On B-curvatures of curves on surfaces of the Euclidean space

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Introduction

For curves in metrical spaces (Euclidean, Riemann-spaces, Finsler-spaces, and the most general spaces introduced by Schouten and Haantjes [5], p. 73-75) it is possible to deduce the formulae of Frenet or their generalizations by means of differentiating unit tangent vectors with respect to the length of the arc as parameter.

Hlavatý has introduced an analogue (not a generalization, however) of these formulas for the case of curves in non-metrical spaces L_n (*i. e.* in the spaces X_n in which the principle of parallel displacement of vectors and affinors has been adopted); for this purpose he introduces the notions of the *affine arc* and *affine curvature*.

Following the idea of Hlavatý, S. Gołąb has defined, for curves lying in p-dimensional ($2 \leqslant p \leqslant n-1$) hypersurfaces embedded in L_n, the so called B-curvatures [1], which reflect the fact that the curve is not considered as situated in L_n directly, but as lying in X_p, which is, in turn, embedded in L_n. If we take the space R_n instead of L_n, *i. e.* replace L_n by the (metrical) Euclidean space, we can use the metrical arc instead of the affine one.

The use of the term *B-curvature* is explained by the fact that in the case of the space L_3 these curvatures are obtained by differentiating the field of the bivectors tangent to the surface along the curve.

The object of this paper is to develop the theory of B-curvatures of curves on developable surfaces in the space R_3, and to ascertain what

information about those B-curvatures is sufficient to determine the type of the surface (plane, cylindrical, conical, a surface with an edge of regression).

1. General definition of B-curvatures for curves φ lying on a hypersurface X_p embedded in L_n

An n-dimensional space X_n will be denoted (after Schouten) by L_n [5] if there is defined in this space a geometrical object with n^3 components, $\Gamma^\lambda_{\mu\nu}$, called the parameters of linear displacement. By aid of this object it is possible to define the parallel displacement of vectors along any curve C, and consequently the parallel displacement of all other quantities of affinory character, in particular of the p-vectors.

In the spaces L_n one can also define the *affine arc* of curves in the sense of Pick and Hlavatý [3]. This arc is defined up to the linear transformations, just as the metrical arc in V_n is determined to within translations of the parameter.

If we are given a hypersurface X_p embedded in the space L_n we can associate with every regular point of X_p a simple p-vector (in the case of $p=2$ a simple bivector) tangent to X_p. Let us denote this p-vector by $\boldsymbol{B}$; it is determined up to a non-vanishing scalar factor σ. In particular we can take as $\boldsymbol{B}$

(1) $$\boldsymbol{B}^{\lambda_1\ldots\lambda_p} = \underset{1}{\boldsymbol{B}}^{[\lambda_1}\ldots\underset{p}{\boldsymbol{B}}^{\lambda_p]}$$

setting

(2) $$\underset{a}{\boldsymbol{B}} \overset{\mathrm{df}}{=} \frac{\partial\xi^\nu}{\partial\eta^a}, \qquad \nu=1,2,\ldots,n, \quad a=1,2,\ldots,p,$$

where

(3) $$\xi^\nu=\xi^\nu(\eta^1,\ldots,\eta^p), \qquad \nu=1,2,\ldots,n$$

denote the parametric equations of our X_p.

Now let C be a curve on X_p parametrized by means of the affine arc s; we set

(4) $$\underset{1}{\boldsymbol{B}}=\sigma\boldsymbol{B}$$

where $\boldsymbol{B}$ is defined by formula (1), and σ is a scalar field, not determined for the moment.

Now we define by induction a sequence $\underset{1}{\boldsymbol{B}}, \underset{2}{\boldsymbol{B}}, \underset{3}{\boldsymbol{B}}, \ldots$ of p-vectors

(5) $$\underset{k}{\boldsymbol{B}}=\frac{D\underset{k-1}{\boldsymbol{B}}}{ds}, \qquad k=2,3,\ldots$$

where D/ds denotes the covariant differentiation with respect to the arc s, determined by aid of the parameters Γ.

Let m be a positive integer defined (uniquely) by the following conditions:

I. $\underset{1}{\boldsymbol{B}},\ldots,\underset{m}{\boldsymbol{B}}$ are linearly independent,

II. $\underset{m+1}{\boldsymbol{B}}$ is a linear combination of the p-vectors $\underset{1}{\boldsymbol{B}},\ldots,\underset{m}{\boldsymbol{B}}$.

It may be shown that number m thus defined satisfies the inequalities

$$1\leqslant m\leqslant p(n-p)+1 \tag{6}$$

and does not depend on the parametric representation of the hypersurface X_p; in other words, it is independent of the scalar factor σ of formula (4). Number m may be called the local (since it may depend on the point of the curve C) order of skewness of C with respect to X_p. If, in particular, it is equal to 1 for every point of the curve C, the curve will be said to be B-straight; if $m=2$ it will be termed B-plane...

We assume in the sequel that m is not constant along C. Since $\underset{m+1}{\boldsymbol{B}}$ is a linear combination of the p-vectors $\underset{1}{\boldsymbol{B}},\ldots,\underset{m}{\boldsymbol{B}}$, we may write

$$\underset{m+1}{\boldsymbol{B}}=\sum_{j=1}^{m}\lambda_j\underset{j}{\boldsymbol{B}} \tag{7}$$

where λ_j are scalar functions of the arc s. The factor σ has not been determined yet — now, we shall do it by adding the condition that

$$\lambda_m\equiv 0 \tag{8}$$

along the curve C. For this purpose let us put

$$D^m\boldsymbol{B}=\sum_{j=0}^{m-1}\mu_j D^j\boldsymbol{B}, \tag{9}$$

taking

$$D^{k+1}=D(D^k),\qquad D^1\boldsymbol{B}=D\boldsymbol{B},\qquad D^0\boldsymbol{B}=\boldsymbol{B}. \tag{10}$$

Let us apply the formula of Leibniz

$$\underset{j+1}{\boldsymbol{B}}=\sum_{k=0}^{j}\binom{j}{k}\sigma^{(j-k)}D^k\boldsymbol{B} \tag{11}$$

where $\sigma^{(j)}$ denotes the jth derivative of the function σ with respect to s, and $\sigma^{(0)}=\sigma$. For $j=m$ this gives

$$\underset{m+1}{\boldsymbol{B}}=D\underset{m}{\boldsymbol{B}}=\sum_{k=0}^{m}\binom{m}{k}\sigma^{(m-k)}D^k\boldsymbol{B}. \tag{12}$$

Computing succesively $D^k\boldsymbol{B}$ from the equations (11) we finally obtain

$$D\underset{m}{\boldsymbol{B}}=\sigma D^m\boldsymbol{B}+m\sigma' D^{m-1}\boldsymbol{B}+\ldots \tag{13}$$

To transform the right-hand side of this formula we express the quantities $D^k\boldsymbol{B}$ by $\underset{k}{\boldsymbol{B}},\ldots,\underset{1}{\boldsymbol{B}}$ and calculate the coefficient at $\underset{m}{\boldsymbol{B}}$; after some computations [2] this gives

$$\lambda_m=\mu_{m-1}+\frac{m\sigma'}{\sigma}, \tag{14}$$

whence in virtue of (8)

$$\mu_{m-1}+\frac{m\sigma'}{\sigma}=0, \tag{15}$$

which, finally, leads to

$$\sigma=C_1\exp\left(-\frac{1}{m}\int\mu_{m-1}ds\right),\qquad C_1=\text{const}. \tag{16}$$

Definition. The coefficients $\lambda_1,\ldots,\lambda_{m-1}$ in (7) are called the *B-curvatures of the curve C*, and the integer m — the *order of B-planity of C* [2].

Since the affine arc in L_n is determined to within affine transformations $\bar{s}=\alpha s+\beta$, we infer by a simple and short computation that the B-curvatures are invariants of such kind that if we pass from one affine arc to another by the above formula, they are multiplied by a constant factor equal to a power of α. In the particular case of $L_n=R_n$, the B-curvatures become absolute invariants if we take the metrical arc instead of the affine one.

2. B-curvatures of curves lying on surfaces embedded in three-dimensional spaces

If $n=3$, then $p=2$ and (in virtue of (6)) the curve C may be B-straight ($m=1$), or B-plane ($m=2$), or B-skew ($m=3$). If, moreover, $L_3=R_3$, then instead of the affine arc defined by the integral

$$s=\int\sqrt[6]{\det\left(\boldsymbol{t},\frac{d\boldsymbol{t}}{du},\frac{d^2\boldsymbol{t}}{du^2}\right)}\,du, \tag{17}$$

it is more convenient to introduce the metrical arc

$$s=\int\sqrt{\boldsymbol{t}\cdot\boldsymbol{t}}\,du \tag{18}$$

(where $\boldsymbol{t}$ denotes the vector tangent to the curve: $\boldsymbol{t}=d\boldsymbol{x}/du$, u being the original parameter on the curve, and $\boldsymbol{x}$—the position-vector of the curve C); it is also convenient to replace the bivector

$$\boldsymbol{B}=\frac{\partial\xi^{[\lambda}}{\partial\eta^1}\cdot\frac{\partial\xi^{\mu]}}{\partial\eta^2} \tag{19}$$

by the vector orthogonal to $\boldsymbol{B}$, *i. e.* normal to the surface X_2. Therefore we set

$$\boldsymbol{B}=\left[\frac{\partial \xi^{\lambda}}{\partial \eta^{1}},\frac{\partial \xi^{\lambda}}{\partial \eta^{2}}\right] \tag{20}$$

(where [,] denotes the cross-product), and then

$$\underset{1}{\boldsymbol{B}}\overset{\text{df}}{=}\sigma\boldsymbol{B},\qquad \underset{2}{\boldsymbol{B}}\overset{\text{df}}{=}\frac{d\underset{1}{\boldsymbol{B}}}{ds},\qquad \underset{3}{\boldsymbol{B}}\overset{\text{df}}{=}\frac{d\underset{2}{\boldsymbol{B}}}{ds},\qquad \underset{4}{\boldsymbol{B}}\overset{\text{df}}{=}\frac{d\underset{3}{\boldsymbol{B}}}{ds} \tag{21}$$

(for $L_3=R_3$ the covariant differentiation is identical with the usual one, whence $D/ds=d/ds$).

The normalization of the vector $\boldsymbol{B}$ [formula (20)] by aid of the factor σ (determined by the condition $\lambda_m=0$) does not lead, however, to the same result as the normalization of the vector orthogonal to the surface. This will be proved in the next chapter.

Now we shall discuss in detail the cases of $m=1$, $m=2$, and $m=3$.

I. Let us suppose that the curve C on V_2 is B-straight, *i. e.* that $m=1$ for every point of the curve.

Let the vector $\boldsymbol{B}$ (normal to V_2 along C) have the components (not all equal to zero): $b_1(s), b_2(s), b_3(s)$. Then in virtue of (7), (8) and (21)

$$\sigma' b_i+\sigma b_i'\equiv 0\qquad (i=1,2,3)\qquad \text{along } C. \tag{22}$$

Formula (22) may be considered as a system of three equations with one unknown σ, a non-null solution being sought. Then

$$\sigma=\frac{c_i}{b_i}\qquad (i=1,2,3) \tag{23}$$

where c_i are constants of integration different from 0 (if any b_i were identically 0 along the curve C, then the corresponding equation (22) would be satisfied for every σ). From (23) it follows that the vector $\boldsymbol{B}$ is parallel along the curve C to a fixed line.

The hypothesis that the curve C is B-straight implies that the vector normal to the surface along this curve is parallel to a fixed line. Conversely, if a non-null vector normal to the surface along a certain curve is constantly parallel to a fixed line, then we can determine from (23) the factor σ so that the equations (22) be satisfied, and this means that the curve is B-straight.

Thus we have proved

THEOREM 1. *A necessary and sufficient condition that a curve C lying on a surface V_2 embedded in R_3 be B-straight is that the vector normal to the surface along this curve be parallel to a fixed line.*

For B-straight curves B-curvatures do not exist.

Let us consider the next case.

II. We suppose that C is B-plane, *i. e.* that $m=2$ at every point of the curve. Then by (21)

$$\underset{1}{\boldsymbol{B}}=\sigma\boldsymbol{B},$$

$$\underset{2}{\boldsymbol{B}}=\frac{d\underset{1}{\boldsymbol{B}}}{ds}=\sigma'\boldsymbol{B}+\sigma\frac{d\boldsymbol{B}}{ds}, \tag{24}$$

$$\underset{3}{\boldsymbol{B}}=\frac{d\underset{2}{\boldsymbol{B}}}{ds}=\sigma''\boldsymbol{B}+2\sigma'\frac{d\boldsymbol{B}}{ds}+\sigma\frac{d^2\boldsymbol{B}}{ds^2},$$

but in virtue of (7) and (8)

$$\underset{3}{\boldsymbol{B}}=\lambda_1\underset{1}{\boldsymbol{B}}+\lambda_2\underset{2}{\boldsymbol{B}}\quad\text{and}\quad \lambda_2\equiv 0; \tag{25}$$

hence (24) and (25) enable us to write

$$\lambda_1\sigma\boldsymbol{B}\equiv\sigma''\boldsymbol{B}+2\sigma'\frac{d\boldsymbol{B}}{ds}+\sigma\frac{d^2\boldsymbol{B}}{ds^2} \tag{26}$$

along the curve C. Hence and by (9)

$$\lambda_1\sigma\boldsymbol{B}\equiv(\sigma''+\sigma\mu_0)\boldsymbol{B}+(2\sigma'+\sigma\mu_1)\frac{d\boldsymbol{B}}{ds}. \tag{27}$$

The vectors $\boldsymbol{B}$ and $d\boldsymbol{B}/ds$ being linearly independent (for C is by hypothesis B-straight), formula (27) gives

$$2\sigma'+\sigma\mu_1=0, \tag{28}$$

$$\sigma''+(\mu_0-\lambda_1)\sigma=0. \tag{29}$$

Formula (28) gives

$$\sigma=k\exp\left(-\frac{1}{2}\int\mu_1\,ds\right) \tag{30}$$

(k is a constant of integration). Differentiating this formula twice we get

$$\sigma'=-\frac{1}{2}k\mu_1\exp\left(-\frac{1}{2}\int\mu_1\,ds\right),$$

$$\sigma''=\frac{1}{4}k\mu_1^2\exp\left(-\frac{1}{2}\int\mu_1\,ds\right)-\frac{1}{2}k\mu_1'\exp\left(-\frac{1}{2}\int\mu_1\,ds\right)$$

the substitution of these quantities into (29) gives

$$\lambda_1 = -\frac{1}{2}\mu_1' + \frac{1}{4}\mu_1^2 + \mu_0. \tag{31}$$

The coefficients μ_0 and μ_1 may be determined by aid of (9). Denoting, as above, the components of the vector $\boldsymbol{B}$ by $b_i(s)$ $(i=1,2,3)$, we infer by (9) that

$$\mu_0 b_i + \mu_1 b_i' \equiv b_i'' \qquad (i=1,2,3) \tag{32}$$

along the curve C. This is a system of three equations with two unknowns; by hypothesis it has a solution, whence the determinant $[b_i, b_i', b_i''] \equiv 0$ along the curve C.

We shall prove now that the rank of the matrix $\|b_i, b_i', b_i''\|$ is equal to 2 along C. Indeed, if it were equal to 1, then the vector $\boldsymbol{B}$ normal to the surface along the curve C would be parallel to a fixed line, and this means that C is B-straight, which contradicts our hypothesis.

Conversely, if the rank of the matrix $\|b_i, b_i', b_i''\|$ is equal to 2, then the system (32) has one solution for μ_0 and μ_1. Then we can determine λ_1 and σ from (29) and (31) and the curve C is B-plane. Hence we get

THEOREM 2. *A necessary and sufficient condition for a curve C lying on the surface V_2 embedded in R_3 to be B-plane is that the vector normal to the surface along the curve C would be constantly parallel to some fixed plane, and would not be parallel to any fixed line.*

Any B-plane curve has only one B-curvature expressed by formula (31) if we substitute in it the solutions of (32). To calculate it, let

$$a \overset{\text{df}}{=} b_1 b_2' - b_2 b_1', \tag{33}$$

and suppose that $a \neq 0$ (this is justified by the fact that the rank of the matrix $\|b_i, b_i', b_i''\|$ is equal to 2). Then

$$\mu_1 = \frac{a'}{a}, \qquad \mu_0 = \frac{a_0}{a} \tag{34}$$

where $a_0 \overset{\text{df}}{=} b_1'' b_2' - b_2'' b_1'$. Introducing the expressions of (34) into (31) we finally obtain

$$\lambda_1 = -\frac{1}{2}\cdot\frac{a''}{a} + \frac{3}{4}\cdot\frac{a'^2}{a^2} + \frac{a_0}{a}. \tag{35}$$

This formula expresses the sought B-curvature of a B-plane curve C, a denotes an arbitrary non-null determinant of degree 2, whose elements are components of the vectors $\boldsymbol{B}$ and $d\boldsymbol{B}/ds$, and a_0—the corresponding determinant formed of the components of the vectors $d\boldsymbol{B}/ds$ and $d^2\boldsymbol{B}/ds^2$.

III. Finally let us consider the case of $m=3$. Since every curve C on the surface V_2 embedded in R_3 is B-straight, or B-plane, or B-skew, we get

THEOREM 3. *A necessary and sufficient condition that a curve C lying on the surface V_2 embedded in R_3 be B-skew is that the vector normal to the surface along the curve C would not be parallel to any fixed plane.*

From the above theorems it follows that the degree m of skewness of the curve C lying on V_2 is equal to the rank of the matrix composed of the components of the vector normal to the surface V_2 along the curve, and of the components of the first and the second derivative of this vector.

As we have observed above, B-skew curves have two B-curvatures, λ_1 and λ_2, which we shall compute now. By (21)

$$\begin{aligned}
\underset{1}{\boldsymbol{B}} &= \sigma\boldsymbol{B},\\
\underset{2}{\boldsymbol{B}} &= \frac{d\underset{1}{\boldsymbol{B}}}{ds} = \sigma'\boldsymbol{B} + \sigma\frac{d\boldsymbol{B}}{ds},\\
\underset{3}{\boldsymbol{B}} &= \frac{d\underset{2}{\boldsymbol{B}}}{ds} = \sigma''\boldsymbol{B} + 2\sigma'\frac{d\boldsymbol{B}}{ds} + \sigma\frac{d^2\boldsymbol{B}}{ds^2},\\
\underset{4}{\boldsymbol{B}} &= \frac{d\underset{3}{\boldsymbol{B}}}{ds} = \sigma'''\boldsymbol{B} + 3\sigma''\frac{d\boldsymbol{B}}{ds} + 3\sigma'\frac{d^2\boldsymbol{B}}{ds^2} + \sigma\frac{d^3\boldsymbol{B}}{ds^3}.
\end{aligned} \tag{36}$$

But in virtue of (7)

$$\underset{4}{\boldsymbol{B}} = \frac{d\underset{3}{\boldsymbol{B}}}{ds} = \lambda_1\underset{1}{\boldsymbol{B}} + \lambda_2\underset{2}{\boldsymbol{B}} + \lambda_3\underset{3}{\boldsymbol{B}} \tag{37}$$

whence (8), (36) and (37) imply

$$\lambda_1\sigma\boldsymbol{B} + \lambda_2\left(\sigma'\boldsymbol{B} + \sigma\frac{d\boldsymbol{B}}{ds}\right) = \sigma'''\boldsymbol{B} + 3\sigma''\frac{d\boldsymbol{B}}{ds} + 3\sigma'\frac{d^2\boldsymbol{B}}{ds^2} + \sigma\frac{d^3\boldsymbol{B}}{ds^3}. \tag{38}$$

The formula (9) yields, however,

$$\frac{d^3\boldsymbol{B}}{ds^3} = \mu_0\boldsymbol{B} + \mu_1\frac{d\boldsymbol{B}}{ds} + \mu_2\frac{d^2\boldsymbol{B}}{ds^2}. \tag{39}$$

Substituting (39) into (38), we get

(40)
$$(\lambda_1\sigma + \lambda_2\sigma')\boldsymbol{B} + \lambda_2\sigma\frac{d\boldsymbol{B}}{ds} = (\sigma''' + \sigma\mu_0)\boldsymbol{B} + (3\sigma'' + \sigma\mu_1)\frac{d\boldsymbol{B}}{ds} + (3\sigma' + \sigma\mu_2)\frac{d^2\boldsymbol{B}}{ds^2}$$

whence, the vectors $\boldsymbol{B}, d\boldsymbol{B}/ds$, $d^2\boldsymbol{B}/ds^2$ being linearly independent (f r $m=3$), we deduce further

$$(41)\qquad \lambda_1\sigma+\lambda_2\sigma'=\sigma'''+\sigma\mu_0, \qquad \lambda_2\sigma=3\sigma''+\sigma\mu_1, \qquad 0=3\sigma'+\sigma\mu_2.$$

From equations (41) we determine (just for B-plane curves) λ_1, λ_2 and σ by aid of μ_0, μ_1 and μ_2; whence

$$(42)\qquad \lambda_2=\frac{3\sigma''}{\sigma}+\mu_1 \quad \text{and} \quad \lambda_1=\frac{\sigma'''}{\sigma}-\frac{3\sigma''\sigma'}{\sigma^2}-\frac{\sigma'}{\sigma}\mu_1+\mu_0,$$

$$(43)\qquad \sigma=k_1\exp\left(-\frac{1}{3}\int\mu_2\,ds\right)$$

where k_1 is a constant of integration. Differentiation of (43) gives

$$(44)\qquad \begin{aligned} \sigma' &=-\frac{1}{3}\mu_2\sigma,\\ \sigma'' &=-\frac{1}{3}\mu_2'\sigma+\frac{1}{9}\mu_2^2\sigma,\\ \sigma''' &=-\frac{1}{3}\mu_2''\sigma+\frac{1}{3}\mu_2'\mu_2\sigma-\frac{1}{27}\mu_2^3\sigma. \end{aligned}$$

Introducing these expressions into (42) we obtain

$$(45)\qquad \lambda_2=-\mu_2'+\frac{1}{3}\mu_2^2+\mu_1, \qquad \lambda_1=-\frac{1}{3}\mu_2''+\frac{2}{27}\mu_2^3+\frac{1}{3}\mu_1\mu_2+\mu_0,$$

and developing formula (9) we get

$$(46)\qquad \mu_0 b_i+\mu_1 b_i'+\mu_2 b_i''=b_i''' \qquad (i=1,2,3).$$

This is a system of three equations with three unknowns μ_0, μ_1, μ_2, which we are seeking.

If the curve C is B-skew the system (46) has only one solution (for the determinant $[b_i, b_i', b_i'']\neq 0$ along the curve) expressed by the formulas

$$(47)\qquad \mu_0=\frac{[b_i''', b_i', b_i'']}{[b_i, b_i', b_i'']}=\frac{W_0}{W}, \quad \mu_1=\frac{[b_i, b_i''', b_i'']}{[b_i, b_i', b_i'']}=\frac{W_1}{W}, \quad \mu_2=\frac{[b_i, b_i', b_i''']}{[b_i, b_i', b_i'']}=-\frac{W'}{W}$$

where

$$W\overset{\text{df}}{=}[b_i, b_i', b_i''], \qquad W_0\overset{\text{df}}{=}[b_i''', b_i', b_i''], \qquad W_1\overset{\text{df}}{=}[b_i, b_i''', b_i''].$$

Now let us calculate μ_2' and μ_2'', which appear in formulas (45)

$$(48)\qquad \begin{aligned} \mu_2' &=-\frac{W''}{W}-\frac{W'^2}{W^2},\\ \mu_2'' &=-\frac{W'''}{W}-\frac{W''W'}{W^2}-\frac{2W''W'}{W^2}+\frac{2W'^3}{W^3}=-\frac{W'''}{W}-\frac{3W''W'}{W^2}+\frac{2W'^3}{W^3}. \end{aligned}$$

From formulas (45), (47) and (48) we finally obtain

$$(49)\qquad \begin{aligned} \lambda_2 &=-\frac{W''}{W}+\frac{4}{3}\cdot\frac{W'^2}{W^2}+\frac{W_1}{W},\\ \lambda_1 &=-\frac{1}{3}\cdot\frac{W'''}{W}+\frac{W''W'}{W^2}-\frac{16}{27}\cdot\frac{W'^3}{W^3}+\frac{1}{3}\cdot\frac{W_1W'}{W^2}+\frac{W_0}{W}. \end{aligned}$$

These formulas express the first and the second B-curvature of B-skew curves lying on sufficiently regular surfaces V_2 mbedded in R_3. W denotes the determinant formed of the components of the vectors $\boldsymbol{B}$, $d\boldsymbol{B}/ds$ and $d^2\boldsymbol{B}/ds^2$; W' and W'' denote the first and the second derivative of this determinant with respect to the arc; W_1 is the determinant whose terms are the components of the vectors $\boldsymbol{B}$, $d^3\boldsymbol{B}/ds^3$, and $d^2\boldsymbol{B}/ds^2$, and W_0 is obtained analogously from the vectors $d^3\boldsymbol{B}/ds^3$, $d\boldsymbol{B}/ds$, $d^2\boldsymbol{B}/ds^2$.

3. B-curvatures and the curvatures α, β, γ

We have stated in chapter 2 that the normalization of the vector $\boldsymbol{B}$ (see formula (20)) performed by aid of the factor σ ($\lambda_m\equiv 0$ along the curve) does not lead to the same result as the normalization of the length of the vector normal to the surface. Now we supply the proof of this fact.

Let us associate with every point of the curve C a system of three mutually orthogonal unit vectors: the tangent vector $\boldsymbol{t}_1$, the vector $\boldsymbol{t}_3$ normal to the surface and the vector $\boldsymbol{t}_2$ directed so that the system $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$ is right-handed. For this system of vectors

$$(50)\qquad \boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3,$$

called the system of Darboux, the following well-known relations hold

$$(51)\qquad \begin{aligned} \frac{d\boldsymbol{t}_1}{ds} &=\alpha\boldsymbol{t}_2+\gamma\boldsymbol{t}_3 && \text{where } \alpha \text{ is the geodesic curvature,}\\ \frac{d\boldsymbol{t}_2}{ds} &=-\alpha\boldsymbol{t}_1+\beta\boldsymbol{t}_3 && \text{where } \beta \text{ is the geodesic torsion,}\\ \frac{d\boldsymbol{t}_3}{ds} &=-\gamma\boldsymbol{t}_1-\beta\boldsymbol{t}_2 && \text{where } \gamma \text{ is the normal curvature.} \end{aligned}$$

Besides this system of Darboux, let us associate with every point of the curve C another system $\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3$ of orthogonal vectors defined (when $\nu_1 > 0$) as:

(52) $$\boldsymbol{T}_1 \overset{\mathrm{df}}{=} \boldsymbol{t}_3, \quad \frac{d\boldsymbol{T}_1}{ds} \overset{\mathrm{df}}{=} \nu_1 \boldsymbol{T}_2 \quad \text{where} \quad \nu_1 = \left|\frac{d\boldsymbol{T}_1}{ds}\right| \quad \text{and} \quad \boldsymbol{T}_3 \overset{\mathrm{df}}{=} \boldsymbol{T}_1 \times \boldsymbol{T}_2.$$

Now we shall deduce equations analogous to the formulas of Frenet by differentiating the field of unit vectors *normal to the surface* with respect to the arc as parameter.

The first of these equations is given already by the definition

(53) $$\frac{d\boldsymbol{T}_1}{ds} = \nu_1 \boldsymbol{T}_2 \quad \text{where} \quad \nu_1 = \left|\frac{d\boldsymbol{T}_1}{ds}\right| \quad \text{and} \quad \nu_1 > 0.$$

Let us set

(54) $$\frac{d\boldsymbol{T}_2}{ds} = \varrho_1 \boldsymbol{T}_1 + \varrho_2 \boldsymbol{T}_2 + \varrho_3 \boldsymbol{T}_3,$$

and let us determine the scalar factors $\varrho_1, \varrho_2, \varrho_3$. Since the derivative of any unit vector with respect to the arc is either a null-vector or orthogonal to the vector itself, we may write ($\varrho_2 = 0$)

(55) $$\frac{d\boldsymbol{T}_2}{ds} = \varrho_1 \boldsymbol{T}_1 + \varrho_3 \boldsymbol{T}_3,$$

but $\boldsymbol{T}_1 \boldsymbol{T}_2 = 0$, whence

(56) $$\frac{d\boldsymbol{T}_1}{ds} \boldsymbol{T}_2 + \frac{d\boldsymbol{T}_2}{ds} \boldsymbol{T}_1 = 0,$$

that is

(57) $$\frac{d\boldsymbol{T}_2}{ds} \boldsymbol{T}_1 = -\frac{d\boldsymbol{T}_1}{ds} \boldsymbol{T}_2 = +\varrho_1 = -\nu_1.$$

Further let us set

(58) $$\varrho_3 \overset{\mathrm{df}}{=} \nu_2.$$

Thus the second equation may be written in the form

(59) $$\frac{d\boldsymbol{T}_2}{ds} = -\nu_1 \boldsymbol{T}_1 + \nu_2 \boldsymbol{T}_3.$$

Setting

(60) $$\frac{d\boldsymbol{T}_3}{ds} \overset{\mathrm{df}}{=} \delta_1 \boldsymbol{T}_1 + \delta_2 \boldsymbol{T}_2 + \delta_3 \boldsymbol{T}_3,$$

we likewise obtain

(61) $$\delta_1 = \delta_3 = 0, \qquad \delta_2 = -\nu_2.$$

Hence, finally,

(62) $$\frac{d\boldsymbol{T}_1}{ds} = \nu_1 \boldsymbol{T}_2, \quad \frac{d\boldsymbol{T}_2}{ds} = -\nu_1 \boldsymbol{T}_1 + \nu_2 \boldsymbol{T}_3, \quad \frac{d\boldsymbol{T}_3}{ds} = -\nu_2 \boldsymbol{T}_2$$

where ν_1 and ν_2 are the analogous of the first and the second curvature in the ordinary formulas of Frenet. Let us express these quantities in terms of the curvatures α, β, γ defined by formula (51). By (51) and (52) it follows that

(63) $$\nu_1 = \sqrt{\beta^2 + \gamma^2}.$$

The second of the equations (62) gives

(64) $$\nu_2 = \frac{d\boldsymbol{T}_2}{ds} \boldsymbol{T}_3;$$

by the first we obtain

(65) $$\boldsymbol{T}_2 = \frac{d\boldsymbol{T}_1}{ds} : \left|\frac{d\boldsymbol{T}_1}{ds}\right| = \frac{-\gamma}{\sqrt{\gamma^2+\beta^2}} \boldsymbol{t}_1 - \frac{\beta}{\sqrt{\gamma^2+\beta^2}} \boldsymbol{t}_2,$$

whence, differentiating with respect to s, we get

(66) $$\frac{d\boldsymbol{T}_2}{ds} = \frac{\beta(\gamma\beta' - \gamma'\beta) + \alpha\beta(\gamma^2+\beta^2)}{(\gamma^2+\beta^2)^{3/2}} \boldsymbol{t}_1 - \frac{\gamma(\gamma\beta' - \beta\gamma') + \alpha\gamma(\gamma^2+\beta^2)}{(\gamma^2+\beta^2)^{3/2}} \boldsymbol{t}_2 - \frac{(\gamma^2+\beta^2)^2}{(\gamma^2+\beta^2)^{3/2}} \boldsymbol{t}_3.$$

By formulas (52) and (65)

(67) $$\boldsymbol{T}_3 = \frac{\beta}{\sqrt{\gamma^2+\beta^2}} \boldsymbol{t}_1 - \frac{\gamma}{\sqrt{\gamma^2+\beta^2}} \boldsymbol{t}_2,$$

whence by (64) and (66) and (67) we finally obtain

(68) $$\nu_2 = \frac{\alpha(\gamma^2+\beta^2) + (\beta'\gamma - \beta\gamma')}{\gamma^2+\beta^2}.$$

By (63) and (68), it follows that the new curvatures are expressed in quite a simple manner by α, β and γ.

Our problem is whether or not, in the case of a B-skew curve, the curvatures ν_1 and ν_2 coincide (are identical) with the B-curvatures λ_1, λ_2 of this curve. Therefore let us express λ_1 and λ_2 in terms of α, β, and γ, and let us compare the results.

Notice first that the vectors $\boldsymbol{B}$ and $\boldsymbol{T}_1$ are parallel and that $\boldsymbol{T}_1$ is a unit vector. Let us denote its coordinates by τ_i $(i=1,2,3)$ and set

(69) $$\boldsymbol{B} \stackrel{\mathrm{df}}{=} \omega \boldsymbol{T}_1$$

whence $b_i = \omega\tau_i$ where ω is the scalar field along the curve C. Further, write

$$U = [\tau_i, \tau_i', \tau_i''] \quad \text{where} \quad \tau_i' = \frac{d\tau_i}{ds}, \quad \tau_i'' = \frac{d^2\tau_i}{ds^2},$$

(70) $$U_1 = [\tau_i, \tau_i''', \tau_i''] \quad \text{where} \quad \tau_i''' = \frac{d^3\tau_i}{ds^3}, \quad U' = \frac{dU}{ds},$$

$$U_0 = [\tau_i''', \tau_i', \tau_i''] \quad \text{where} \quad U'' = \frac{d^2U}{ds^2}, \quad U''' = \frac{d^3U}{ds^3}.$$

If the curve C is B-skew, we may write the formulas for the first and the second B-curvature in the form

(71)
$$\lambda_2 = -\frac{U''}{U} + \frac{4}{3}\cdot\frac{U'^2}{U^2} + \frac{U_1}{U},$$
$$\lambda_1 = -\frac{1}{3}\cdot\frac{U'''}{U} + \frac{U''U'}{U^2} - \frac{16}{27}\cdot\frac{U'^3}{U^3} + \frac{1}{3}\cdot\frac{U_1U'}{U^2} + \frac{U_0}{U}.$$

Let the vectors $\boldsymbol{t}_k$ of formula (50) have the coordinates t_{k1}, t_{k2}, t_{k3} $(k=1,2,3)$; then by (51), (52) and (70)

(72)
$$\begin{aligned}
\tau_i &= t_{3i},\\
\tau_i' &= -\gamma t_{1i} - \beta t_{2i},\\
\tau_i'' &= (-\gamma' + \alpha\beta)t_{1i} + (-\beta' - \alpha\gamma)t_{2i} + (-\gamma^2 - \beta^2)t_{3i},\\
\tau_i''' &= (\alpha'\beta + 2\alpha\beta' + \alpha^2\gamma + \beta^2\gamma - \gamma'' + \gamma^3)t_{1i} +\\
&\quad + (-\alpha'\gamma - 2\alpha\gamma' + \alpha^2\beta - \beta'' + \beta\gamma^2 + \beta^3)t_{2i} + (-3\beta'\beta - 3\gamma'\gamma)t_{3i}.
\end{aligned}$$

By aid of these relations, we can compute the values of the determinants U, U_1 and U_0. Thus

(73)
$$\begin{aligned}
U &= \alpha(\beta^2+\gamma^2) + (\beta'\gamma - \beta\gamma'),\\
U' &= \alpha'(\beta^2+\gamma^2) + 2\alpha(\beta'\beta+\gamma'\gamma) + (\beta''\gamma - \beta\gamma''),\\
U'' &= \alpha''(\beta^2+\gamma^2) + 4\alpha'(\beta'\beta+\gamma'\gamma) + 2\alpha(\beta''\beta + \beta'^2 + \gamma''\gamma + \gamma'^2) +\\
&\quad + (\beta'''\gamma + \beta''\gamma' - \beta'\gamma'' - \beta\gamma'''),\\
U''' &= \alpha'''(\beta^2+\gamma^2) + 6\alpha''(\beta'\beta+\gamma'\gamma) + 6\alpha'(\beta''\beta+\beta'^2+\gamma''\gamma+\gamma'^2) +\\
&\quad + 2\alpha(\beta'''\beta + 3\beta''\beta' + \gamma'''\gamma + 3\gamma'\gamma'') +\\
&\quad + (\beta^{\mathrm{IV}}\gamma + 2\beta'''\gamma' - 2\beta'\gamma''' - \beta\gamma^{\mathrm{IV}}),\\
U_1 &= -[\alpha^2+\beta^2+\gamma^2][\alpha(\beta^2+\gamma^2) + (\gamma\beta' - \beta\gamma')] - (\beta''\gamma' - \gamma''\beta') +\\
&\quad + \alpha(\gamma''\gamma + \beta''\beta) - \alpha'(\beta'\beta + \gamma'\gamma) - 2\alpha(\beta'^2 + \gamma'^2) +\\
&\quad + 2\alpha^2(\beta\gamma' - \beta'\gamma),\\
U_0 &= [\alpha'(\beta^2+\gamma^2) + 2\alpha(\beta'\beta+\gamma'\gamma) + (\beta''\gamma - \beta\gamma'')](\beta^2+\gamma^2) -\\
&\quad - 3(\beta'\beta + \gamma'\gamma)[\alpha(\beta^2+\gamma^2) + (\beta\gamma' - \beta'\gamma)].
\end{aligned}$$

Finally, using the formulas (71) and (73), we can write down the relations holding between the B-curvatures of a B-skew curve and the curvatures α, β and γ of formulas (51).

Comparing formulas (63) and (68) with (71), account being taken of (73), we find that the B-curvatures are not the curvatures which may be obtained by the normalization (with respect to the lenght) of the vector normal to the surface. The difference of these two normalizations can be shown in a much simples way than has been done here. We have chosen this way in order to show simultaneously how the B-curvatures of B-skew curves may be expressed by the curvatures α, β and γ.

4. The investigation of the rank of skewness of curves lying on developable surfaces

From Theorem 1 (chapter 2) we know that a curve C lying on a surface V_2 embedded in R_3 is B-straight if and only if the vector $\boldsymbol{B}$ normal to the surface V_2 along this curve is constantly parallel to a fixed line, *i. e.* if the vector $\boldsymbol{T}_1$ (see formula (69)) is constant along C. Hence, by (51) and (52), *a curve C lying on a surface V_2 embedded in R_3 is B-straight if and only if it is the line of curvature and simultaneously the asymptotic line* ($\beta\equiv 0$, $\gamma\equiv 0$) *of the surface* V_2.

Hence we deduce the following corollaries:

a. *All curves lying on a plane are B-straight.*

b. *The generatrices are the only B-straight lines on developable surfaces (the cylinder, the cone, the surface with an edge of regression).*

From Theorem 2 (chapter 2) we know that the curve C lying on a surface V_2 embedded in R_3 is B-plane if and only if the vector $\boldsymbol{B}$ or $\boldsymbol{T}_1$ (formula (69)) normal to the surface V_2 along this curve is parallel to a fixed plane, and is parallel to no fixed line, that is if the vector $\boldsymbol{T}_1$ ($\neq$const) is constantly orthogonal to a fixed vector $\boldsymbol{D}\neq 0$. It follows that the *envelope of the planes tangent to the surface V_2 along B-plane curve C is a cylinder circumscribed on V_2, since every two planes of this family intersect along a line parallel to a fixed vector* $\boldsymbol{D}$.

The above proposition implies the following corollaries:

c. *On a cylindrical surface every curve different from a generatrix is B-plane.*

d. *There are no B-straight lines on a conical surface and on a surface with an edge of regression, since the envelope of planes tangent along an arbitrary curve, different from a generatrix, is identical with the surface itself. All curves lying on these surfaces (except the generatrices) are B-skew.*

From the above considerations, we see that the knowledge of the rank of B-skewness of a curve lying on developable surfaces does not enable us to decide whether the curve lies on a conical surface or on a surface with an edge of regression.

The subject of this paper has been suggested to me by S. Gołąb. I wish to express my sincere gratitude to him for his helpful advice and, criticism during the preparation of this paper.

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Appréciation du domaine d'existence de l'intégrale d'un système involutif d'équations aux dérivées partielles du premier ordre

par W. Pawelski (Gdańsk)

Considérons le système d'équations

(1) $$\frac{\partial z}{\partial t_\alpha}+H_\alpha\left(t_\beta,x_i,\frac{\partial z}{\partial x_i},z\right)=0 \qquad (\alpha,\beta=1,2,\ldots,m;\quad i=1,2,\ldots,n)$$

pour lequel a lieu la condition de compatibilité

(2) $$\frac{\partial H_\beta}{\partial t_\alpha}-H_\alpha\frac{\partial H_\beta}{\partial z}-\frac{\partial H_\alpha}{\partial t_\beta}+H_\beta\frac{\partial H_\alpha}{\partial z}+$$
$$+\sum_{i=1}^{n}\left\{\frac{\partial H_\alpha}{\partial q_i}\left(\frac{\partial H_\beta}{\partial x_i}+q_i\frac{\partial H_\beta}{\partial z}\right)-\frac{\partial H_\beta}{\partial q_i}\left(\frac{\partial H_\alpha}{\partial x_i}+q_i\frac{\partial H_\alpha}{\partial z}\right)\right\}\equiv 0.$$

Supposons que les fonctions H_α de variables réelles t_α, x_i, q_i, z soient de classe C^2 dans l'ensemble

(3) $$|t_\alpha-t_\alpha^0|\leqslant c,\qquad |x_i-x_i^0|\leqslant c,\qquad |z-z_0|\leqslant c,\qquad |q_i-q_i^0|\leqslant c.$$

Soit

(4) $$\omega(x_1,\ldots,x_n)$$

une fonction de classe C^1 dans le cube $|x_i-x_i^0|\leqslant c$. Désignons par M le nombre constant positif tel que

$$1^0\quad |H_\alpha|<M,\qquad \left|\frac{\partial H_\alpha}{\partial x_i}\right|<M,\qquad \left|\frac{\partial H_\alpha}{\partial z}\right|<M,$$

$$\left|\frac{\partial H_\alpha}{\partial q_i}\right|<M,\qquad \left|\frac{\partial^2 H_\alpha}{\partial x_i\partial x_j}\right|<M,\qquad \left|\frac{\partial^2 H_\alpha}{\partial z^2}\right|<M,$$

(5) $$\left|\frac{\partial^2 H_\alpha}{\partial q_i\partial q_j}\right|<M,\qquad \left|\frac{\partial^2 H_\alpha}{\partial x_i\partial z}\right|<M,\qquad \left|\frac{\partial^2 H_\alpha}{\partial z\partial q_i}\right|<M,$$

$$\left|\frac{\partial^2 H_\alpha}{\partial x_i\partial q_j}\right|<M,$$