varrho-\frac{\varrho}{4}.$$

But (5) gives

$$\frac{\sum_{n=1}^{x}\log(A+a_n^2)}{a_x\log a_x}\leqslant 1+O\left(\frac{1}{\log a_x}\right),$$

and for x sufficiently large

$$\frac{\sum_{n=1}^{x}\log(A+a_n^2)}{a_x\log a_x}\leqslant 1+\frac{\varrho}{2}.$$

Hence $1+2\varrho-\varrho/2<1+\varrho/2$, $2\varrho<\varrho$. The theorem is thus proved.

Proof of (3). Up to (4) we can repeat the reasoning without changes. In (4) a_x is to be replaced by x. In all the next formulas we have the sums and products for $p\leqslant gx$.

The modification of (5): $\sum_{n=1}^{x}\log(A+a_n^2)\leqslant a_x\log x+O(a_x)$. The proof is to be finished analogously to (2).

Example. Every sequence $a_n=[\tau n]$ $(1<\tau<2)$ satisfies the conditions of (3). $\{a_n\}$ is a monotonous sequence $a_n\leqslant\tau n<\tau(n+1)-1<a_{n+1}$. Besides

$$\frac{\log n!}{\tau n\log n}\leqslant\frac{\log([\tau\cdot 1][\tau\cdot 2]\ldots[\tau n])}{[\tau n]\log n}\leqslant\frac{\log(\tau^n n!)}{(\tau n-1)\log n}$$

$$=\frac{n\log\tau}{(\tau n-1)\log n}+\frac{\log n!}{(\tau n-1)\log n}.$$

Hence $\lim_{n\to\infty}\frac{\log(a_1a_2\ldots a_n)}{a_n\log n}=\frac{1}{\tau}>\frac{1}{2}$, q. e. d.

For instance $a_n=\left[\frac{3}{2}n\right]$ (1,3,4,6,7,9,10,12,...) satisfies theorem 1 (3).

Theorem 2 (Stormer's case: $a_n=n$). *Let a sequence of integers* $\{a_n\}$ $0<a_1<a_2<\ldots<a_n<\ldots$ *satisfy the condition*

1^0 $$\lim_{n\to\infty}\frac{\log(a_1a_2\ldots a_n)}{a_n\log a_n}>\frac{1}{2}$$

or

2^0 $$\lim_{n\to\infty}\frac{\log(a_1a_2\ldots a_n)}{a_n\log n}>\frac{1}{2}\quad\text{and}\quad a_n=O(n).$$

Then, among the numbers $i(i-a_1)(i-a_2)\ldots(i-a_x)$ (*x-integer and* $\geqslant 1$), *at most finitely many are* .*eal or pure imaginary.*

Proof. For a x let the number $i(i-a_1)(i-a_2)\dots(i-a_x)$ be real or pure imaginary.

Then $-i(-i-a_1)(-i-a_2)\dots(-i-a_x)$ and also $(i+a_1)(i+a_2)\dots(i+a_x)$ are real or pure imaginary numbers. We have

$$(i+a_1)(i+a_2)\dots(i+a_x)=\pm(a_1-i)(a_2-i)\dots(a_x-i).$$

(6)
$$\begin{aligned} Q_{nx} &= (a_1+a_n)(a_2+a_n)\dots(a_x+a_n) \\ &\equiv (a_1-i)(a_2-i)\dots(a_x-i)(\bmod a_n+i) \\ &\equiv \pm(a_1+i)(a_2+i)\dots(a_x+i)(\bmod a_n+i) \\ &\equiv 0(\bmod a_n+i) \qquad (1\leqslant n\leqslant x). \end{aligned}$$

If $p>2$ is a real prime and $p|(1+a_1^2)(1+a_2^2)\dots(1+a_x^2)$, then there exists a positive integer n $(1\leqslant n\leqslant x)$, so that $p\,|\,1+a_n^2=(a_n+i)(a_n-i)$. We represent p as a product of a complex prime and its conjugate (see *e. g.* [4], p. 393)

$$p=\mathfrak{p}\bar{\mathfrak{p}}.$$

We can order these $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in such a way that $\mathfrak{p}|a_n+i$. (6) gives $\mathfrak{p}|Q_{nx}=(a_1+a_n)(a_2+a_n)\dots(a_x+a_n)$. Hence $p|Q_{nx}$,

(7)
$$p\leqslant a_x+a_n\leqslant 2a_x.$$

If a_n satisfies the condition 1°, then for x sufficiently large we have $P_x/a_x>2$. But (7) gives $P_x/a_x\leqslant 2$.

If a_n satisfies the condition 2°, then for x sufficiently large we have $P_x/x>2C$. But (7) gives $P_x\leqslant 2a_x$ $(a_n\leqslant Cn)$.

We can apply theorem 1 also in another way. Namely we take such a sequence $\{a_n\}$ that $P_x/a_x\not\to\infty$. Then we know that

$$\lim_{n\to\infty}\frac{\log(a_1a_2\dots a_n)}{a_n\log a_n}\leqslant\frac{1}{2}.$$

For instance

THEOREM 3. *Let $\{x_n\}$ be a sequence of integers $0<x_1<x_2<\dots<x_n<\dots$ satisfying the following conditions:*

1° *There exists a sequence $\{q_n\}$ of integers $0<q_1<q_2<\dots<q_n<\dots$ such that*

$$x_n^2+A\equiv 0(\bmod q_n) \qquad (A \text{ denotes a positive integer}),$$

2°
$$\frac{q_n}{2}\leqslant x_n\leqslant q_n.$$

Then

$$\lim_{n\to\infty}\frac{\log(x_1x_2\dots x_n)}{x_n\log x_n}\leqslant\frac{1}{2}.$$

Proof. The greatest prime factor of $\prod_{k=1}^{n}(A+x_k^2)$ we denote by P_n. We have

$$P_n\leqslant\max\left[\frac{A+x_1^2}{q_1},\frac{A+x_2^2}{q_2},\dots,\frac{A+x_n^2}{q_n},q_1,q_2,\dots,q_n\right],$$

$$\frac{A+x_k^2}{q_k}\leqslant\frac{A+q_k^2}{q_k}\leqslant A+q_n \qquad (k=1,2,\dots,n).$$

Hence

$$P_n\leqslant A+q_n, \qquad \frac{P_n}{x_n}\leqslant\frac{A+q_n}{\frac{q_n}{2}}=\frac{2A}{q_n}+2\leqslant 2A+2, \qquad \frac{P_n}{x_n}\not\to\infty.$$

Theorem 1 gives $\lim\limits_{n\to\infty}\dfrac{\log(x_1x_2\dots x_n)}{x_n\log x_n}\leqslant\dfrac{1}{2}$, q. e. d.

(*e. g.* $\{x_n\}$: 3, 7, 8, 13, 18, 21, 31, ... $\{q_n\}$: 5, 10, 13, 17, 25, 26, 37, ...).

THEOREM 4. *Let $\{\tau_n\}$ be a sequence of integers $0<\tau_1<\tau_2<\dots<\tau_n<\dots$, $\tau_n=O(n)$. Let $a>1$ be an integer. Denoting the greatest prime factor of $\prod_{n=1}^{x}(a^{\tau_n}+1)$ by P_x (x-integer and $\geqslant 1$), we have*

$$\lim_{x\to\infty}\frac{P_x}{x^{\alpha}}=\infty \qquad \textit{for every } \alpha<1.$$

Proof. Let p be a prime and $(a,p)=1$. Let δ_m be the least positive exponent such that $a^{\delta_m}\equiv 1(\bmod p^m)$. Denoting the number of those numbers among $a^1,a^2,\dots,a^{\tau_x}$ which are congruent to $-1(\bmod p^m)$ by $D_{p^m\tau_x}$, we have $D_{p^m\tau_x}\leqslant\tau_x/\delta_m+1$. In fact, if τ is the least positive exponent such that $a^{\tau}\equiv -1(\bmod p^m)$, we have

$$D_{p^m\tau_x}\leqslant h+1, \qquad \text{where} \qquad h=\left[\frac{\tau_x-\tau}{\delta_m}\right].$$

Hence $D_{p^m\tau_x}\leqslant\tau_x/\delta_m+1$. Thus, if $\tau_x/\delta_m\geqslant 1$,

(8)
$$D_{p^m\tau_x}\leqslant 2\frac{\tau_x}{\delta_m}.$$

If $\tau_x/\delta_m<1$, (8) follows from the remark that if for some $\tau\leqslant\tau_x$ we have $a^{\tau}\equiv -1(\bmod p^m)$, then $a^{2\tau}\equiv 1(\bmod p^m)$, whence $\delta_m\leqslant 2\tau$ and $2\tau_x/\delta_m\geqslant 1$.

Thus, in both cases we have $D_{p^m\tau_x}\leqslant 2\tau_x/\delta_m$.

Suppose that the theorem is false. Thus there exists such $g>0$ and such a sequence $x\to\infty$ that $P_x\leqslant gx^{\alpha}$. By (8) we have for these x

$$a^{\frac{x(x+1)}{2}}\leqslant\prod_{n=1}^{x}a^{\tau_n}\leqslant\prod_{n=1}^{x}(a^{\tau_n}+1)\leqslant\prod_{\substack{p\leqslant gx^{\alpha}\\(p,a)=1}}p^{2\tau_x K_{px}},$$

where $K_{px}=\sum_{m=1}^{m_0}\frac{1}{\delta_m}$, $m_0=\left[\frac{\tau_x\log a^2}{\log p}\right]$; in fact

$$p^m\,|\,1+a^{\tau_n},\qquad p^m\leqslant 1+a^{\tau_x}<a^{\tau_x+1},\qquad m\leqslant\left[\frac{\log a^{\tau_x+1}}{\log p}\right]\leqslant m_0.$$

Hence

$$\frac{x(x+1)}{2}\log a\leqslant\sum_{\substack{p\leqslant gx^{\alpha}\\(p,a)=1}}2\tau_x K_{px}\log p.$$

There exists such a positive integer k that $1+\alpha+1/k\leqslant 2$. There exists such a positive integer b that $\sqrt[b]{a}\leqslant 2$. Let

$$K_{px}=\sum_{m=1}^{[\sqrt{m_0}]}\frac{1}{\delta_m}+\sum_{m=[\sqrt[k]{m_0}]+1}^{[\sqrt[k]{m_0^2}]}\frac{1}{\delta_m}+\ldots+\sum_{m=[\sqrt[k]{m_0^{k-1}}]+1}^{m_0}\frac{1}{\delta_m}.$$

If $m\geqslant\sqrt[k]{m_0^r}$ $(1\leqslant r\leqslant k)$, then $\delta_m\geqslant\sqrt[k]{m_0^r}/b$. Supposing conversely that $\delta_m<\sqrt[k]{m_0^r}/b$, we have

$$p^m\geqslant p^{\sqrt[k]{m_0^r}},\qquad p^m\,|\,a^{\delta_m}-1,\qquad p^{\sqrt[k]{m_0^r}}\leqslant p^m\leqslant a^{\delta_m}-1<a^{\delta_m}<a^{\sqrt[k]{m_0^r}/b}\leqslant 2^{\sqrt[k]{m_0^r}}\leqslant p^{\sqrt[k]{m_0^r}},$$

which is a contradiction. Hence

$$K_{px}\leqslant\sqrt[k]{m_0}\cdot 1+\sqrt[k]{m_0^2}\,\frac{1}{\sqrt[k]{m_0}}\,b+\sqrt[k]{m_0^3}\,\frac{1}{\sqrt[k]{m_0^2}}\,b+\ldots+m_0\,\frac{1}{\sqrt[k]{m_0^{k-1}}}\,b\leqslant kb\sqrt[k]{m_0},$$

$$K_{px}\leqslant kb\,\frac{\sqrt[k]{\log a^2}}{\sqrt[k]{\log p}}\,\sqrt[k]{\tau_x},$$

$$\frac{x(x+1)}{2}\log a\leqslant\sum_{p\leqslant gx^{\alpha}}2kb\tau_x^{1+1/k}\sqrt[k]{\log a^2}\,\sqrt[k]{\log^{k-1}p}$$

$$=2kb\tau_x^{1+1/k}\sqrt[k]{\log a^2}\sum_{p\leqslant gx^{\alpha}}\sqrt[k]{\log^{k-1}p}\,.$$

But, in view of $\pi(gx^{\alpha})=O(gx^{\alpha}/\log gx^{\alpha})=O(x^{\alpha}/\log x)$,

$$\sum_{p\leqslant gx^{\alpha}}\sqrt[k]{\log^{k-1}p}\leqslant\pi(gx^{\alpha})\sqrt[k]{\log^{k-1}gx^{\alpha}}=O\left(\frac{x^{\alpha}}{\sqrt[k]{\log x}}\right).$$

Hence, by virtue of $\tau_x=O(x)$

$$\frac{x(x+1)}{2}\log a=O\left(\tau_x^{1+\alpha+1/k}\,\frac{1}{\sqrt[k]{\log x}}\right).$$

Therefore

$$\frac{x(x+1)}{2}\log a=O\left(x^{1+\alpha+1/k}\,\frac{1}{\sqrt[k]{\log x}}\right),\qquad\frac{x(x+1)}{2x^2}\log a\leqslant Cx^{1+\alpha+1/k-2}\,\frac{1}{\sqrt[k]{\log x}}.$$

As $1+\alpha+1/k-2\leqslant 0$, we have $\log a/2\leqslant 0$ — which is a contradiction. The proof is thus finished.

References

[1] P. Erdös, *On the greatest prime factor of* $\prod_{k=1}^{x}f(k)$, J. London Math. Soc. 27 (1952), p. 379-384.

[2] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd. 1, Leipzig u. Berlin 1909.

[3] P. G. Lejeune-Dirichlet, *Vorlesungen über Zahlentheorie*, Braunschweig 1894.

[4] W. Sierpiński, *Teoria liczb*, Warszawa-Wrocław 1950.

[5] И. М. Виноградов, *Основы теории чисел*, Москва - Ленинград 1949.