Travaux cités

[1] Л. В. Канторович и Г. П. Акилов, *Функциональный анализ в нормированных пространствах*, Москва 1959.

[2] М. К. Гаврурин, *Нелинейные функциональные уравнения и непрерывные аналоги итеративных методов*, Известия высших учебных заведений (Математика) 5 (6) (1958), p. 18-30.

[3] В. В. Немыцкий и В. В. Степанов, *Качественная теория дифференциальных уравнений*, Москва-Ленинград 1949.

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A connection between two certain methods of successive approximations in differential equations

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1. The methods we are going to consider are that of Picard and the following one.

Suppose we have a system of ordinary differential equations

$$x' = F(t, x) \tag{1}$$

where $x = (x_1, \dots, x_n)$ and $F(t, x) = \big(f_1(t, x), \dots, f_n(t, x)\big)$ is a continuous vector-function.

Let $x_0(t)$ be the solution of (2)

$$x' = A(t)x \tag{2}$$

satisfying the initial condition

$$x_0(0) = c \quad (c \text{ is a constant vector}), \tag{3}$$

and let $x_m(t)$ $(m = 1, 2, \dots)$ fulfill the conditions

$$x'_m(t) = A(t)x_m(t) + F\big(t, x_{m-1}(t)\big) - A(t)x_{m-1}(t), \qquad x_m(0) = c. \tag{4}$$

$A(t)$, in (2) and (4), is a square matrix and it may be arbitrary.

The sequence

$$x_0(t), x_1(t), \dots \tag{5}$$

defined here is the sequence of successive approximations of the solution of (1) and if it is uniformly convergent in $\langle 0, T)$ then, owing to (4), it tends to the solution of (1) satisfying the initial condition (3).

Notice that if each element of $A(t)$ is identically equal to zero, then the method just defined becomes the well-known Picard method of successive approximations. Thus the method given above may be considered as a generalization or a modification of Picard's method. We will use the abbreviations M.P.M. (modified Picard's method) if we speak about the scheme defined above and P.M. (Picard's method) if we have in mind the classical method.

The P.M. is often practically useless because in many cases the first approximations given by it are too far from the real solution and one must do a lot of computations (however simple) to obtain a sufficiently accurate result. These troubles may be avoided, under favourable circumstances, by applying the M.P.M. provided that $A(t)$ is suitably chosen. Hence in some cases the results obtained by the M.P.M. are much better than those given by the P.M.

The M.P.M. was considered, in much more general form, by T. Ważewski [3]. Ważewski in his note has formulated a general condition guaranteeing the convergence of the M.P.M. S. A. Schelkunoff [2] has discussed the M.P.M. in the case when (1) is linear and he has pointed out that this method is useful in some technical problems. Recently, M. Kwapisz has obtained some results connected with the convergence of the M.P.M. and with the estimate of the error (see [1]).

The author should like to give his thanks to Professor J. Lenkowski who called his attention to the M.P.M.

2. THEOREM 1. *Suppose* (5) *is defined by* (2), (3) *and* (4), *and suppose the matrix* $U(t)$ *is given by*

$$U(0)=I\,, \quad dU(t)/dt=A(t)\,U(t)\,, \tag{6}$$

where I *is the unit matrix.*

Then the one-to-one mapping

$$x=U(t)z\,, \quad t=t \tag{7}$$

carries (5) *into the sequence* $z_0(t), z_1(t), \ldots,$ *where*

$$x_m(t)=U(t)z_m(t) \quad (m=0,1,2,\ldots)\,, \tag{8}$$

and $z_m(t)$ *is the sequence of successive approximations given by the Picard's method for the system resulting from* (1) *by* (7).

Proof. From (6) and (8) we get by derivation

$$x_m'(t)=A(t)\,U(t)z_m(t)+U(t)z_m'(t) \quad (m=0,1,2,\ldots)\,. \tag{9}$$

By (9), (7) and (4) we have

$$U(t)z_m'(t)+A(t)\,U(t)z_m(t)=A(t)\,U(t)z_m(t)+ \\ +F\big(t,\,U(t)z_{m-1}(t)\big)-A(t)\,U(t)z_{m-1}(t).$$

Hence

$$z_m'(t)=U^{-1}(t)\Big(F\big(t,\,U(t)z_{m-1}(t)\big)-A(t)\,U(t)z_{m-1}(t)\Big) \tag{10}$$

and

$$z_0(t)=c\,, \quad z_m(0)=c \quad (m=1,2,\ldots)\,. \tag{11}$$

Now let us put

$$G(t,z)=U^{-1}(t)\big(F(t,\,U(t)z)-A(t)\,U(t)z\big)\,. \tag{12}$$

Using this notation we get, owing to (10) and (11) that

$$z_m'(t)=G\big(t,z_{m-1}(t)\big)\,, \quad z_m(0)=c \quad (m=1,2,\ldots)\,.$$

This together with (11) show that $z_m(t)$ is a sequence of successive approximations obtained by the P.M. for the equation

$$z'=G(t,z)\,. \tag{13}$$

It is easy to verify that system (13) results from (1) by (7). So we find theorem 1 completely proved.

3. It follows from theorem 1 that the M.P.M. is convergent for system (1) provided that the P.M. is convergent for (13).

Schelkunoff, describing the M.P.M., said that in this method we consider a solution of (1) as a solution of a perturbated linear system (2). Owing to our result we may state that the M.P.M. can be considered as the P.M. provided that in the space (t, x) we introduce a system of curvilinear coordinates based on n linearly independent solutions of (2).

4. The first term in (5) is defined as a solution of (2) with the initial condition (3). However, as may be easily seen, theorem 1 remains valid if $x_0(t)$ is an arbitrary vector-function satisfying (3). Thus we can reformulate theorem 1 in a more general way.

THEOREM 2. *Suppose* $x(t)$ *and* $y(t)$ *satisfy the following condition*

$$x(0)=y(0) \quad \text{and} \quad x'(t)=A(t)x(t)+F\big(t,y(t)\big)-A(t)y(t) \tag{14}$$

and suppose the matrix $U(t)$ *is defined by* (6).

Then the one-to-one mapping

$$x=U(t)z\,, \quad t=t$$

carries any pair $x(t), y(t)$ *fulfilling* (14) *into the pair* $u(t), v(t)$, *where*

$$x(t)=U(t)u(t) \quad \text{and} \quad y(t)=U(t)v(t)\,, \tag{15}$$

satisfying the conditions

$$u(0)=v(0) \quad \text{and} \quad u'(t)=G\big(t,v(t)\big)\,, \tag{16}$$

where $G(t,z)$ *is defined by* (12).

5. Now we are going to discuss the general case of the M.P.M. We suppose in the following for the sake of simplicity that x is a real variable, that is $n=1$.

Consider the sequence $x_m(t)$ of successive approximations defined as follows: $x_0(t)$ is an arbitrary function, $x_m(t)$ $(m = 1, 2, \ldots)$ is defined inductively by

$$x_m'(t) = H\big(t, x_m(t), x_{m-1}(t)\big), \qquad x_m(0) = x_0(0) = c \tag{17}$$

where $H(t, x, y)$ is a suitably regular function and c is constant.

Under suitable assumptions concerning function $H(t, x, y)$ the sequence $x_m(t)$ defined by (17) tends to the solution of

$$x' = H(t, x, x). \tag{18}$$

The question arises:

QUESTION 1. Does there exist a one-to-one mapping

$$x = T(t, z), \qquad t = t \tag{19}$$

of the half-plane $t \geqslant 0$, $-\infty < z < +\infty$ onto itself and the equation

$$z' = h(t, z) \tag{20}$$

such that (20) results from (18) by (19), and that the sequence $z_m(t)$ $(m = 0, 1, 2, \ldots)$ determined by

$$x_m(t) = T\big(t, z_m(t)\big), \tag{21}$$

where $x_m(t)$ is defined by (17), is a sequence of successive approximations for (20) given by the P.M., for each constant c and each function $x_0(t)$?

The above question is equivalent to the following one.

QUESTION 2. Does there exist a one-to-one mapping (19) and the function $h(t, z)$ such that for any pair $x(t), y(t)$ satisfying

$$x(0) = y(0) \quad \text{and} \quad x'(t) = H\big(t, x(t), y(t)\big) \tag{22}$$

the functions $u(t)$ and $v(t)$ determined by

$$x(t) = T\big(t, u(t)\big) \quad \text{and} \quad y(t) = T\big(t, v(t)\big) \tag{23}$$

satisfy the conditions

$$u(0) = v(0) \quad \text{and} \quad u'(t) = h\big(t, v(t)\big)? \tag{24}$$

6. In this section we give a result concerning questions 1 and 2. First we prove two lemmas.

LEMMA 1. *Suppose $h(t, z)$ is continuous for $0 \leqslant t$ and $-\infty < z < +\infty$. Then to any numbers u_0, v_0 and $t_0 > 0$ there exist two functions $u(t)$ and $v(t)$ determined on $\langle 0, t_0\rangle$ such that*

$$u(t_0) = u_0, \qquad v(t_0) = v_0 \quad \textit{and} \quad u(0) = v(0) \tag{25}$$

and

$$u'(t) = h\big(t, v(t)\big) \quad \textit{for} \quad 0 \leqslant t \leqslant t_0. \tag{26}$$

Proof. Let $M(c) = \max\limits_{\substack{|z| \leqslant |c| \\ 0 \leqslant t \leqslant t_0}} |h(t, z)|$. $M(c)$ is evidently continuous for $-\infty < c < +\infty$. Consider the one-parameter family of functions $v_c(t)$ $(-\infty < c < +\infty)$ defined as follows

$$v_c(t) = c + (v_0 - c)t/t_0 P(c) \quad \text{for} \quad 0 \leqslant t \leqslant P(c)t_0,$$
$$v_c(t) = v_0 \quad \text{for} \quad P(c)t_0 < t \leqslant t_0,$$

where $P(c) = 1/\big(M(c)+1\big) \leqslant 1$.

Put

$$I(c) = \int_0^{t_0} h\big(t, v_c(t)\big)\, dt.$$

$I(c)$ is continuous and we prove that it is bounded also. Indeed, if $|c| > |v_0|$ then $|v_c(t)| < |c|$ and

$$I(c) \leqslant \int_0^{P(c)t_0} \big|h\big(t, v_c(t)\big)\big| dt + \int_{P(c)t_0}^{t_0} |h(t, v_0)|\, dt \leqslant P(c)t_0 M(c) + \int_0^{t_0} |h(t, v_0)|\, dt$$
$$\leqslant t_0 + \int_0^{t_0} |h(t, v_0)|\, dt.$$

If $|c| \leqslant |v_0|$ then $|v_c(t)| \leqslant |v_0|$ and $I(c)$ is bounded by $t_0 M(v_0)$. Hence $I(c)$ is bounded for all c. Now consider the continuous function

$$J(c) = c + I(c) \quad (-\infty < c < +\infty).$$

Since $I(c)$ is bounded, $\lim\limits_{c \to -\infty} J(c) = -\infty$ and $\lim\limits_{c \to +\infty} J(c) = +\infty$. Therefore the equation

$$J(c) = u_0 \tag{27}$$

has at least one solution. Let c_0 be a solution of (27). Put

$$v(t) = v_{c_0}(t) \quad \text{and} \quad u(t) = c_0 + \int_0^t h\big(t, v_{c_0}(t)\big)\, dt.$$

It is easy to verify that function $u(t)$ and $v(t)$ so defined satisfy (25) and (26). Thus lemma 1 is proved.

LEMMA 2. *Let the function $H(t, x, y)$ be continuous for $t \geqslant 0$, $-\infty < x < +\infty$, $-\infty < y < +\infty$. Suppose there exist functions $T(t, z)$ of class C^1 and $h(t, z)$ continuous for $t \geqslant 0$ and $-\infty < z < +\infty$ such that*

$$x = T(t, z), \qquad t = t$$

is a one-to-one mapping of half-plane $t \geqslant 0$, $-\infty < z < +\infty$ onto itself and that for arbitrary functions $x(t), y(t)$ satisfying (22) the functions $u(t), v(t)$ obtained by (23) satisfy (24).

Then $T(t,z)$ fulfills the following equation

$$H\big(t, T(t,u), T(t,v)\big) = T_t(t,u) + T_z(t,u)h(t,v) \tag{28}$$

for arbitrary u, v and $t > 0$.

Proof. Let u_0, v_0 and t_0 ($t_0 > 0$) be arbitrary constants. By lemma 1 there exist functions $u(t)$, $v(t)$ satisfying (24). Let us put

$$x(t) = T\big(t, u(t)\big) \quad \text{and} \quad y(t) = T\big(t, v(t)\big). \tag{29}$$

Then $x(t)$, $y(t)$ satisfy (22). Indeed, let $x^*(t)$ be a solution of

$$x^{*\prime} = H\big(t, x^*, y(t)\big), \quad x^*(0) = y(0).$$

Evidently $x^*(t)$ exists in some interval $\langle 0, t^*)$. Let $u^*(t)$ be defined by

$$x^*(t) = T\big(t, u^*(t)\big).$$

Functions $x^*(t)$, $y(t)$ satisfy (22), therefore by the assumption concerning $T(t,z)$ functions $u^*(t)$, $v(t)$ satisfy (24). Hence

$$u^{*\prime}(t) = h\big(t, v(t)\big) \quad \text{and} \quad u^*(0) = v(0).$$

The last formulas and the definition of $u(t)$ and $v(t)$ prove that $u^*(t) = u(t)$ and, moreover, that $u^*(t)$ as well as $x^*(t)$ may be continued over the interval $\langle 0, t_0\rangle$ and in consequence

$$x^*(t) = x(t).$$

Thus $x(t)$, $y(t)$ satisfy (22).

Taking now the derivatives of two sides of the first equation of (29) we have

$$x'(t) = T_t\big(t, u(t)\big) + T_z\big(t, u(t)\big)u'(t). \tag{30}$$

By (30), (22) and (24) we get

$$H\Big(t, T\big(t, u(t)\big), T\big(t, v(t)\big)\Big) = T_t\big(t, u(t)\big) + T_z\big(t, u(t)\big)h\big(t, v(t)\big) \quad \text{for} \quad t > 0$$

and therefore

$$H\big(t, T(t,u), T(t,v)\big) = T_t(t,u) + T_z(t,u)h(t,v)$$
$$\text{for} \quad t = t_0, \quad u = u_0, \quad v = v_0.$$

Since t_0, u_0, v_0 are arbitrary constants, the last formula proves lemma 2 completely.

Now we prove the following theorem.

THEOREM 3. *Let the functions $\varphi(t,x)$ and $\psi(t,y)$ be of class C^1 and let for every $t_0 > 0$ $\psi_y(t,y)$ be not identically zero. Suppose there exist functions $T(t,z)$ and $h(t,z)$ of class C^1 and suppose*

$$T_z(t,z) \neq 0 \quad \text{for} \quad t \geqslant 0 \quad \text{and} \quad -\infty < z < +\infty.$$

Further suppose that the one-to-one mapping

$$x = T(t,z), \quad t = t$$

carries the half-plane $t \geqslant 0$, $-\infty < z < +\infty$ onto itself as well as any pair of functions $x(t)$, $y(t)$ such that

$$x(0) = y(0) \quad \text{and} \quad x'(t) = \varphi\big(t, x(t)\big) + \psi\big(t, y(t)\big)$$

into the pair $u(t)$, $v(t)$ satisfying the conditions

$$u(0) = v(0) \quad \text{and} \quad u'(t) = h\big(t, v(t)\big).$$

Then the function $T(t,z)$ is linear with respect to z as well as $\varphi(t,x)$ is linear with respect to x.

Proof. By the assumptions of theorem 3 and by lemma 2 we learn that $T(t,z)$ fulfills the equation

$$\varphi\big(t, T(t,u)\big) + \psi\big(t, T(t,v)\big) = T_t(t,u) + T_z(t,u)h(t,v) \tag{31}$$

for any u, v and $t > 0$.

Let us fix $t = t_0$. Denote by

$$s(v) = \psi\big(t_0, T(t_0, v)\big), \quad r(v) = h(t_0, v), \tag{32}$$

$$A(u) = T_z(t_0, u), \quad B(u) = T_t(t_0, u) - \varphi\big(t_0, T(t_0, u)\big). \tag{33}$$

Using the above notations, (31) may be rewritten in the following form

$$s(v) = A(u)r(v) + B(u). \tag{34}$$

Because $\psi_y(t_0, y)$ is not identically equal to zero, therefore $s(v)$ is not constant. Hence the diagram of

$$s = s(v), \quad r = r(v), \quad -\infty < v < +\infty \tag{35}$$

contains at least two different points. It follows from (34) that the diagram of (35) lies on the straight line. But the set containing at least two different points cannot lie simultaneously on two different straight lines, thus

$$A(u) = \text{const} \quad \text{and} \quad B(u) = \text{const}.$$

This and (33) imply that $T_z(t,z)$ as well as $T_t(t,z) - \varphi\big(t, T(t,z)\big)$ does not depend on z. Hence

$$T_z(t,z) = \alpha(t), \quad T_t(t,z) - \varphi\big(t, T(t,z)\big) = \beta(t), \tag{36}$$

and in consequence

$$T(t, z) = a(t)z + \gamma(t) . \tag{37}$$

By (37) and by the second condition of (36) we get that $\varphi(t, x)$ is linear with respect to x. So theorem 3 is completely proved.

7. Remark 1. Notice that if $H(t, x, y) = a(t)x + \psi(t, y)$, then the function

$$T(t, z) = z\exp\Big(\int_0^t a(t)\,dt\Big)$$

fulfills (28) for any v, u if we put $h(t, z) = \exp\big(-\int_0^t a(t)\,dt\big)\psi\big(t, z\exp\big(\int_0^t a(t)\,dt\big)\big)$. Hence it follows from theorem 3, if we restrict ourselves to the case when $H(t, x, y) = \varphi(t, x) + \psi(t, y)$, where $\psi(t, y)$ depends essentially on y, that the answer to question 2 is positive if and only if $\varphi(t, x) = a(t)x$. Therefore we may suspect that there does not exist any theorem analogous to theorem 2 concerning the general case of the M.P.M.

8. The example we are going to present in this section shows that the case $H(t, x, y) = a(t)x + \psi(t, y)$ is not the only one when the answer to question 2 is positive; in other words, we give a function $H(t, x, y)$ different from that of the form $a(t)x + \psi(t, y)$ for which equation (28) admits a solution.

Suppose

$$H(t, x, y) = \varphi(x)\psi(t, y)$$

where $\varphi(x)$ is continuous and positive for $-\infty < x < +\infty$, and

$$\int_0^{+\infty} dx/\varphi(x) = \int_{-\infty}^0 dx/\varphi(x) = +\infty .$$

$\psi(t, y)$ is continuous for $t \geqslant 0$ and $-\infty < y < +\infty$.

Define $T(z)$ as a solution of the equation

$$T'(z) = \varphi\big(T(z)\big) , \qquad T(0) = 0$$

and put

$$h(t, z) = \psi\big(t, T(z)\big) .$$

Then one can easily verify that $T(z)$ and $h(t, z)$ satisfy the following equation, the special case of (28),

$$\varphi\big(T(u)\big)\psi\big(t, T(v)\big) = T'(u)h(t, v) .$$

Therefore, the sequence of successive approximations $x_m(t)$ obtained by the M.P.M. for the differential equation

$$x' = \varphi(x)\psi(t, x)$$

such that

$$x_m'(t) = \varphi\big(x_m(t)\big)\psi\big(t, x_{m-1}(t)\big) , \qquad x_m(0) = x_0(0) \qquad (m = 0, 1, \ldots)$$

$x_0(t)$—arbitrary function, may be carried by the one-to-one mapping

$$x = T(z) , \qquad t = t$$

into the sequence of successive approximations given by the P.M. for the equation

$$z' = h(t, z) = \psi\big(t, T(z)\big) .$$

Some of these results have been presented in a talk given by the author at the Polish Mathematical Society, Kraków Branch, on January 20, 1959.

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References

[1] M. Kwapisz, *O zmodyfikowanej metodzie kolejnych przybliżeń rozwiązywania zwyczajnych równań różniczkowych liniowych rzędu n* (in Polish), Zeszyty Naukowe Politechniki Gdańskiej, Łączność, nr 3 (1960).

[2] S. A. Schelkunoff, *Solution of linear and slightly non-linear differential equations*, Quart. of Appl. Math. 3 (1949), p. 348-355.

[3] T. Ważewski, *Sur la méthode des approximations successives*, Annal. de la Soc. Polon. de Math. 16 (1937), p. 214-215.

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